

ON FINITE TIMES DEGENERATE HIGHER-ORDER CAUCHY NUMBERS AND POLYNOMIALS

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ABSTRACT. Cauchy polynomials are also called Bernoulli polynomials of the second kind and these polynomials are very important to study mathematical physics. D. S. Kim et al. have studied some properties of Bernoulli polynomials of the second kind associated with special polynomials arising from umbral calculus.

T. Kim introduced the degenerate Cauchy numbers and polynomials which are derived from the degenerate function e^t . Recently J. Jeong, S. H. Rim and B. M. Kim studied on finite times degenerate Cauchy numbers and polynomials.

In this paper we consider finite times degenerate higher-order Cauchy numbers and polynomials, and give some identities and properties of these polynomials.

1. Introduction

In the book [2] Comtet introduced Cauchy numbers, denoted by C_n , by the integral of the following formula:

$$(1) \quad \begin{aligned} C_n &= \int_0^1 (x)_n dx = \int_0^1 x(x-1)\cdots(x-n+1)dx \\ &= n! \int_0^1 \binom{x}{n} dx. \end{aligned}$$

From (1), we can derive the generating function as follows:

$$(2) \quad \begin{aligned} \sum_{n=0}^{\infty} \frac{C_n}{n!} t^n &= \sum_{n=0}^{\infty} \int_0^1 \binom{x}{n} dx t^n = \int_0^1 \sum_{n=0}^{\infty} \binom{x}{n} t^n dx \\ &= \int_0^1 (1+t)^x dx = \frac{t}{\log(1+t)} \quad (\text{see [2, 4, 6, 13–15]}). \end{aligned}$$

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As is well known, the *higher-order Cauchy polynomials* are given by the generating function.

$$(3) \quad \left(\frac{t}{\log(1+t)} \right)^r (1+t)^x = \sum_{n=0}^{\infty} C_n^{(r)}(x) \frac{t^n}{n!}.$$

Thus we note that

$$(4) \quad \int_0^1 \cdots \int_0^1 (1+t)^{x_1+\cdots+x_r+x} dx_1 \cdots dx_r = \sum_{n=0}^{\infty} C_n^{(r)}(x) \frac{t^n}{n!} \quad (\text{see [2, 4]}).$$

From (4), we have

$$(5) \quad \int_0^1 \cdots \int_0^1 (x_1 + \cdots + x_r + x)_n dx_1 \cdots dx_r = \sum_{n=0}^{\infty} C_n^{(r)}(x) \quad (n \geq 0).$$

It is known that

$$(6) \quad \left(\frac{t}{\log(1+t)} \right)^r (1+t)^{x-1} = \sum_{n=0}^{\infty} B_n^{(n-r+1)}(x) \frac{t^n}{n!} \quad (\text{see [7, 9, 12]}),$$

where $B_n^{(r)}(x)$ are called the *higher-order Bernoulli polynomials*.

The Stirling number of the first kind is defined by

$$(7) \quad (x)_n = x(x-1)\cdots(x-n+1) = \sum_{l=0}^n S_1(n, l) x^l \quad (\text{see [1, 4, 12]}),$$

and the Stirling number of the second kind is given by

$$(8) \quad x^n = \sum_{l=0}^n S_2(n, l) (x)_l \quad (\text{see [4, 7, 8, 12]}).$$

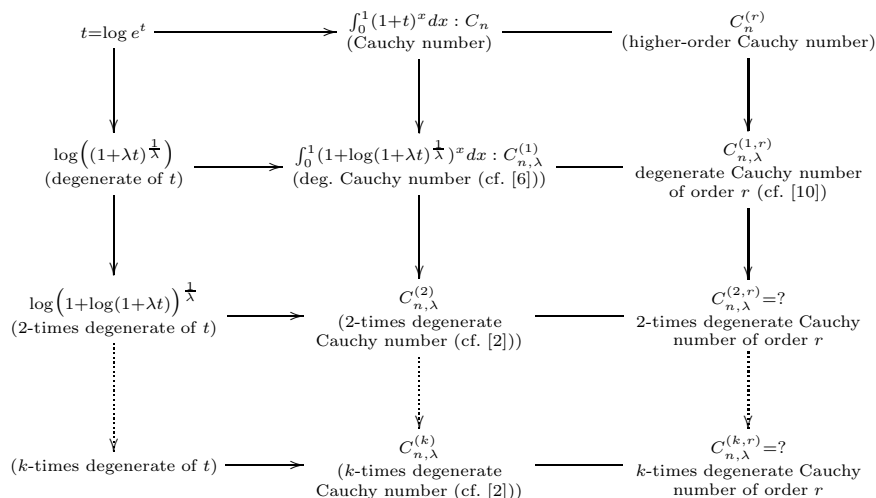
Cauchy polynomials are also called Bernoulli polynomials of the second kind and these polynomials are very important to study mathematical physics (see [5, 13]). In [5, 13], D. S. Kim, T. Kim and T. Mansour have studied some properties of Bernoulli polynomials of the second kind associated with special polynomials arising from umbral calculus.

