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A NOTE ON GORENSTEIN PRÜFER DOMAINS

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Abstract. In this note, we mainly discuss the Gorenstein Prüfer domains. It is shown that a domain is a Gorenstein Prüfer domain if and only if every finitely generated ideal is Gorenstein projective. It is also shown that a domain is a PID (resp., Dedekind domain, Bézout domain) if and only if it is a Gorenstein Prüfer UFD (resp., Krull domain, GCD domain).

1. Introduction

Throughout this note, all rings are commutative with identity element and all modules are unitary. Since the concept of Gorenstein homological algebra has been introduced, it has been attempted to prove the following (provable) meta-theorem:

Meta-theorem. *Every result in classical homological algebra has a counterpart in Gorenstein homological algebra.*

In this vein, it is proved in [3, Theorem 2.6] that a ring R is G-semihereditary if and only if every finitely generated submodule of any projective R-module is G-projective. Also it is shown in [8, Theorem 4.2] that a domain is a G-Prüfer domain if and only if it is coherent and any finitely generated ideal is G-projective. However the latter one is not a *perfect* counterpart to the well-known fact that a domain is a Prüfer domain if and only if any finitely generated ideal is projective. One of purposes of this paper is to show that the coherence condition in [8, Theorem 4.2] is superfluous. In order to do so, we first prove that if any finitely generated ideal of a domain R is G-projective, then R is coherent. So, it can be seen that the notion of G-Prüfer domains is a natural generalization of that of Prüfer domains and hence also a natural generalization of that of *Dedekind domains* [9].

On the other hand, it is well known that a PID is necessarily a UFD and a UFD is necessarily a GCD domain. It is also well known that a Dedekind domain is a UFD if and only if it is a PID [9, Theorem 4.26], a domain is a

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Bézout domain if and only if it is a Prüfer GCD domain. As a generalization of Dedekind domains, are there any similar results about G-Prüfer domains? Thus the others are to show that three results in classical ideal theory have counterparts in "Gorenstein ideal theory". More precisely, it is also shown that a domain is a PID (resp., Dedekind domain, Bézout domain) if and only if it is a Gorenstein Prüfer UFD (resp, Krull domain, GCD domain).

Next we introduce some definitions and notations. For an R-module M, the dual module $\mathrm{Hom}_R(M,R)$ and the double dual module $\mathrm{Hom}_R(\mathrm{Hom}_R(M,R),R)$ are denoted by M^* and M^{**} respectively. Recall that an R-module M is called *Gorenstein projective* (*G*-projective for short) in [1] if there exists an exact sequence

$$\cdots \to P_1 \to P_0 \to P^0 \to P^1 \to \cdots$$

of projective R-modules with $M = \ker(P^0 \to P^1)$ such that $\mathrm{Hom}_R(-,Q)$ leaves the sequence exact whenever Q is a projective module. An R-module M is called *Gorenstein flat* [2] if there exists an exact sequence

$$\cdots \to F_1 \to F_0 \to F^0 \to F^1 \to \cdots$$

of flat R-modules with $M = \ker(F^0 \to F^1)$ such that $E \otimes_R -$ leaves the sequence exact whenever E is an injective R-module.

A ring R is called *coherent* (resp. *semihereditary*) if every finitely generated ideal of R is finitely presented (resp., projective). Then it is well known that a ring R is a semihereditary ring if and only if every finitely generated submodule of any projective R-module is projective. A semihereditary domain is called a *Prüfer domain*, in other words, a domain is a Prüfer domain if and only if every finitely generated ideal is projective. Recall from [7] that a ring R is called *Gorenstein semihereditary* (*G-semihereditary* for short) if it is coherent and every submodule of a flat R-module is Gorenstein flat. Similarly, an integral domain R is called a *Gorenstein Prüfer domain* (*G-Prüfer domain* for short) [8] if it is a G-semihereditary domain. The definition of a *greatest common divisor domain* (*GCD domain* for short) and a *unique factorization domain* (UFD for short) can be found in [5]. A domain R is called a *principal ideal domain* [9] (PID for short) if every ideal of R is principal. If any finitely generated ideal of R is principal, then R is called a *Bézout domain* [5].

