

## RIGIDITY THEOREMS OF SOME DUALLY FLAT FINSLER METRICS AND ITS APPLICATIONS

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ABSTRACT. In this paper, we study a class of Finsler metric. First, we find some rigidity results of the dually flat  $(\alpha, \beta)$ -metric where the underline Riemannian metric  $\alpha$  satisfies nonnegative curvature properties. We give a new geometric approach of the Monge-Ampère type equation on  $R^n$  by using those results. We also get the non-existence of the compact globally dually flat Riemannian manifold.

### 1. Introduction

Dually flat metrics, which first appeared in information geometry, has been introduced into the Finsler geometry by Z. Shen in 2006 [8]. After that, X. Cheng, Z. Shen and Y. Zhou gave equivalent equations of the locally dually flat Randers metric and the classification of locally dually flat Rander metrics with isotropic S-curvature or weakly isotropic flag curvature [5]. Meanwhile, C. Yu studied it from the navigation point of view. With some special deformations, he pointed out that every dually flat Randers metric always arises from some locally dually flat Riemannian metric and a 1-form that is dually related to this Riemannian metric [11]. On the other hand, Qiaoling Xia has studied this problem for  $(\alpha, \beta)$ -metrics. She proved the following:

**Theorem 1.1** ([10]). *Let  $F = \alpha\phi(s)$ ,  $s = \frac{\beta}{\alpha}$ , be a non-Riemannian  $(\alpha, \beta)$ -metric on an  $n$ -dimensional manifold  $M^n (n \geq 3)$ , where  $\alpha = \sqrt{a_{ij}(x)y^i y^j}$  and  $\beta = b_i(x)y^i \neq 0$ . If  $\phi$  satisfies  $\phi'(0) \neq 0$ ,  $s(k_2 - k_3 s^2)(\phi\phi' - s\phi'^2 - s\phi\phi'') - (\phi'^2 + \phi\phi'') + k_1\phi(\phi - s\phi') \neq 0$ , or  $\phi'(0) = \phi''(0) = 0$  or  $\phi(s)$  is a polynomial*

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of  $s$  with  $\phi'(0) = 0$ . Then  $F$  is locally dually flat if and only if  $\alpha$  and  $\beta$  satisfy

$$(1) \quad G_\alpha^m = \frac{1}{3}[2\theta y^m + \theta^m \alpha^2],$$

$$(2) \quad r_{00} = \frac{2}{3}[\theta\beta - \theta_l b^l \alpha^2],$$

$$(3) \quad s_{i0} = \frac{1}{3}(\beta\theta_i - \theta b_i),$$

where  $\theta = \theta_i(x)y^i$  is a 1-form on  $M$  and  $\theta^m = a^{lm}\theta_l$ .

Note that such metric class excludes the Randers metric, but include some other famous  $(\alpha, \beta)$ -metrics. We need to remark here that there is another theorem in [10] including the Randers case but we are not concerned with it in this paper. Recently, the authors focused on generalized Kropina metrics which can be considered as some special kinds of Finsler metrics with singularity [9], which are not considered in [10].

Based on [10] and [9], we are going to investigate the PDEs in Theorem 1.1. These equations can be improved to get more information. Firstly, we can rewrite those equations to describe the Riemannian metric and the 1-form respectively. Then we can easily get that the 1-form in Theorem 1.1 is closed and it is indeed completely determined by the Riemannian metric  $\alpha$  once it's given. See Lemma 3.4 in Section 3. Secondly, curvature conditions of  $\alpha$  give more informations about rigidity properties of  $(\alpha, \beta)$ -metrics. Some rigidity theorems have been discussed in [9] already, by using different conditions of  $\beta$ . In this paper, we concern how the Riemannian metric  $\alpha$  influence the dually flat property. Such as:

**Theorem 1.2.** *Let  $F = \alpha\phi(s)$  be a non-Riemannian  $(\alpha, \beta)$ -metric on an  $n$ -dimensional manifold  $M$  as described in Theorem 1.1. If the scalar curvature of Riemannian metric  $\alpha$  is nonnegative, then  $F$  is locally dually flat if and only if  $\alpha$  is an Euclidean metric and  $\beta$  is a constant 1-form.*

More theorems and details will be given in Section 4.

At last, we apply dually flat  $(\alpha, \beta)$ -metrics to solve the Monge-Ampère type equation on  $\mathbb{R}^n$  with the assumption of the solution being convex.

**Theorem 1.3** (Bernstein type theorem). *Any convex solution  $f \in C^2(\mathbb{R}^n)$  of the Monge-Ampère equation  $\det(\frac{\partial^2 f}{\partial x^i \partial x^j}) = C$  on  $\mathbb{R}^n$  with  $C$  being a non-zero constant is given by*

$$(4) \quad f(x) = x^T A x + \langle B, x \rangle + D,$$

where  $A$  is a constant positive definite matrix,  $B$  and  $D$  are two constant vectors on  $\mathbb{R}^n$  and  $\langle \cdot, \cdot \rangle$  is the standard Euclidean inner product.