T. Kim introduced the degenerate Cauchy numbers and polynomials which are derived from the degenerate function e^t (see [10, 13]). In this paper, we try to degenerate higher-order Cauchy numbers and polynomials k -times and investigate some properties of these k -times degenerate higher-order Cauchy numbers and polynomials.

Recently J. Jeong, S. H. Rim and B. M. Kim have studied on finite times degenerate Cauchy numbers and polynomials (see [3]).

In this paper we consider finite times degenerate higher order Cauchy numbers and polynomials. We give some identities and properties of these polynomials.

2. k -times degenerate higher-order Cauchy numbers



$C_{n,\lambda}^{(k,r)}$ means k -times degenerate Cauchy numbers of order r .

In [10], T. Kim considered the degenerate Cauchy numbers of order r which are defined by the generating function

$$\begin{aligned}
 \sum_{n=0}^{\infty} C_{n,\lambda}^{(1,r)} \frac{t^n}{n!} &= \int_0^1 \cdots \int_0^1 (1 + \log(1 + \lambda t)^{\frac{1}{x}})^{x_1 + \cdots + x_r + x} dx_1 \cdots dx_r \\
 (9) \qquad \qquad \qquad &= \left(\frac{\log(1 + \lambda t)^{\frac{1}{x}}}{\log(1 + \log(1 + \lambda t)^{\frac{1}{x}})} \right)^r (1 + \log(1 + \lambda t)^{\frac{1}{x}})^x.
 \end{aligned}$$

When $x = 0$, $C_{n,\lambda}^{(1,r)} = C_{n,\lambda}^{(1,r)}(0)$ are called the degenerate higher-order Cauchy numbers. And the following observations are given by T. Kim (see [10]).

Theorem 1. For $n \geq 0$, we have

- (i) $C_{n,\lambda}^{(1,r)}(x) = \sum_{l=0}^n \sum_{j=0}^l S_1(n, l) S_1(l, j) \lambda^{n-l} B_j^{(r)}(x).$
- (ii) $C_{n,\lambda}^{(1,r)}(x) = \sum_{m=0}^n \int_0^1 \cdots \int_0^1 (x_1 + \cdots + x_r)_m dx_1 \cdots dx_r S_1(n, m) \lambda^{n-m}.$
- (iii) $C_{n,\lambda}^{(1,r)}(x) = \sum_{m=0}^n B_m^{(m-r+1)}(x) S_1(n, m) \lambda^{n-l}.$
- (iv) $C_{n,\lambda}^{(1,r)}(x) = \sum_{l=0}^n \sum_{m=0}^l \binom{n}{l} S_1(n, m) \lambda^{l-m}(x)_m C_{n-l,\lambda}^{(1,r)}.$

$$(v) \quad C_{n,\lambda}^{(1,r)}(x) = \sum_{l=0}^n C_l^{(1,r)}(x)\lambda^{n-l}S_1(n,l).$$

We degenerate one more time in (9), i.e., 2-times degenerate Cauchy polynomials of order r :

$$\begin{aligned} (10) \quad & \sum_{n=0}^{\infty} C_{n,\lambda}^{(2,r)}(x)\frac{t^n}{n!} \\ &= \int_0^1 \cdots \int_0^1 \left(1 + \log(1 + \log(1 + \lambda t))^{\frac{1}{\lambda}}\right)^{x_1+\cdots+x_r+x} dx_1 \cdots dx_r \\ &= \left(\frac{\log(1 + \log(1 + \lambda t))^{\frac{1}{\lambda}}}{\log(1 + \log(1 + \log(1 + \lambda t))^{\frac{1}{\lambda}})}\right)^r \left(1 + \log(1 + \log(1 + \lambda t))^{\frac{1}{\lambda}}\right)^x \\ & \quad \text{(see [11]).} \end{aligned}$$

When $x = 0$, $C_{n,\lambda}^{(2,r)} = C_{n,\lambda}^{(2,r)}(0)$ are called the 2-times degenerate Cauchy numbers of order r .