2. A characterization of G-Prüfer domains

We say that $a \in R$ is a *zero divisor* [9] if there is a nonzero element $b \in R$ such that $ab = 0$; in the case $a \neq 0$ we say that a is a *nontrivial zero divisor*. If a is not a zero divisor, we say a to be a *regular element* [9]. An ideal I of R is called *regular* if it contains a regular element. It can be seen that any nonzero ideal of a domain is regular.

Lemma 2.1. *Let R be a ring and I be a regular ideal of R. Then $I^* = \mathrm{Hom}_R(I,R)$ is isomorphic to an ideal of R.*

Proof. Since I is regular, it must contain a regular element, say a. Consider the following homomorphism $\theta : I^* = \mathrm{Hom}_R(I, R) \to R$ which is defined by $\theta(f) = f(a)$ for $f \in I^*$. If $f(a) = 0$, then for any element $b \in I$, we have $f(ab) = af(b) = bf(a) = 0$. Because a is regular, we surely have $f(b) = 0$, this means that θ is a monomorphism. So I^* is isomorphic to the image of θ which is an ideal of R. □

Lemma 2.2. *Let R be a ring and let M be a finitely generated G-projective R-module. Then there exists the following short exact sequence $0 \longrightarrow M \longrightarrow F \longrightarrow G \longrightarrow 0$, where F is a finitely generated projective module and G is G-projective.*

Proof. Just see [13, Proposition 2.6]. □

Now, we consider the following evaluation map $\mu_M : M \to M^{**}$, which is defined by $\mu_M(x)(f) = f(x)$ for $x \in M$ and $f \in M^*$. Then M is called *reflexive* if μ_M is an isomorphism.

Lemma 2.3. *Let R be a ring, $0 \longrightarrow M \longrightarrow P \longrightarrow C \longrightarrow 0$ be a short exact sequence such that P is a finitely generated projective R-module and C is a G-projective R-module. Then both M and C are reflexive. In particular, finitely generated G-projective R-modules are reflexive.*

Proof. Taking into consideration of Lemma 2.2, we only need to prove the first claim. Since G-projective modules are projective resolving, both M and C are G-projective modules. So, by [13, Lemma 3.1], both M and C are reflexive. □

Theorem 2.4. *Let R be a ring such that every finitely generated ideal of R is G-projective. If I is a finitely generated regular ideal of R, then I is finitely presented.*

Proof. Since I is finitely generated, I is G-projective from the assumption. So, by Lemma 2.3, $I \cong I^{**}$. Also notice that I^* is finitely generated by [13, Proposition 2.6]. Because I is regular, $I^* = \mathrm{Hom}_R(I, R)$ is isomorphic to an ideal of R by Lemma 2.1. Therefore I^* is a finitely generated G-projective R-module. Thus, by Lemma 2.2, we have the following short exact sequence:

$$0 \longrightarrow I^* \longrightarrow F' \longrightarrow G' \longrightarrow 0,$$

where F' is finitely generated free, G' is a finitely generated G-projective R-module, and $\mathrm{Ext}^1_R(G', R) = 0$. Further, by applying the functor $\mathrm{Hom}_R(-, R)$ to this sequence, we get the following exact sequence:

$$0 \longrightarrow G'^* \longrightarrow F'^* \longrightarrow I^{**} \longrightarrow 0.$$

Since $I \cong I^{**}$, we surely have an exact sequence:

$$0 \longrightarrow G'^* \longrightarrow F'^* \longrightarrow I \longrightarrow 0.$$

Since by [13, Proposition 2.6], G'^* is finitely generated, it can be seen from this short exact sequence that I is finitely presented. □

Trivially we have the following:

Corollary 2.5. *Let R be a domain such that every finitely generated ideal of R is G-projective. Then R is coherent.*

Corollary 2.6. *Let R be a domain. Then R is a G-Prüfer domain if and only if every finitely generated ideal of R is G-projective.*

Proof. Just notice that, by [8, Theorem 4.2], a domain is a G-Prüfer domain if and only if it is coherent and every finitely generated ideal is G-projective. □

It is worth to point out that, by [8, Theorem 4.2] and [14, Theorem 1.5], a domain is a Prüfer domain if and only if it is an integral closed G-Prüfer domain.