The convexity assumption is natural in geometry and somehow accustomed in the research of nonlinear partial differential equations. Such rigidity theorem

is called Bernstein type theorem for the Monge-Ampère equation which was first proved by Jorgens in 1954 in the case of dimension 2 and by Calabi in 1958 for dimension less than 6 [2]. Then Pogorelov solved it in full generality in 1972 [7]. So this result is also called the J-C-P Theorem of Monge-Ampère equation. It was then re-proved and generalized by some other people [1, 6]. Here we just consider the basic J-C-P Theorem case and give a new approach to the Bernstein type problem. It is the first time that the Finsler geometry is used to solve a nonlinear equation problem. Meanwhile we get a non-existence theorem on compact globally dually flat manifolds.

**Theorem 1.4.** *There is no compact globally dually flat Riemannian manifold.*

This shows that such concept is strongly dependent on the topology of the manifold.

### 2. Preliminaries

Let  $(M, F)$  be an oriented Finsler manifold.  $F$  is actually defined on  $TM$  and smooth on  $TM \setminus \{0\}$ , i.e.,  $F = F(x, y)$ . We call  $F$  a Riemannian metric if  $F = \sqrt{a_{ij}(x)y^i y^j}$ , where the fundamental tensor  $a_{ij} = \frac{1}{2} \frac{\partial^2 F}{\partial y^i \partial y^j}$  is independent of the tangent coordinates  $y$ . If  $F = F(y)$  is independent of the point coordinates  $x$ ,  $F$  is called a Minkowskian metric. Moreover, if  $a_{ij}$  is independent of both the point  $x$  and  $y$ , we say  $F$  is an Euclidean metric.

The  $(\alpha, \beta)$ -metric is important in Finsler geometry. It is not only computable, but also has fascination relationships with the Riemannian metric and the symmetry. The expression of such metric is

$$F = \alpha\phi(s), \quad s := \frac{\beta}{\alpha}.$$

In local coordinates,  $\alpha = \sqrt{a_{ij}(x)y^i y^j}$  is a positive definite Riemannian metric and  $\beta = b_i(x)y^i$  is a 1-form.  $\phi(s)$  is a  $C^\infty$  positive function on an open interval  $(-b_0, b_0)$  satisfying the following inequality to make sure the positivity of the metric.

$$\phi(s) - s\phi'(s) + (b^2 - s^2)\phi''(s) > 0, \quad |s| \leq b < b_0,$$

where  $b := \|\beta\|_\alpha$ . It is known that a Finsler metric  $F = \alpha\phi(s)$  is positive definite if and only if  $\|\beta\|_\alpha < b_0$  for any  $x \in M$ . However, the condition excludes some important metrics with singularity such as the Kropina metric  $F = \frac{\alpha^2}{\beta}$ , and the generalized Kropina metric  $F = \frac{\alpha^{m+1}}{\beta^m}$ . So in this paper the  $(\alpha, \beta)$ -metric also includes the non-positive definite generalized Kropina metric.

As pointed out in [8], a Finsler metric  $F = F(x, y)$  is dually flat if and only if on an open subset  $U \subset \mathbb{R}^m$

$$(5) \quad (F^2)_{x^l y^k} y^l - 2(F^2)_{x^k} = 0.$$

Equation (5) is also considered as the definition of locally dually flat Finsler metrics in local coordinates. Using it, one can deduce equivalent PDEs that

characterize locally dually flat Randers metrics [5], Kropina metrics [9] and even  $(\alpha, \beta)$ -metrics [10]. This leads to Theorem 1.1.

Associated with any Finsler metric  $F$ , there is a refer field on  $TM$ ,

$$G = y^i \frac{\partial}{\partial x^i} - 2G^i \frac{\partial}{\partial y^i},$$

which is locally defined by

$$G^i(x, y) = \frac{1}{4} g^{ij} \{ [F^2]_{x^k y^j} y^k - [F^2]_{x^j} \}.$$

$G$  is called the spray of  $F$ . By the spray coefficients, one can define the geodesic of  $F$  by

$$\frac{d^2 x^i}{dt^2} + 2G^i(x, \frac{dx}{dt}) = 0.$$

So  $G^i$  are also called geodesic coefficients of  $F$  sometimes. As well known, if  $F$  is a Riemannian metric, i.e.,  $F = \sqrt{a_{ij}(x)y^i y^j}$ , then  $G^i = \frac{1}{2} \Gamma_{jk}^i(x) y^j y^k$ , where  $\Gamma_{jk}^i$  are components of the second Christoff symbol of the Riemannian metric  $a_{ij}$ .

Following symbols are usually used in the discussion of  $(\alpha, \beta)$ -metrics.

$$r_{ij} = \frac{1}{2}(b_{i|j} + b_{j|i}), \quad s_{ij} = \frac{1}{2}(b_{i|j} - b_{j|i}),$$

where “|” denotes the covariant derivative with respect to the Levi-Civita connection of  $\alpha$ . Denote

$$\begin{aligned} r_j^i &:= a^{ik} r_{kj}, & r_j &:= b^i r_{ij}, & r_0 &:= r_j y^j = r_{ij} b^i y^j, & r_{00} &:= r_{ij} y^i y^j, \\ s_j^i &:= a^{ik} s_{kj}, & s_j &:= b^i s_{ij}, & s_0 &:= s_j y^j = s_{ij} b^i y^j, \end{aligned}$$

where  $(a^{ij}) := (a^{-1})_{ij}$  are components of the inverse matrix, and  $b^i := a^{ij} b_j$ .

We call  $\beta$  is a closed 1-form if  $s_{ij} = 0$  and a parallel 1-form if  $r_{ij} + s_{ij} = 0$ , for all  $i, j \in \{1, 2, \dots, n\}$ .