Now we observe that

$$\begin{aligned} (11) \quad & \int_0^1 \cdots \int_0^1 \left(1 + \log(1 + \log(1 + \lambda t))^{\frac{1}{\lambda}}\right)^{x_1+\cdots+x_r+x} dx_1 \cdots dx_r \\ &= \sum_{n=0}^{\infty} \int_0^1 \cdots \int_0^1 (x_1 + \cdots + x_r + x)_n dx_1 \cdots dx_r \\ & \quad \times \frac{1}{n!} \left(\log(1 + \log(1 + \lambda t))^{\frac{1}{\lambda}}\right)^n \\ &= \sum_{n=0}^{\infty} \int_0^1 \cdots \int_0^1 (x_1 + \cdots + x_r + x)_n dx_1 \cdots dx_r \\ & \quad \times \sum_{m=n}^{\infty} \lambda^{-n} S_1(m,n) \frac{\log(1 + \lambda t)^m}{m!} \\ &= \sum_{j=0}^{\infty} \left(\sum_{m=0}^j \sum_{n=0}^m \int_0^1 \cdots \int_0^1 (x_1 + \cdots + x_r + x)_n dx_1 \cdots dx_r \right. \\ & \quad \left. \times \lambda^{j-n} S_1(j,m) S_1(m,n) \right) \frac{t^j}{j!}. \end{aligned}$$

Therefore, by (10) and (11), we obtain the following theorem.

Theorem 2. For $j \geq 0$, we have

$$C_{j,\lambda}^{(2,r)}(x) = \sum_{m=0}^j \sum_{n=0}^m \int_0^1 \cdots \int_0^1 (x_1 + \cdots + x_r + x)_n dx_1 \cdots dx_r S_1(j,m) S_1(n,m) \lambda^{j-n}.$$

Inductively we try k -times degenerate Cauchy polynomials of order r . Then we have the following result.

Theorem 3. For $n_i \geq 0, i = 1, \dots, k + 1$,

$$\begin{aligned}
 C_{k+1,\lambda}^{(k,r)} &= \sum_{n_k=0}^{n_{k+1}} \sum_{n_{k-1}=0}^{n_k} \cdots \sum_{n_1=0}^{n_2} \int_0^1 \cdots \int_0^1 (x_1 + \cdots + x_r + x)_n dx_1 \cdots dx_r \\
 &\quad \times \prod_{i=1}^k S_1(n_{i+1}, n_i) \lambda^{n_{k+1}-n_1} \\
 &= \sum_{n_k=0}^{n_{k+1}} \sum_{n_{k-1}=0}^{n_k} \cdots \sum_{n_1=0}^{n_2} C_{n_1}^{(1,r)}(x) \prod_{i=1}^k S_1(n_{i+1}, n_i) \lambda^{n_{k+1}-n_1}.
 \end{aligned}$$

Specially for the case $k = 1$ and $r = 1$, we have the following known result.

Corollary 4. For $m \geq 0$, and each $n_i \geq 0$ where $i = 0, 1, \dots, k + 1$, we have

- (i) $\sum_{n=0}^m \int_0^1 \cdots \int_0^1 (x_1 + \cdots + x_r + x)_n dx_1 \cdots dx_r S_1(m, n) \lambda^{m-n} = C_{m,\lambda}^{(r)}(x).$
- (ii) $C_{n_{k+1},\lambda}^{(k)}(x) = \sum_{n_k=0}^{n_{k+1}} \cdots \sum_{n_1=0}^{n_2} \lambda^{n_{k+1}-n_1} \left(\prod_{i=1}^k S_1(n_{i+1}, n_i) \right) C_{n_1}(x).$

Proof. (i) See [10, Theorem 2.1].

(ii) See [3, Theorem 7].

Remark 5. The following is well known in [10].