3. G-Prüfer domains and UFDs

It is well known that a Dedekind domain is a UFD if and only if it is a PID. In this section, we will prove that a domain is a PID (resp., Dedekind domain, Bézout domain) if and only if it is a Gorenstein Prüfer UFD (resp., Krull domain, GCD domain).

Let R be a domain with quotient field K and I be a fractional ideal of R. Then I^{-1} is defined as follow:

$$I^{-1} = \{x \in K | xI \subset R\}.$$

An ideal J of R is called a *Glaz-Vasconcelos ideal* (for short, GV-ideal, denoted by $J \in GV(R)$) if J is a finitely generated ideal of R with $J^{-1} = R$. A torsion-free R-module M is called a *w-module* if $Jx \subset M$ for $J \in GV(R)$ and $x \in M \otimes K$ imply that $x \in M$. It is easy to see that free modules are w-modules. For a torsion-free R-module M, Wang and McCasland defined the w-envelope of M in [11] as follows:

$$M_w = \{x \in M \otimes K | Jx \subset M \text{ for some } J \in GV(R)\}.$$

So, a torsion-free module M is a w-module if and only if $M = M_w$. A torsion-free module M is said to be *of finite type* if there exists a finitely generated submodule N of M such that $M_w = N_w$. In particular, if I is a nonzero fractional ideal of R, then

$$I_w = \{x \in K | Jx \subset I \text{ for some } J \in GV(R)\}.$$

The canonical map $I \to I_w$ on $F(R)$ (the set of fractional ideals of R) is a star-operation, denoted by w. An ideal I is called a w-ideal if and only if $I = I_w$. It can be seen that if $J \in GV(R)$, then $J_w = R$.

Lemma 3.1. *Let R be a GCD domain and $a_1, \ldots, a_n \in R$ $(n \geq 2)$. If $a_1, \ldots, a_n$ are relatively prime, then the ideal $J = (a_1, \ldots, a_n)$ is a GV-ideal and any finitely generated ideal which contains J is also a GV-ideal.*

Proof. It will suffice to prove that $J^{-1} = R$. Let $\frac{u}{v} \in J^{-1}$ (where $u, v \in R$ are relatively prime). Then $\frac{u}{v} a_i \in R$ and this means that $v | a_i$ for each $i = 1, \dots, n$. So v is a common divisor of $a_1, \dots, a_n$ and must be a unit. Therefore $\frac{u}{v} \in R$. □

An R-module M is called *GV-torsion-free* if for any $J \in GV(R)$ and $x \in M$, $Jx = 0$ implies $x = 0$. It can be seen that a torsion-free module is GV-torsion-free. The following lemma is [10, Theorem 8.2.5].

Lemma 3.2. *Let M be a torsion-free module. The following statements are equivalent.*

(1) *M is a w-module.*

(2) *For any exact sequence $0 \longrightarrow M \longrightarrow F \longrightarrow N \longrightarrow 0$ where F is a w-module, N is GV-torsion-free.*

(3) *There exists an exact sequence $0 \longrightarrow M \longrightarrow F \longrightarrow N \longrightarrow 0$ such that F is a w-module and N is GV-torsion-free.*

Since free modules are w-modules, it can be seen from this theorem that projective modules are w-modules.