### 3. Dually flat $(\alpha, \beta)$ -metrics

In this section, we will deduce some properties of locally dually flat  $(\alpha, \beta)$ -metrics.

In an adapted coordinate system, the fundamental tensor of a *Locally dually flat Riemannian metrics* can be expressed in terms of a smooth function, i.e.,

$$g_{ij} = \frac{\partial^2 \varphi}{\partial x^i \partial x^j}(x),$$

where the function  $\varphi = \varphi(x)$  is a smooth function on the manifold  $M$  [4]. A *Finsler metric  $F$  is locally dually flat* if in an adopted coordinate system, the spray coefficients are given by

$$G^i = -\frac{1}{2} g^{ij} H_{y^i},$$

where  $H = H(x, y)$  is a  $C^\infty$  function on the punched manifold  $TM \setminus \{0\}$ , with the homogeneity of degree three:  $H(x, \lambda y) = \lambda^3 H(x, y)$ . The *Funk metric*, which is a special Randers metric, defined on the unit ball is the first non-Riemannian dually flat metrics' example

$$F = \frac{\sqrt{(1 - |x|^2)|y|^2 + \langle x, y \rangle^2}}{1 - |x|^2} \pm \frac{\langle x, y \rangle}{1 - |x|^2}.$$

There are many results and examples of Randers metrics, see [4, 11].

From Theorem 1.1 and [9], we can obtain that if an  $(\alpha, \beta)$ -metric  $F = \alpha\phi(s)$  satisfies

$$s(k_2 - k_3 s^2)(\phi\phi' - s\phi'^2 - s\phi\phi'') - (\phi'^2 + \phi\phi'') + k_1\phi(\phi - s\phi') \neq 0 \text{ and } \phi'(0) \neq 0,$$

or

$$\phi'(0) = \phi''(0) = 0,$$

or

$$\phi(s) \text{ is a polynomial of } s \text{ with } \phi'(0) = 0,$$

then it must admit the equations (1), (2) and (3). Combining (2) and (3) we can get

$$b_{i|j} = \frac{2}{3}b_j\theta_i - \frac{2}{3}\theta_m b^m a_{ij},$$

where  $\theta_i = \theta_i(x) = \frac{\partial\theta}{\partial y^i}$  and  $b^i = a^{ij}(x)b_j$ . It follows by taking the derivation of (1) that

$$(6) \quad \Gamma_{ij}^l = \frac{2}{3}(\theta^l a_{ij} + \theta_i \delta_j^l + \theta_j \delta_i^l).$$

Rotating the indexes and adding them up, we obtain

$$\frac{\partial a_{ij}}{\partial x^l} = \frac{4}{3}(\theta_l a_{ij} + \theta_i a_{lj} + \theta_j a_{li}).$$

So it is obviously to get that:

**Lemma 3.1.** *The results in Theorem 1.1 is equal to in a local adapt coordinates,  $\alpha$  and  $\beta$  satisfy*

$$(7) \quad \frac{\partial a_{ij}}{\partial x^l} = \frac{4}{3}(\theta_l a_{ij} + \theta_i a_{lj} + \theta_j a_{li}),$$

$$(8) \quad b_{i|j} = \frac{2}{3}b_j\theta_i - \frac{2}{3}\theta_m b^m a_{ij},$$

where  $\theta = \theta_i(x)y^i$  is a 1-form on  $M$  and  $b^m = a^{lm}b_l$ .

*Remark 3.2.* The function  $\phi$  given in Theorem 1.1 satisfied by a lot of well known metrics. For example the Matsumoto metric  $F = \frac{\alpha^2}{\alpha - \beta}$ , the quadratic metric  $\frac{(\alpha + \beta)^2}{\alpha}$ , the Kropina metric  $\frac{\alpha^2}{\beta}$ , most polynomial metrics, the exponential metric and so on. But the Randers metric is excluded.

It's obviously that the equivalent equations of dually flat  $(\alpha, \beta)$ -metrics are independent of  $\phi$  from (7) and (8). The fact means the dually flat property is independent of  $\phi$  once it satisfies the condition in Theorem 1.1. From (8), as discussing in [9], we get the following:

**Corollary 3.3.** *Let  $F = \alpha\phi(s)$  be a non-Riemannian  $(\alpha, \beta)$ -metric as described in Theorem 1.1. If  $\beta$  is a conformal 1-form,  $F$  is locally dually flat if and only if  $\alpha$  is an Euclidean metric and  $\beta$  is a constant 1-form.*

By Lemma 3.1, one can get the following property by contracting (7) with  $a^{ij}$ .  $\theta_i$  comes from not only the derivation of 1-form  $\theta$  with respect to  $y^l$  but also the determination of the metric matrix on manifold.

