

ON THE TRANSFORMATION FORMULA OF THE SLICE BERGMAN KERNELS IN THE QUATERNION VARIABLES

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ABSTRACT. In complex analysis, the Bergman kernels for two biholomorphically equivalent complex domains satisfy the transformation formula. Recently new Bergman theory of slice regular functions of the quaternion variables has been investigated. In this paper we construct the transformation formula of the slice Bergman kernels under slice biregular functions in the setting of the quaternion variables.

1. Introduction

Let Ω be a bounded domain in \mathbb{C}^n and consider the Bergman space $L_a^2(\Omega)$ defined by the set of all square-integrable holomorphic functions on Ω . For any $z \in \Omega$, $\Phi_z : L_a^2(\Omega) \rightarrow \mathbb{C}$ given by $\Phi_z(f) = f(z)$ is a bounded linear functional on $L_a^2(\Omega)$. By Riesz representation theorem, there exists an element $K_z(\cdot) \in L_a^2(\Omega)$ such that $\Phi_z(f) = \langle f(\cdot), K_z(\cdot) \rangle$, namely

$$f(z) = \int_{\Omega} f(\zeta) \overline{K_z(\zeta)} dV(\zeta)$$

for all $f \in L_a^2(\Omega)$. In [1] the Bergman kernel for Ω is defined by

$$K_{\Omega}(z, \zeta) := \overline{K_z(\zeta)}$$

for $z, \zeta \in \Omega$. It plays an important role in studying the geometry of bounded domain in several complex variables. It has the nice transformation formula under biholomorphic mappings.

Proposition 1.1 ([8]). *Let Ω^1 and Ω^2 be domains in \mathbb{C}^n and $f : \Omega^1 \rightarrow \Omega^2$ be biholomorphic mapping. Then for any $z, \zeta \in \Omega^1$, it holds that*

$$(1.1) \quad K_{\Omega^1}(z, \zeta) = \det J_{\mathbb{C}}f(z) K_{\Omega^2}(f(z), f(\zeta)) \overline{\det J_{\mathbb{C}}f(\zeta)},$$

where $J_{\mathbb{C}}f(z)$ is the complex Jacobian matrix of f at z .

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Recently new Bergman theory of slice regular functions of a quaternion variable has been introduced and many interesting and significant functional properties on the setting of a quaternion variable have been studied. See the references [2, 3, 5, 6, 7].

We now collect some definitions on quaternions. By \mathbb{H} we denote the algebra of real quaternions and each element $q \in \mathbb{H}$ has the form $q = q_0 + q_1e_1 + q_2e_2 + q_3e_3$, where e_1, e_2, e_3 satisfy

- (i) $e_ie_j + e_je_i = -2\delta_{ij}$ for $1 \leq i, j \leq 3$,
- (ii) $e_1e_2 = e_3, e_2e_3 = e_1, e_3e_1 = e_2$.

In this case, we write the imaginary part (vector part) of $q \in \mathbb{H}$ by $\underline{q} := q_1e_1 + q_2e_2 + q_3e_3$. Also the conjugate of $q \in \mathbb{H}$ is denoted by $\bar{q} := q_0 - \underline{q}$. For the study of slice regular functions, we define two-dimensional sphere

$$\mathbb{S}^2 := \{\underline{q} \in \mathbb{R}^3 : q_1^2 + q_2^2 + q_3^2 = 1\}.$$

Given $I \in \mathbb{S}^2$, we define $L_I = \{x + yI : x, y \in \mathbb{R}\}$ the real linear space generated by 1 and I . Since $I^2 = -1$ for any $I \in \mathbb{S}^2$, we see that $L_I \cong \mathbb{C}$.

Similarly as holomorphic functions of a complex variable, we define the slice regular function as the function which is holomorphic on each complex plane L_I for all $I \in \mathbb{S}^2$. A real differentiable quaternion-valued function f defined on $\Omega \subset \mathbb{H}$ is called (left) slice regular on Ω if for any $I \in \mathbb{S}^2$, the restriction $f|_{\Omega_I}$ satisfies standard Cauchy-Riemann equations as following:

$$\bar{\partial}_I f(q) := \frac{1}{2} \left(\frac{\partial}{\partial x} + I \frac{\partial}{\partial y} \right) f|_{\Omega_I}(x + yI) = 0 \text{ on } \Omega_I,$$

where $q = x + yI$ for some $I \in \mathbb{S}^2$. The set of all slice regular functions on Ω is denoted by $\mathcal{SR}(\Omega)$. For two domains Ω^1 and Ω^2 in \mathbb{H} , the function $f : \Omega^1 \rightarrow \Omega^2$ is called a *slice biregular function* if $f \in \mathcal{SR}(\Omega^1)$ and there exists $f^{-1} \in \mathcal{SR}(\Omega^2)$ such that $f^{-1} \circ f$ and $f \circ f^{-1}$ are identities on Ω^1 and Ω^2 , respectively. In the setting of quaternion variables, when we consider the composition and the product of two slice regular functions, we restrict our attention to functions in

$$\mathcal{N}(\Omega) := \{f \in \mathcal{SR}(\Omega) : f(\Omega \cap L_I) \subset L_I \text{ for all } I \in \mathbb{S}^2\},$$

where Ω is a domain in \mathbb{H} .

In the theory of slice regular functions, it is natural to consider the domains $\Omega \in \mathbb{H}$ satisfying the following conditions.

Definition 1.2 ([2]). Let $\Omega \subset \mathbb{H}$ be a domain.