Proposition 3.3. *If R is a G-Prüfer domain, then any finitely generated ideal I of R is a w-ideal.*

Proof. It will suffice to prove that I is a w-module. Since R is a G-Prüfer domain and I is finitely generated, I must be G-projective. Therefore there exists a short exact sequence $0 \longrightarrow I \longrightarrow P \longrightarrow G \longrightarrow 0$ such that P is projective and G is G-projective. Because projective modules are w-modules and G is GV-torsion-free (as a submodule of a free module), by Lemma 3.2, I is a w-module. □

Corollary 3.4. *If R is a G-Prüfer domain, then any ideal I of R is a w-ideal.*

Proof. Let $J \in GV(R)$ and $x \in K$, the quotient field of R, such that $Jx \subset I$. Since J is finitely generated, Jx is a finitely generated ideal. So it must be a w-ideal. Therefore $Jx = (Jx)_w = x(J)_w = (x)$. Hence $x \in Jx \subset I$. This means that I is a w-ideal. □

Remark that an alternative proof of Corollary 3.4 can be given by Corollary 3.3 and the facts that every ideal of R is the direct limit of its finitely generated subideals and [15] the direct limit commutes with w-operation.

Recall that a domain R is called a *Krull domain* if R_P is a DVR for each $P \in X^{(1)}(R)$, the set of height-one prime ideals of R, $R = \bigcap_{P \in X^{(1)}(R)} R_P$, and each nonzero $r \in R$ is a unit in all but a finite number of R_P's (i.e., the intersection is "locally finite"). The following result strengthens the well-known result that a domain R is a Dedekind domain if and only if R is a Prüfer Krull domain.

Corollary 3.5. *A domain R is a Dedekind domain if and only if R is a G-Prüfer Krull domain.*

Proof. Clearly a Dedekind domain is both a G-Prüfer and a Krull domain. Conversely assume that R is a G-Prüfer Krull domain. By Corollary 3.4, $d = w$. Now the assertion follows from the fact that R is a Krull domain if and only if every nonzero ideal of R is w-invertible, i.e., $(II^{-1})_w = R$ for any nonzero ideal I of R [12, Theorem 2.8]. □

Let R be a domain. Recall that for any fractional ideal I of R, $I_v := (I^{-1})^{-1}$ and $I_t := \bigcup\{J_v \mid J$ is a finitely generated subideal of $I\}$. Also recall that for $* = v$ or t, a fractional ideal I of R is called a $*$*-ideal* if $I_* = I$ and R is called a *Prüfer v-multiplication domain* (*PvMD* for short) if $(AA^{-1})_t = R$ for any finitely generated ideal A of R.

In the rest, we will show that (i) a domain R is a UFD if and only if any w-ideal of R is principal and (ii) if R is a GCD domain, then every w-ideal of R of finite type is principal. These can be proved easily by using the facts that (i) a domain R is a UFD if and only if any t-ideal of R is principal [4], (ii) every t-ideal is a w-ideal (Actually in a PvMD, $t = w$), (iii) a GCD domain (and hence a UFD) is a PvMD, and (iv) if R is a GCD domain, then every v-ideal of R of finite type is principal. However we give their proofs in the w-theoretic context.

Lemma 3.6. *If R is a UFD, then every prime w-ideal P of R is principal.*

Proof. We can assume that P is nonzero. Let a be a nonzero element of P. Then $a = up_1 \cdots p_n$ where u is a unit and p_i is an irreducible element for each $i = 1, \ldots, n$. Since P is a prime ideal, one of the $p_i's$, say, p_1 must be contained in P. If $(p_1) \neq P$, there exists some $b \in P$ but b is not inside (p_1). So p_1 and b are relatively prime and the ideal (p_1, b) is a GV-ideal contained in P. This contradicts the fact that P is a w-ideal. Therefore $P = (p_1)$ is principal. □

Lemma 3.7 ([6, Corollary 2.2])**.** *Let R be a domain and $a \in R$ is a nonzero element. If I is a w-ideal of R, then the ideal $(I : a) = \{x \in R \,|\, ax \in I\}$ is also a w-ideal.*