**Lemma 3.4.**  *$\theta$  in Theorem 1.1 or Theorem 3.1 is a globally closed 1-form. More precisely, in an adapted coordinate system*

$$(9) \quad \theta_l = \frac{3}{2(n+2)} \frac{\partial}{\partial x^l} \log \sqrt{\det(a_{ij})}.$$

The 1-form  $\theta$  given in Theorem 1.1 or Theorem 3.1 is obviously locally defined. One can easily verify that the expression in Lemma 3.4 is independent of coordinate transformation. This gives us a way to define globally dually flat metrics. Although  $\theta$  is a global 1-form, we just express it as the logarithm of the determination of  $a_{ij}$  in the adapted coordinates. This lemma indicates that the factor  $\theta$  is already determined once the  $(\alpha, \beta)$ -metric is given. It only depends on the Riemannian metric  $\alpha$ . Since  $\theta$  is a global form, we can define the globally dually flat as a concept mentioned in the introduction. More precisely,

**Definition 3.5.** A Riemannian metric  $(M, g)$  is called globally dually flat if there is a function  $f \in C^2(M)$  such that at any point  $p \in M$ , the metric can be expressed by

$$g = Hess(f).$$

We also get the following lemma from Theorem 3.1 and (6).

**Lemma 3.6.** *In local coordinates, it follows from the direct computation that*

$$(10) \quad \frac{\partial b_i}{\partial x^j} = \frac{2}{3}(2b_j\theta_i + \theta_j b_i),$$

$$(11) \quad \frac{\partial b^i}{\partial x^j} = -\frac{2}{3}(\theta_j b^i + 2(\theta_m b^m)\delta_j^i),$$

$$(12) \quad \frac{\partial \theta_l}{\partial x^j} = \theta_{l|j} + \frac{2}{3}(\theta_m \theta^m a_{lj} + 2\theta_l \theta_j),$$

$$(13) \quad \frac{\partial \theta^i}{\partial x^i} = \theta^i_{|i} - \frac{2}{3}(n+2)\theta_m \theta^m.$$

4. Rigidity theorems on dually flat  $(\alpha, \beta)$ -manifolds

In this section we deduce some rigidity theorems of  $(\alpha, \beta)$ -metrics satisfying the inequality in Theorem 1.1. Let us first consider the scalar curvature of the underline Riemannian metric.

**Theorem 4.1.** *Let  $F = \alpha\phi(s)$  be a non-Riemannian  $(\alpha, \beta)$ -metric on an  $n$ -dimensional manifold  $M$  as described in Theorem 1.1. If the scalar curvature of Riemannian metric  $\alpha$  is nonnegative, then  $F$  is locally dually flat if and only if  $\alpha$  is an Euclidean metric and  $\beta$  is a constant 1-form.*

By (6) and the expression of the sectional curvature of a Riemannian metric, we have

$$\begin{aligned}
 R_{jkl}^i &= \frac{\partial \Gamma_{jl}^i}{\partial x^k} - \frac{\partial \Gamma_{jk}^i}{\partial x^l} + \Gamma_{hk}^i \Gamma_{jl}^h - \Gamma_{hl}^i \Gamma_{jk}^h \\
 &= \frac{2}{3} \left( \frac{\partial \theta^i}{\partial x^k} a_{jl} + \theta^i \frac{\partial a_{jl}}{\partial x^k} + \frac{\partial \theta_j}{\partial x^k} \delta_l^i + \frac{\partial \theta_l}{\partial x^k} \delta_j^i \right) \\
 &\quad - \frac{2}{3} \left( \frac{\partial \theta^i}{\partial x^l} a_{jk} + \theta^i \frac{\partial a_{jk}}{\partial x^l} + \frac{\partial \theta_j}{\partial x^l} \delta_k^i + \frac{\partial \theta_k}{\partial x^l} \delta_j^i \right) \\
 (14) \quad &+ \frac{4}{9} (\theta^i a_{hk} + \theta_h \delta_k^i + \theta_k \delta_h^i) (\theta^h a_{jl} + \theta_j \delta_l^h + \theta_l \delta_j^h) \\
 &\quad - \frac{4}{9} (\theta^i a_{hl} + \theta_h \delta_l^i + \theta_l \delta_h^i) (\theta^h a_{jk} + \theta_j \delta_k^h + \theta_k \delta_j^h) \\
 &= \frac{2}{3} \left[ \frac{\partial \theta^i}{\partial x^k} a_{jl} - \frac{\partial \theta^i}{\partial x^l} a_{jk} + \left( \frac{\partial \theta_j}{\partial x^k} \delta_l^i - \frac{\partial \theta_j}{\partial x^l} \delta_k^i \right) + \delta_j^i \left( \frac{\partial \theta_l}{\partial x^k} - \frac{\partial \theta_k}{\partial x^l} \right) \right] \\
 &\quad + \frac{4}{9} [(\theta_h \theta^h) (a_{jl} \delta_k^i - a_{jk} \delta_l^i) + \theta_j (\theta_l \delta_k^i - \theta_k \delta_l^i) + \theta^i (\theta_k a_{jl} - \theta_l a_{jk})].
 \end{aligned}$$

With the definition of Ricci curvature we obtain

$$\begin{aligned}
 R_{jl} &= R_{jl}^i \\
 &= \frac{2}{3} \left[ \frac{\partial \theta^i}{\partial x^i} a_{jl} + \frac{\partial \theta_l}{\partial x^j} - n \frac{\partial \theta_j}{\partial x^l} - \frac{\partial \theta^i}{\partial x^l} a_{ij} \right] \\
 (15) \quad &+ \frac{4}{9} [n(\theta_m \theta^m) a_{jl} + (n - 2)\theta_j \theta_l] \\
 &= \frac{2}{3} \left[ \frac{\partial \theta^i}{\partial x^i} a_{jl} + \frac{\partial \theta_l}{\partial x^j} - (n + 1) \frac{\partial \theta_j}{\partial x^l} \right] + \frac{4}{9} [(n + 2)(\theta_m \theta^m) a_{jl} + (n + 2)\theta_j \theta_l],
 \end{aligned}$$

where we have already used (7) at the second equality.