- (i) $\int_0^1 \cdots \int_0^1 (x_1 + \cdots + x_r + x)_n dx_1 \cdots dx_r$
 $= \sum_{l=0}^n S_1(n, l) \int_0^1 \cdots \int_0^1 (x_1 + \cdots + x_r + x)^l dx_1 \cdots dx_r$
 $= \sum_{l=0}^n S_1(n, l) B_l^{(r)}(x).$
- (ii) $\int_0^1 \cdots \int_0^1 e^{(x_1+\cdots+x_r+x)} dx_1 \cdots dx_r = \left(\frac{t}{e^t - 1} \right)^r e^{xt} = \sum_{n=0}^{\infty} B_n^{(r)}(x) \frac{t^n}{n!}.$
- (iii) $\int_0^1 \cdots \int_0^1 (x_1 + \cdots + x_r + x)^n dx_1 \cdots dx_r = B_n^{(r)}(x) \quad (n \geq 0).$

Therefore, from Theorem 3 and Remark 5(i) and (iii), we have the following corollary. □

Corollary 6. For $n_i \geq 0, i = 1, \dots, k + 1$, we have the following identity.

$$C_{n_{k+1}, \lambda}^{(k,r)} = \sum_{n_k=0}^{n_{k+1}} \sum_{n_{k-1}=0}^{n_k} \dots \sum_{n_1=0}^{n_2} \lambda^{n_{k+1}-n_1} \prod_{i=1}^k S_1(n_{i+1}, n_i) B_{n_1}^{(r)}(x).$$

Specially for the case $k = 1$, we have the following corollary.

Corollary 7 ([10, Corollary 2.2]). For $n \geq 0$, we have

$$C_{n, \lambda}^{(r)}(x) = \sum_{k=0}^n \sum_{j=0}^k S_1(n, k) \lambda^{n-k} S_1(k, j) B_j^{(r)}(x).$$

By replacing t by $\frac{1}{\lambda}(e^{\lambda t} - 1)$ in (10), we get

$$(12) \quad \sum_{n=0}^{\infty} C_{n, \lambda}^{(2,r)}(x) \frac{1}{n!} \left(\frac{e^{\lambda t} - 1}{\lambda} \right)^n = \left(\frac{\log(1 + \lambda t)^{\frac{1}{\lambda}}}{\log(1 + \log(1 + \lambda t)^{\frac{1}{\lambda}})} \right)^r (1 + \log(1 + \lambda t)^{\frac{1}{\lambda}})^x.$$

From (6), the right hand side of (12) becomes as follows:

$$(13) \quad \begin{aligned} & \left(\frac{\log(1 + \lambda t)^{\frac{1}{\lambda}}}{\log(1 + \log(1 + \lambda t)^{\frac{1}{\lambda}})} \right)^r (1 + \log(1 + \lambda t)^{\frac{1}{\lambda}})^x \\ &= \sum_{m=0}^{\infty} B_m^{(m-r+1)}(x+1) \frac{1 + \log(1 + \lambda t)^{\frac{1}{\lambda}}}{m!} \\ &= \sum_{m=0}^{\infty} B_m^{(m-r+1)}(x+1) \sum_{l=m}^{\infty} S_1(l, m) \frac{\lambda^l t^l}{l!} \lambda^{-m} \\ &= \sum_{l=0}^{\infty} \left(\sum_{m=0}^{\infty} B_m^{(m-r+1)}(x+1) \lambda^{l-m} S_1(l, m) \right) \frac{t^l}{l!}. \end{aligned}$$

The left hand side of (12) becomes

$$(14) \quad \begin{aligned} & \sum_{n=0}^{\infty} C_{n, \lambda}^{(2,r)}(x) \frac{1}{n!} \lambda^{-n} \left(\frac{e^{\lambda t} - 1}{\lambda} \right)^n \\ &= \sum_{n=0}^{\infty} C_{n, \lambda}^{(2,r)}(x) \lambda^{-n} \sum_{m=n}^{\infty} S_2(m, n) \frac{\lambda^m t^m}{m!} \\ &= \sum_{m=0}^{\infty} \left(\sum_{n=0}^m C_{n, \lambda}^{(2,r)}(x) \lambda^{m-n} S_2(m, n) \right) \frac{t^m}{m!}. \end{aligned}$$

By comparing the coefficients of (13) and (14) we obtain the following result.