Proof. First, we prove that the module $\frac{R}{I}$ is GV-torsion-free. Just notice the short exact sequence $0 \longrightarrow I \longrightarrow R \longrightarrow \frac{R}{I} \longrightarrow 0$. Since I and R are w-modules, by Lemma 3.2, the module $\frac{R}{I}$ is GV-torsion-free. Secondly, we look at the short exact sequence $0 \longrightarrow (I : a) \longrightarrow R \longrightarrow \frac{Ra+I}{I} \longrightarrow 0$. Since R is a w-module and $\frac{Ra+I}{I}$ is GV-torsion-free (as a submodule of $\frac{R}{I}$), the ideal $(I : a) = \{x \in R | ax \in I\}$ is also a w-ideal by Lemma 3.2. □

Theorem 3.8. *Let R be a domain. If any prime w-ideal of R is principal, then any w-ideal of R is principal.*

Proof. Let Γ be the set of non-principal w-ideals of R. We will prove that the set Γ is empty. Suppose that $A_1 \subseteq A_2 \subseteq \cdots \subseteq A_n \subseteq \cdots$ is an ascending chain in Γ. The ideal $\bigcup A_i$ is also a w-ideal. Further, it is also non-principal. By Zorn's lemma, Γ has a maximal element. Let P be a maximal element in Γ and

$r, s \in R$. Suppose that $rs \in P$ and s is not inside P. By Lemma 3.7, the ideal $(P : r)$ is a w-ideal. Since $P \subsetneqq (P : r)$ ($s \in (P : r)$ but s is not inside P) and P is maximal in Γ, $(P : r)$ must be a principal ideal of R, say, $(P : r) = (q)$. Therefore, $P \subseteq (q)$ and $\frac{P}{q}$ is also a non-principal w-ideal. By the maximality of P, we get that $P = \frac{P}{q}$. Since $rq \in P$, we have $r \in \frac{P}{q} = P$. This means that P is a prime ideal and must be principal. This contradiction shows that the set Γ must be empty. □

A domain R is said to satisfy *ACCP* (the ascending chain condition on principal ideals) if every ascending chain of principal ideals is stationary, that is, if $(a_1) \subseteq (a_2) \subseteq (a_3) \subseteq \cdots \subseteq (a_n) \subseteq \cdots$ is any chain of principal ideals, then there is a positive integer m such that $(a_n) = (a_m)$ for all $m \leqslant n$.

Theorem 3.9. *A domain R is a UFD if and only if any w-ideal of R is principal.*

Proof. The "only if" part of the proof can be seen from Lemma 3.6 and Theorem 3.8. For the "if" part, we prove two facts: (1) R satisfies ACCP; (2) R is a GCD domain. So, by [10, Theorem 1.7.6], R is a UFD. If $(a_1) \subseteq (a_2) \subseteq (a_3) \subseteq \cdots \subseteq (a_n) \subseteq \cdots$ is an ascending chain of principal ideals in R, then this chain is also a chain of w-ideals. So the ideal $\bigcup (a_i)$ is also a w-ideal and must be principal, say, $\bigcup (a_i) = (b)$. Therefore, $b \in (a_j)$ for some j. This leads to the fact that $(a_j) = (a_{j+1}) = \cdots = (b)$, that is, this chain is stationary. For the second fact, let (a) and (b) be any two principal ideals of R. Then $(a) \cap (b)$ is a w-ideal (as an intersection of w-ideals). From the condition, $(a) \cap (b)$ is also principal. Therefore, by [10, Theorem 1.7.3], R is a GCD domain. □

The following result strengthens the well-known result that a domain R is a Prüfer UFD if and only if it is a PID.