Noticing  $R_{jl} = R_{lj}$ , we get  $\frac{\partial \theta_l}{\partial x^j} = \frac{\partial \theta_j}{\partial x^l}$ , which also can be obtained from Lemma 3.4. Then (15) becomes

$$(16) \quad R_{jl} = \frac{2}{3} \left( \frac{\partial \theta^i}{\partial x^i} a_{jl} - n \frac{\partial \theta_l}{\partial x^j} \right) + \frac{4(n + 2)}{9} (\theta_m \theta^m a_{jl} + \theta_j \theta_l).$$

The scalar curvature of  $\alpha$  is

$$\begin{aligned}
 (17) \quad R = R_j^j &= \frac{2}{3} \left( n \frac{\partial \theta^i}{\partial x^i} - n \frac{\partial \theta_l}{\partial x^j} a^{lj} \right) + \frac{4(n+2)(n+1)}{9} \theta_m \theta^m \\
 &= -\frac{4(n-1)(n+2)}{9} \theta_m \theta^m.
 \end{aligned}$$

So, if  $R \geq 0$ , then  $\theta_m \theta^m = 0$ , i.e.,  $\theta = 0$ . Plugging it back into (7) and (8), one can obtain  $\frac{\partial a_{ij}}{\partial x^i} = 0$  and  $b_{i|j} = 0$  for any  $i, j, l \in \{1, 2, \dots, n\}$ . The first equality means  $a_{ij}$  are independent of  $x$ , hence  $\alpha$  is an Euclidean metric. The second one means that  $\beta$  is parallel with respect to  $\alpha$ , hence it is a constant 1-form here.  $\square$

Theorem 4.1 means the scalar curvature condition can restrict the form of  $(\alpha, \beta)$ -metrics sharply. Now we turn to the space form. We need to constrain the norm of the gradient of the function  $f = \log \det a_{ij}$ . Before presenting the next theorem, we cite the following proposition [3], which is used in the next theorem. In what follows,  $L^1(M)$  stands for the space of Lebesgue integrable functions on  $M$ .

**Proposition 4.2** ([3]). *Let  $X$  be a smooth vector field on the  $n$  dimensional complete, noncompact, oriented Riemannian manifold  $M^m$ , such that  $\operatorname{div} X$  does not change sign on  $M$ . If  $|X| \in L^1(M)$ , then  $\operatorname{div} X = 0$  on  $M$ .*

**Theorem 4.3.** *Let  $F = \alpha\phi(s)$  be a non-Riemannian  $(\alpha, \beta)$ -metric on an  $n$ -dimensional manifold  $M$  as described in Theorem 1.1. Suppose the Riemannian metric  $\alpha$  has constant curvature and the gradient of  $f = \log \det(a_{ij})$  satisfies  $|\nabla f| \in L^1(M)$ , then  $F$  is locally dually flat if and only if  $\alpha$  is an Euclidean metric and  $\beta$  is a constant 1-form.*

*Proof.* Suppose that the Riemannian metric has constant curvature  $K$ , i.e.,  $R_j^i{}_{kl} = K(a_{jl}\delta_k^i - a_{jk}\delta_l^i)$ , where  $K$  is a constant. Then

$$(18) \quad R_{jl} = K(n-1)a_{jl};$$

$$(19) \quad R = n(n-1)K.$$

By (17) and (19), it follows that

$$(20) \quad \theta_m \theta^m = -\frac{9n}{4(n+2)}K,$$

By (16), (18) and (20), we have

$$\left( \frac{2}{3} \frac{\partial \theta^i}{\partial x^i} - (2n-1)K \right) a_{jl} - \frac{2}{3} n \frac{\partial \theta_l}{\partial x^j} + \frac{4(n+2)}{9} \theta_j \theta_l = 0.$$

Using Lemma 3.6 and (20) again, one can get

$$(21) \quad \left( \frac{2}{3} \theta^i{}_{|i} - \frac{2-n}{2+n} K \right) a_{jl} - \frac{2}{3} n \theta_{l|j} - \frac{4(n-2)}{9} \theta_j \theta_l = 0.$$



Contracting it with  $\theta^l$ , one can obtain

$$\left[ \frac{2}{3}\theta^i{}_{|i} + \frac{(n-1)(n-2)}{n+2}K \right] \theta_j = 0,$$

which can be contracted again by  $\theta^j$  to get

$$(22) \quad -\frac{9n}{4(n+2)}K \left[ \frac{2}{3}\theta^i{}_{|i} + \frac{(n-1)(n-2)}{n+2}K \right] = 0.$$

This implies that  $K = 0$  or  $\frac{2}{3}\theta^i{}_{|i} + \frac{(n-1)(n-2)}{n+2}K = 0$ . Without loss of generality, one may assume that  $K < 0$  if  $K \neq 0$ . The latter equation shows that  $\Delta \log \det(a_{ij}) = -\frac{3(n-1)(n-2)}{n+2}K > 0$  on  $M$ , which implies that  $f$  is subharmonic. By Proposition 4.2 and  $f \in L^1(M)$ ,  $f$  is harmonic. Then  $\frac{3(n-1)(n-2)}{n+2}K = 0$  implies  $K = 0$  for  $n \geq 3$ . From (20), the result holds.  $\square$

From equation (22), one can get the following corollary with integral of both side on the manifold.