Theorem 8. For $m \geq 0$, we have

$$C_{m, \lambda}^{(1,r)}(x) = \sum_{n=0}^m B_n^{(n-r+1)}(x+1) \lambda^{m-n} S_1(m, n) = \sum_{n=0}^m C_{n, \lambda}^{(2,r)}(x) \lambda^{m-n} S_1(m, n).$$

From 2-times degenerate Cauchy polynomials of order r in (10), we have

$$\begin{aligned}
 (15) \quad & \left(\frac{\log(1 + \log(1 + \lambda t))^{\frac{1}{\lambda}}}{\log(1 + \log(1 + \log(1 + \lambda t))^{\frac{1}{\lambda}})} \right)^r \left(1 + \log(1 + \log(1 + \lambda t))^{\frac{1}{\lambda}} \right)^x \\
 &= \sum_{m=0}^{\infty} B_m^{(m-r+1)}(x+1) \frac{1}{m!} \left(\log(1 + \log(1 + \lambda t))^{\frac{1}{\lambda}} \right)^m \\
 &= \sum_{m=0}^{\infty} B_m^{(m-r+1)}(x+1) \sum_{l=m}^{\infty} \lambda^{-m} S_1(l, m) \frac{\log(1 + \lambda t)^m}{m!} \\
 &= \sum_{m=0}^{\infty} B_m^{(m-r+1)}(x+1) \sum_{l=m}^{\infty} \lambda^{-m} S_1(l, m) \sum_{n=l}^{\infty} S_1(n, l) \frac{\lambda^n t^n}{n!} \\
 &= \sum_{n=0}^{\infty} \left(\sum_{l=0}^n \sum_{m=0}^l B_m^{(m-r+1)}(x+1) \lambda^{n-m} S_1(n, l) S_1(l, m) \right) \frac{t^n}{n!}.
 \end{aligned}$$

Thus by comparing the coefficients in (10) and (15), we have the following identity.

$$C_{n,\lambda}^{(2,r)}(x) = \sum_{l=0}^n \sum_{m=0}^l B_m^{(m-r+1)}(x+1) \lambda^{n-m} S_1(n, l) S_1(l, m).$$

Inductively we have the following theorem, which represents k -times degenerated Cauchy polynomials of order r .

Theorem 9. For $n_i \geq 0, i = 1, 2, \dots, k + 1$, we have

$$C_{n_{k+1},\lambda}^{(k,r)}(x) = \sum_{n_k=0}^{n_{k+1}} \dots \sum_{n_1=0}^{n_2} B_{n_1}^{(n_1-r+1)}(x+1) \lambda^{n_{k+1}-n_1} \prod_{i=1}^k S_1(n_{i+1}, n_i).$$

From above Theorem 9, specially for the case $k = 1$ or $r = 1$, we have the following well-known results.

Corollary 10. For $n \geq 0$ and $n_i \geq 0, i = 1, \dots, k + 1$, we have

$$\begin{aligned}
 (i) \quad & C_{n,\lambda}^{(r)}(x) = \sum_{m=0}^n B_m^{(m-r+1)}(x+1) S_1(n, m) \lambda^{n-m}. \\
 (ii) \quad & C_{n_{k+1},\lambda}^{(k)}(x) = \sum_{n_k=0}^{n_{k+1}} \dots \sum_{n_1=0}^{n_2} B_{n_1}^{(n_1)}(x+1) \lambda^{n_{k+1}-n_1} \times \prod_{i=1}^k S_1(n_{i+1}, n_i).
 \end{aligned}$$

In particular,

$$C_{n_{k+1},\lambda}^{(k)} = \sum_{n_k=0}^{n_{k+1}} \dots \sum_{n_1=0}^{n_2} B_{n_1}^{(n_1)}(1) \lambda^{n_{k+1}-n_1} \times \prod_{i=1}^k S_1(n_{i+1}, n_i).$$

Proof. (i) See [10, Theorem 2.4].