Theorem 3.10. *A domain R is a G-Prüfer UFD if and only if it is a PID.*

Proof. The fact that a PID is a UFD is well known. Since every ideal of a PID is a free module, a PID must be a G-Prüfer domain. For the "only if" part, by Lemma 3.4, every ideal of R is a w-ideal. If R is a UFD, then every w-ideal of R is principal by Theorem 3.9. Therefore, every ideal of R is principal and R is a PID. □

Lemma 3.11. *If R is a GCD domain, then every w-ideal of R of finite type is principal.*

Proof. Let I be a w-ideal of finite type. We can assume that $I = (a_1, \dots, a_n)_w$. Let the greatest common divisor of $a_1, \dots, a_n$ is d. So $a_i = db_i$ for some $b_i \in R$ and $i = 1, \dots, n$. $b_1, \dots, b_n$ are relatively prime. By Lemma 3.1, the ideal $(b_1, \dots, b_n)$ is a GV-ideal of R. So $I = (a_1, \dots, a_n)_w = d(b_1, \dots, b_n)_w = (d)$ is principal. □

Recall that a domain is called a *Bézout domain* if every finitely generated ideal is principal. The following result strengthens the well-known result that a domain R is a Prüfer GCD domain if and only if R is a Bézout domain.

Theorem 3.12. *A domain R is a G-Prüfer GCD domain if and only if R is a Bézout domain.*

Proof. Since principal ideals are free modules, every finitely generated ideal of a Bézout domain is free and must be G-projective. Therefore a Bézout domain is a G-Prüfer domain. If a, b are any two nonzero elements of a Bézout domain R, then the ideal (a, b) is principal, say, $(a, b) = (c)$ for some $c \in R$. It can be seen that c is the greatest common divisor of a, b. So a Bézout domain is also a GCD domain. For the "only if" part, by Lemma 3.4, every ideal of R is a w-ideal. If R is a GCD domain, then every w-ideal of R of finite type is principal by Lemma 3.11. Therefore, every finitely generated ideal of R is principal, and so R is a Bézout domain. □

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References

[1] E. Enochs and O. Jenda, *Gorenstein injective and projective modules*, Math. Z. **220** (1995), no. 4, 611–633.

[2] E. Enochs, O. Jenda, and B. Torrecillas, *Gorenstein flat modules*, Nanjing Daxue Xuebao Shuxue Bannian Kan **10** (1993), no. 1, 1–9.

[3] Z. Gao and F. Wang, *All Gorenstein hereditary rings are coherent*, J. Algebra Appl. **13** (2014), no. 4, 1350140, 5 pages.

[4] B. G. Kang, *On the converse of a well-known fact about Krull domains*, J. Algebra **124** (1989), no. 2, 284–299.

[5] I. Kaplansky, *Commutative Rings*, Revised Edition, Univ. Chicago Press, Chicago, 1974.

[6] H. Kim, *Module-theoretic characterizations of t-linkative domains*, Comm. Algebra **36** (2008), no. 5, 1649–1670.

[7] N. Mahdou and M. Tamekkante, *On (strongly) Gorenstein (semi)hereditary rings*, Arab. J. Sci. Eng. **36** (2011), no. 3, 431–440.

[8] L. Qiao and F. Wang, *A Gorenstein analogue of a result of Bertin*, J. Algebra Appl. **14** (2015), no. 2, 1550019, 13 pages.

[9] J. J. Rotman, *An Introduction to Homological Algebra*, Second Ed., New York: Springer Science+Business Media, LLC, 2009.

[10] F. Wang, *Commutative Rings and Star-Operation Theory*, Beijing: Sicence Press. (in Chinese), 2006.

[11] F. Wang and R. L. McCasland, *On w-modules over strong Mori domains*, Comm. Algebra **25** (1997), no. 4, 1285–1306.

[12] ——, *On strong Mori domains*, J. Pure Appl. Algebra **135** (1999), no. 2, 155–165.

[13] F. Wang, L. Qiao, and H. Kim, *Super finitely presented modules and Gorenstein projective modules*, Comm. Algebra. to appear.

[14] L. Xu, K. Hu, S. Zhao, and F. Wang, *A chcaracterization of Prüfer domains*, Comm. Algebra **44** (2016), no. 1, 135–140.

[15] D. C. Zhou and F. Wang, *The direct and inverse limits of w-modules*, Comm. Algebra. to appear.

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