**Corollary 4.4.** *Let  $F = \alpha\phi(s)$  be a non-Riemannian  $(\alpha, \beta)$ -metric on an  $n$ -dimensional orientable compact manifold  $M$  as described in Theorem 1.1. If the Riemannian metric  $\alpha$  has constant curvature, then  $F$  is locally dually flat if and only if  $\alpha$  is an Euclidean metric and  $\beta$  is a constant 1-form.*

**Example 4.5.** The flat torus is a compact manifold with vanishing curvature which satisfies the conditions in Theorem 4.3. It's curvature is a constant and  $f = \log \det(a_{ij})$  is Lebesgue integrable, since the manifold is compact. The theorem or corollary implies the dually flat  $(\alpha, \beta)$ -metrics on it must with  $\beta$  is a constant 1-form.

The following theorem are more useful in applications.

**Theorem 4.6.** *Let  $F = \alpha\phi(s)$  be a non-Riemannian  $(\alpha, \beta)$ -metric on an  $n$ -dimensional orientable compact manifold  $M$  as described in Theorem 1.1. If  $\theta_m b^m = 0$  holds on the whole manifold  $M$ , then  $F$  is locally dually flat if and only if  $\alpha$  is an Euclidean metric and  $\beta$  is a constant 1-form.*

*Proof.* By (8), we have

$$b_{j|k|l} = \frac{2}{3}(b_{k|l}\theta_j + b_k\theta_{j|l} - b_m\theta^m{}_{|l}a_{jk} - \theta^m b_{m|l}a_{jk}).$$

By (8) and Lemma 3.6, one can deduce

$$(23) \quad b_{j|k|l} - b_{j|l|k} = \frac{2}{3} \left[ 2\theta_j(b_l\theta_k - b_k\theta_l) + \frac{2}{3}(\theta_m\theta^m)(b_k a_{jl} - b_l a_{jk}) \right. \\ \left. + \left( \frac{\partial\theta_j}{\partial x^l} b_k - \frac{\partial\theta_j}{\partial x^k} b_l \right) + b_m \left( \frac{\partial\theta^m}{\partial x^k} a_{jl} - \frac{\partial\theta^m}{\partial x^l} a_{jk} \right) \right].$$

On the other hand, one can get the following equation from (14),

$$\begin{aligned}
 (24) \quad b_i R_j^i{}_{kl} &= \frac{2}{3} \left[ b_i \left( \frac{\partial \theta^i}{\partial x^k} a_{jl} - \frac{\partial \theta^i}{\partial x^l} a_{jk} \right) + \left( b_l \frac{\partial \theta_j}{\partial x^k} - b_k \frac{\partial \theta_j}{\partial x^l} \right) \right. \\
 &\quad + \frac{2}{3} (\theta_m \theta^m) (b_k a_{jl} - b_l a_{jk}) + \frac{2}{3} \theta_j (\theta_l b_k - \theta_k b_l) \\
 &\quad \left. + (b_m \theta^m) (\theta_k a_{jl} - \theta_l a_{jk}) \right].
 \end{aligned}$$

By the Ricci identity  $b_{j|k|l} - b_{j|l|k} = b_i R_j^i{}_{kl}$ , the following equation holds,

$$(25) \quad 2 \left( \frac{\partial \theta_j}{\partial x^l} b_k - \frac{\partial \theta_j}{\partial x^k} b_l \right) + \frac{8}{3} \theta_j (b_l \theta_k - b_k \theta_l) - (b_m \theta^m) (\theta_k a_{jl} - \theta_l a_{jk}) = 0.$$

With the assumption  $b_m \theta^m = 0$ , the equation becomes

$$(26) \quad 3 \left( \frac{\partial \theta_j}{\partial x^l} b_k - \frac{\partial \theta_j}{\partial x^k} b_l \right) + 4 \theta_j (b_l \theta_k - b_k \theta_l) = 0.$$

Taking the trace of  $j, l$ , we get

$$(27) \quad 3 \left( b_k \frac{\partial \theta_j}{\partial x^l} a^{jl} - b^j \frac{\partial \theta_j}{\partial x^k} \right) = 4 b_k \theta_m \theta^m.$$

By the assumption and Lemma 3.6, it follows that

$$b^j \frac{\partial \theta_j}{\partial x^k} = \frac{\partial (b^j \theta_j)}{\partial x^k} - \theta_j \frac{\partial b^j}{\partial x^k} = 2 (\theta_m b^m) \theta_k = 0.$$

Then (27) implies

$$(28) \quad 3 b_k \frac{\partial \theta_j}{\partial x^l} a^{jl} = 4 b_k \theta_m \theta^m.$$

Again, by using Lemma 3.6, it follows that

$$(29) \quad 3 \theta^i{}_{|i} = -2 n \theta_m \theta^m \leq 0.$$

Integrating on the whole compact manifold  $M$ , we get  $\theta_m = 0$  for any  $m \in \{1, 2, \dots, n\}$ . By (7) and (8),  $\alpha$  must be an Euclidean metric and  $\beta$  is a constant 1-form on  $(M, \alpha)$ . □

One of the key conditions in Theorem 4.6 is  $\theta_m b^m = 0$ . One can construct some manifolds with this property.