(ii) See [3, Theorem 8]. □

Observe that the generating function of 2-times degenerate Cauchy polynomials of order r in (10), we have the following identity.

$$\begin{aligned}
 (16) \quad & \sum_{n=0}^{\infty} C_{n,\lambda}^{(2,r)}(x) \frac{t^n}{n!} \\
 &= \left(\sum_{l=0}^{\infty} C_{l,\lambda}^{(2,r)} \frac{t^l}{l!} \right) \left(1 + \log(1 + \log(1 + \lambda t))^{\frac{1}{\lambda}} \right)^x \\
 &= \left(\sum_{l=0}^{\infty} C_{l,\lambda}^{(2,r)} \frac{t^l}{l!} \right) \left(\sum_{m=0}^{\infty} (x)_m \frac{1}{m!} \left(1 + \log(1 + \log(1 + \lambda t))^{\frac{1}{\lambda}} \right)^m \right) \\
 &= \left(\sum_{l=0}^{\infty} C_{l,\lambda}^{(2,r)} \frac{t^l}{l!} \right) \left(\sum_{m=0}^{\infty} (x)_m \sum_{j=m}^{\infty} S_1(j, m) \lambda^{-m} \log(1 + \lambda t)^j \frac{1}{j!} \right) \\
 &= \left(\sum_{l=0}^{\infty} C_{l,\lambda}^{(2,r)} \frac{t^l}{l!} \right) \left(\sum_{m=0}^{\infty} (x)_m \sum_{j=m}^{\infty} S_1(j, m) \lambda^{-m} \sum_{i=j}^{\infty} S_1(i, j) \frac{\lambda^i t^i}{i!} \right) \\
 &= \left(\sum_{l=0}^{\infty} C_{l,\lambda}^{(2,r)} \frac{t^l}{l!} \right) \left(\sum_{i=0}^{\infty} \left(\sum_{j=0}^i \sum_{m=0}^j (x)_m \lambda^{i-m} S_1(i, j) S_1(j, m) \right) \frac{t^i}{i!} \right).
 \end{aligned}$$

Therefore by comparing the coefficients of (10) and (16), we obtain the following theorem.

Theorem 11. *For $n \geq 0$, we have*

$$C_{n,\lambda}^{(2,r)}(x) = \sum_{i=0}^n \sum_{j=0}^i \sum_{m=0}^j (x)_m \binom{n}{i} \lambda^{i-m} S_1(i, j) S_1(j, m) C_{n-i,\lambda}^{(2,r)}.$$

We try k -times degenerate Cauchy polynomial of order r , inductively we have the following theorem.

Theorem 12. *For $n_{k+1} \geq 0$, we have*

$$C_{n_{k+1},\lambda}^{(k,r)}(x) = \sum_{n_k=0}^{n_{k+1}} \sum_{n_{k-1}=0}^{n_k} \cdots \sum_{n_0=0}^{n_1} (x)_{n_1} \binom{n_{k+1}}{n_k} \lambda^{n_k-n_1} C_{n_{k+1}-n_k,\lambda}^{(k,r)} \prod_{i=0}^k S_1(n_{i+1}, n_i).$$

Specially for the case $k = 1$ or $r = 1$ we have the following well-known result.

Corollary 13. *For $n \geq 0$, and $n_i \geq 0$, $i = 1, 2, \dots, k + 1$, we have*

- (i) $C_{n,\lambda}^{(r)}(x) = \sum_{k=0}^n \sum_{m=0}^k \binom{n}{k} S_1(k, m) (x)_m C_{n-k,\lambda}^{(r)}$, and
- (ii) $C_{n_{k+1},\lambda}^{(k)}(x) \sum_{n_k=0}^{n_{k+1}} \cdots \sum_{n_0=0}^{n_1} \binom{n_{k+1}}{n_k} (x)_{n_1} \lambda^{n_k-n_1} \prod_{j=0}^k S_1(n_{j+1}, n_j) C_{n_{k+1}-n_k,\lambda}^{(k)}$

$$\begin{aligned}
 &= \sum_{n_k=0}^{n_{k+1}} \cdots \sum_{n_0=0}^{n_1} \binom{n_{k+1}}{n_k} (x)_{n_1} \lambda^{n_k-n_1} \prod_{j=0}^k S_1(n_{j+1}, n_j) \times \alpha, \text{ where} \\
 \alpha &= \sum_{m_k=0}^{n_{k+1}-n_k} \sum_{m_{k-1}=0}^{m_k} \cdots \sum_{m_0=0}^{m_1} \lambda^{n_{k+1}-n_k-m_k} S_1(n_{k+1}-n_k, m_k) \prod_{i=0}^{k-1} S_1(m_{i+1}, m_i) \frac{1}{m_0+1}.
 \end{aligned}$$

Proof. (i) See [10, Theorem 2.5].

(ii) See [3, Theorem 6]. □

T. Kim defined the degenerate Cauchy polynomials of the second kind of order r as follows:

$$\begin{aligned}
 (17) \quad &\int_0^1 \cdots \int_0^1 \left(1 + \log(1 + \lambda t)^{\frac{1}{\lambda}}\right)^{-x_1 - \cdots - x_r + x} dx_1 \cdots dx_r \\
 &= \left(\frac{\log(1 + \lambda t)^{\frac{1}{\lambda}}}{\log\left(1 + \log(1 + \lambda t)^{\frac{1}{\lambda}}\right) \left(\log(1 + \lambda t)^{\frac{1}{\lambda}} + 1\right)} \right)^r \left(\log(1 + \lambda t)^{\frac{1}{\lambda}} + 1\right)^x \\
 &= \sum_{n=0}^{\infty} \tilde{C}_{n,\lambda}^{(r)}(x) \frac{t^n}{n!} \quad (\text{see [10], equation (2.10)}).
 \end{aligned}$$

Now we degenerate one more time the above (17) degenerated Cauchy polynomials of the second kind as follows:

$$\begin{aligned}
 (18) \quad &\int_0^1 \cdots \int_0^1 \left(1 + \log(1 + \log(1 + \lambda t))^{\frac{1}{\lambda}}\right)^{-x_1 - \cdots - x_r + x} dx_1 \cdots dx_r \\
 &= \left(\frac{\log(1 + \log(1 + \lambda t))^{\frac{1}{\lambda}}}{\log\left(1 + \log(1 + \log(1 + \lambda t))^{\frac{1}{\lambda}}\right) \left(\log(1 + \log(1 + \lambda t))^{\frac{1}{\lambda}} + 1\right)} \right)^r \\
 &\quad \times \left(\log(1 + \log(1 + \lambda t))^{\frac{1}{\lambda}} + 1\right)^x \\
 &= \sum_{n=0}^{\infty} \tilde{C}_{n,\lambda}^{(2,r)}(x) \frac{t^n}{n!}.
 \end{aligned}$$

Thus, by (18) and (6), we get

$$\tilde{C}_{n,\lambda}^{(2,r)}(x) = \sum_{l=0}^n \sum_{m=0}^l \lambda^{n-m} B_m^{(m-r+1)}(x-r+1) S_1(n, l) S_1(l, m).$$

Inductively we degenerate k -times the degenerate Cauchy polynomials of the second kind with order r , we get the following theorem.

Theorem 14. For $n_i \geq 0$, where $i = 0, 1, \dots, k + 1$, we have

$$\tilde{C}_{n_{k+1},\lambda}^{(k,r)}(x) = \sum_{n_k=0}^{n_{k+1}} \cdots \sum_{n_1=0}^{n_2} \lambda^{n_{k+1}-n_1} B_{n_1}^{(n_1-r+1)}(x-r+1) \prod_{i=1}^k S_1(n_{i+1}, n_i).$$

In case $k = 1$ in Theorem 14, we have ([10], equation(2.10)) as follows.

$$\tilde{C}_{n,\lambda}^{(r)}(x) = \sum_{l=0}^n \lambda^{n-l} B_l^{(l-r+1)}(x-r+1) S_1(n, l).$$

In case $r = 1$ in Theorem 14 we have ([3], Theorem 11) as follows.

$$\begin{aligned} \tilde{C}_{n_{k+1},\lambda}^{(k)}(x) &= \sum_{n_k=0}^{n_{k+1}} \sum_{n_{k-1}=0}^{n_k} \cdots \sum_{n_1=0}^{n_2} \lambda^{n_{k+1}-n_1} \left(\prod_{i=1}^k S_1(n_{i+1}, n_i) \right) \tilde{C}_{n_1,\lambda}(x) \\ &= \sum_{n_k=0}^{n_{k+1}} \sum_{n_{k-1}=0}^{n_k} \cdots \sum_{n_1=0}^{n_2} \lambda^{n_{k+1}-n_1} \left(\prod_{i=1}^k S_1(n_{i+1}, n_i) \right) B_{n_1}^{(n_1)}(x). \end{aligned}$$

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