**Example 4.7.** Let  $M = M_1 \times M_2$  be a product manifold.  $M_1$  is a  $m_1$ -dim flat manifold  $R^{m_1}$  with the standard Euclidean metric and  $M_2$  is an arbitrary Riemannian manifold in dimension  $m_2$ .  $\beta$  is a 1-form limited on  $M_1$ , i.e.,  $\beta = (\beta_1, 0)$ . From Lemma 3.4,  $\theta = (0, \theta_2)$  with  $\theta_2$  is a 1-form depend on the metric of  $M_2$ . It obviously obtains that  $\theta_m b^m = 0$ .

Then if we need a compact manifold, we can take  $M_1$  in Example 1 to be  $S^1$  and  $M_2$  to be compact. Hence we have:

**Example 4.8.** For any compact manifold  $M$ , the product manifold  $M \times S^1$  with 1-form  $\beta$  on  $S^1$  is a compact manifold with  $\theta_m b^m = 0$ .

### 5. Monge-Ampère type equation

We now deal with the Monge-Ampère equation in  $\mathbb{R}^n$ . Such kind of non-linear equation is important and famous in both mathematics and physics. By using dually flat metrics, we can solve the following Monge-Ampère type equation under the convexity assumption and prove the non-existence of compact globally dually flat manifold. We first give the proof respectively. Moreover, we can give both proofs simultaneously.

*Proof of Theorem 1.3.* Suppose  $f$  is such a solution, then  $f$  defines a locally dually flat Riemannian metric  $\alpha$  by  $\alpha^2 = a_{ij}y^i y^j$  with  $a_{ij} = \frac{\partial^2 f}{\partial x^i \partial x^j}$  since  $f$  is convex. Let  $(\mathbb{R}^n, \alpha\phi(\frac{\beta}{\alpha}))$  be an  $(\alpha, \beta)$ -manifold with  $\alpha$  defined as above and  $\beta$  being a parallel 1-form. Moreover,  $\phi$  is a function satisfies the condition in Theorem 1.1. It can be chosen to make the Finsler metric to be a Matsumoto metric for example. By (7) and (8),  $(\mathbb{R}^n, \alpha\phi(\frac{\beta}{\alpha}))$  is a locally dually flat manifold and  $\theta_i = 0$ . Then plugging it into (7), we obtain  $\frac{\partial^2 f}{\partial x^i \partial x^j} = const..$  So we get Theorem 1.3. □

*Proof of Theorem 1.4.* If such manifold exists, we can set  $M = M_1 \times M_2$  with  $M_1 = (\mathbb{R}^n, \phi(s))$  is a dually flat  $(\alpha, \beta)$ -manifold and  $M_2 = (M, \omega)$  is a  $m_2$ -dim compact globally dually flat Riemannian manifold. By the definition,  $\omega$  is defined from a smooth function  $l$  on  $M_2$ , i.e.,  $\omega_{\mu\nu} = l_{|\mu|\nu}$ . Here we use  $x^\mu, x^\nu$  to denote coordinates on  $M_2$ . The determinant is only determined by  $\omega$  for  $f$  is the solution of the Monge-Ampère equation. Hence  $\theta$  is only depend on the coordinates on  $M_2$ . It's easy to see  $\theta_i \beta^i = 0$ . By the above Theorem 4.6, such metric is rigidity. Therefore,  $l_{|\mu|\nu}$  is a constant. By taking the integral of  $\omega^{\mu\nu} l_{\mu\nu}$  on compact manifold  $M_2$ , we get

$$0 = \int_{M_2} \Delta l = \int_{M_2} m_2 = m_2 \text{Vol}M_2,$$

which is a contradiction. □

Now we present the simultaneous proof of Theorem 1.3 and Theorem 1.4 by using the product manifold. This proof may give us the inspiration to prove the Monge-Ampère equation on  $\mathbb{R}^n$  with boundary conditions in the further research.

*Proof of Theorems 1.3 and 1.4 simultaneously.* Let  $f$  be the solution of the Monge-Ampère equation on  $\mathbb{R}^n$ . Denote  $a_{ij} = \frac{\partial^2 f}{\partial x^i \partial x^j}$ . Since  $f$  is convex,  $a_{ij}$  can be considered as a metric matrix on  $\mathbb{R}^n$ .

Let  $M$  be a product manifold, i.e.,  $M = M_1 \times M_2$ , with  $M_1 = \mathbb{R}^n$  and  $M_2$  compact. Moreover, let  $(M, \alpha\phi(s))$  be a dually flat  $(\alpha, \beta)$ -manifold with  $\alpha = \sqrt{a_{ij}y^i y^j}$  defined from  $f$ , i.e.,  $a_{ij} = \frac{\partial^2 f}{\partial x^i \partial x^j}$ , and  $\beta$  is a 1-form. We assume  $M_2$  is a globally dually flat Riemannian manifold with metric defined from a

function  $l$ , i.e.,  $\omega_{\mu\nu} = l_{\mu|\nu}$ . It's easy to see that  $\tilde{f} = f + l$  defines a globally dually flat metric on  $M = M_1 \times M_2$ .

The metric  $\alpha\phi(s)$  satisfying the conditions in Theorem 1.1 must admit equations (1), (2) and (3). Combining (2) and (3) we can get

$$b_{i|j} = \frac{2}{3}b_j\theta_i - \frac{2}{3}\theta_m b^m a_{ij},$$

as described in Theorem 1.1. Especially, one can just choose a Matsumoto metric for example. The underline Riemannian manifold  $(M, \alpha)$  is a product manifold  $M_1^{m_1} \times M_2^{m_2}$ , with  $M_1 = \mathbb{R}^n$  and  $m_1 = n$ . We use the indexes  $i, j$  for the first  $m_1$  coordinates in  $M_1$  and  $\mu, \nu$  for the last  $m_2$  coordinates on  $M_2$ . Then

$$(g_{AB}) = (\tilde{f}_{A|B}) = \begin{pmatrix} a_{ij} & 0 \\ 0 & \omega_{\mu\nu} \end{pmatrix},$$

where  $a_{ij}$  are components of the metric of  $M_1$ ,  $\omega_{\mu\nu}$  are components of the metric of  $M_2$  and  $A, B$  are from 1 to  $m_1 + m_2$ . The dually flat property of  $(\alpha, \beta)$ -metrics implies the Riemannian metric  $\alpha$  must be dually flat from (7). Hence  $a_{ij} = \frac{\partial^2 f}{\partial x^i \partial x^j}$  in the adapted coordinate system.  $f$  is just the solution of Monge-Ampère function. The determinant of  $g_{AB}$  is  $\det(g_{AB}) = \det(a_{ij}) \det(\omega_{\mu\nu})$ . Suppose  $\det(a_{ij}) = C$ , where  $C$  is a constant, then  $\det(g_{AB})$  is only depend on  $\det(\omega_{\mu\nu})$ . Therefore,  $\theta$  is defined on  $M_2$  by Lemma 3.4. Since the 1-form  $\beta$  is defined on  $M_1$ , we have  $\theta_m b^m = 0$  in this case.

In such case, we don't need to ask the whole manifold  $M$  to be compact. Instead of the condition in Theorem 4.6, we just need  $M_2$  to be compact now. Since the connection of the product metric is also block diagonal, the condition of the product metric implies  $\theta_i = 0$  on  $M_1$  and hence  $\theta^i|_i = 0$  automatically on  $M_1$  by (29). Since  $M_1$  is an Euclidean space, the metric is also Euclidean if  $M_2$  is compact, i.e.,

$$\begin{aligned} \frac{\partial^2 f}{\partial x^i \partial x^j} &= a_{ij} = \text{constant}, \\ \tilde{f}_{\mu|\nu} &= l_{\mu|\nu} = \omega_{\mu\nu} = \text{constant}. \end{aligned}$$

On  $M_1$ , the solution is just in the form of (4). We get Theorem 1.3.

On  $M_2$ , we know from the definition that  $\Delta \tilde{f} = \omega^{\mu\nu} \tilde{f}_{\mu|\nu} = m_2 \neq 0$ . One can integrate the equation on the compact manifold  $M_2$  if the globally dually flat metric exists to get

$$m_2 \text{Vol}M_2 = \int \Delta \tilde{f} = 0,$$

which leads to a contradiction. □

By the same method, one can obtain the following:

**Corollary 5.1.** *There is no convex function  $f \in C^2(N)$  on compact differentiable manifold  $N$  such that in any local coordinates,  $\det(\frac{\partial^2 f}{\partial x^i \partial x^j}) = C$ , where  $C$  is a non-zero constant.*

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### References

- [1] L. Caffarelli and Y. Y Li, *An extension to a theorem of Jörgens, Calabi, and Pogorelov*, Comm. Pure Appl. Math. **56** (2003), no. 5, 549–583.
- [2] E. Calabi, *Improper affine hyperspheres of convex type and a generalization of a theorem by K. Jörgens*, Michigan Math. J. **5** (1958), 105–126.
- [3] A. Caminha, P. Souza, and F. Camargo, *Complete foliations of space forms by hypersurfaces*, Bull. Braz. Math. Soc. (N.S.) **41** (2010), no. 3, 339–353.
- [4] X. Cheng and Z. Shen, *Finsler Geometry—An approach via Randers spaces*, Springer, Heidelberg, Science Press, Beijing, 2012.
- [5] X. Cheng, Z. Shen, and Y. Zhou, *On locally dually flat Randers metrics*, Internat. J. Math. **21** (2010), 1531–1543.
- [6] J. Jost and Y. L. Xin, *Some aspects of the global geometry of entire space-like submanifolds*, Dedicated to Shiing-Shen Chern on his 90th birthday, Results Math. **40** (2001), no. 1-4, 233–245.
- [7] A. V. Pogorelov, *Improper convex affine hyperspheres*, Dokl. Akad. Nauk SSSR **202** (1972), 1008–1011.
- [8] Z. Shen, *Riemannian-Finsler geometry with applications to information geometry*, Chinese Ann. Math. Ser. B **27** (2006), no. 1, 73–94.
- [9] B. Shen and Y. Tian, *Dually Flat Kropina Metrics*, Advances in Mathematics(China), to be appear.
- [10] Q. Xia, *On locally dually flat  $(\alpha, \beta)$ -metrics*, Differential Geom. Appl. **29** (2011), no. 2, 233–243.
- [11] C. Yu, *On dually flat Randers metrics*, Nonlinear Anal. **95** (2014), 146–155.

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