- (i) We say that Ω is a slice domain if $\Omega \cap \mathbb{R}$ is non-empty and $\Omega_I := \Omega \cap L_I$ is a domain in L_I for all $I \in \mathbb{S}^2$.
- (ii) We say that Ω is axially symmetric if for every $x + yI \in \Omega$, we have $x + yJ \in \Omega$ for all $J \in \mathbb{S}^2$.

Each regular function on a slice domain has the unique regular extension on the axially symmetric slice domain (see Theorem 4.1 in [2]). Thus axially

symmetric slice domains play, for slice regular functions, the similar role as the domains of holomorphy for holomorphic functions on \mathbb{C}^n .

Now we deal with the relation between slice Bergman kernels of two domains Ω^1 and Ω^2 when there exists a slice biregular function $\phi \in \mathcal{N}(\Omega^1)$ with $\phi(\Omega^1) = \Omega^2$. In 2012, Fabrizio Colombo, J. Oscar Gonzalez-Cervantes, and Irene Sabadini [5] proved the relation between the Bergman kernels for Ω_I^1 and Ω_I^2 as following.

Proposition 1.3 ([5]). *Let Ω^1 and Ω^2 be axially symmetric slice domains and there exists a slice biregular function $\phi \in \mathcal{N}(\Omega^1)$ with $\phi(\Omega^1) = \Omega^2$. Then*

$$K_{\Omega_I^1}(q, r) = K_{\Omega_I^2}(\phi(q), \phi(r)) * \phi'(q)\overline{\phi'(r)}$$

for all $q \in \Omega^1$ and all $r \in \Omega_I^1$.

Here the $*$ -product will be explained in Section 2. The above result is very interesting, but it is a transformation formula of the Bergman kernels for the slice domains Ω_I^1 and Ω_I^2 (not for the whole domains Ω^1 and Ω^2), although it works with the extension of the Bergman kernel in the first coordinate on the whole domain Ω^1 .

In this paper we prove the transformation formula of the slice Bergman kernels K_{Ω^1} and K_{Ω^2} as following.

Main Theorem. *Let Ω^1 and Ω^2 be axially symmetric slice domains and there exists a slice biregular function $\phi \in \mathcal{N}(\Omega^1)$ with $\phi(\Omega^1) = \Omega^2$. Then for any $q \in \Omega^1 \cap L_I$ and $r \in \Omega^1 \cap L_J$, it holds that*

$$\begin{aligned} & K_{\Omega^1}(q, r) \\ = & \frac{1}{4}((1 - IJ)\phi'(q^*)(1 - JI) + (1 + IJ)\phi'(\overline{q^*})(1 + JI))K_{\Omega^2}(\phi(q), \phi(r))\overline{\phi'(r)} \\ & + \frac{1}{4}((1 - IJ)\phi'(q^*)(1 + JI) + (1 + IJ)\phi'(\overline{q^*})(1 - JI))K_{\Omega^2}(\phi(\overline{q}), \phi(r))\overline{\phi'(r)}. \end{aligned}$$

In Section 2, we review the concepts of the Bergman kernel K_{Ω_I} and the slice Bergman kernel K_{Ω} . In Section 3, we prove our Main Theorem using complex transformation formula (Proposition 1.1). In the setting of slice monogenic functions with values in the Clifford algebra \mathbb{R}_n , our Main Theorem also holds, since our proof relies on the fact that \mathbb{H} is a division algebra.

2. Slice Bergman kernels

We begin this section with introducing two properties which are basic and useful materials for studying the regular functions. Let Ω be a domain in \mathbb{H} . Each slice regular function on Ω is represented as the direct sum of two complex holomorphic functions on each slice $\Omega_I = \Omega \cap L_I$.

Proposition 2.1 (Splitting lemma). *Let $f \in \mathcal{SR}(\Omega)$. For all $I \in \mathbb{S}^2$ and for all $J \in \mathbb{S}^2$ with $I \perp J$, there exist two holomorphic functions $F, G : \Omega \cap L_I \rightarrow L_I$*

such that

$$f|_{\Omega_I} = F + GJ.$$

If we restrict our attention to an axially symmetric slice domain, then the values of any slice of slice regular functions are extended to other slice.

Proposition 2.2 (Representation Formula [4, 6]). *Let f be a slice regular function on an axially symmetric slice domain $\Omega \subset \mathbb{H}$. Choose any $J \in \mathbb{S}^2$. Then for all $q = x + yI \in \Omega$, it holds that*

$$f(x + yI) = \frac{1}{2}(1 - IJ)f(x + yJ) + \frac{1}{2}(1 + IJ)f(x - yJ).$$

Now we review the Bergman theory on axially symmetric slice domains. Let Ω be an axially symmetric slice domain in \mathbb{H} and $f \in \mathcal{SR}(\Omega)$. For any $I \in \mathbb{S}^2$, define the Bergman space by

$$\mathcal{A}(\Omega_I) := \left\{ f \in \mathcal{SR}(\Omega) : \|f\|_{\mathcal{A}(\Omega_I)}^2 := \int_{\Omega_I} |f|^2 d\sigma_I < \infty \right\}$$

and the scalar product

$$\langle f, g \rangle_{\mathcal{A}(\Omega_I)} = \int_{\Omega_I} \bar{f}g d\sigma_I.$$

Then it is known that $(\mathcal{A}(\Omega_I), \|\cdot\|_{\mathcal{A}(\Omega_I)})$ is complete for every $I \in \mathbb{S}^2$. Similarly as the complex Bergman space, for every $q \in \Omega_I$, the evaluation map $\Phi_q : \mathcal{A}(\Omega_I) \rightarrow \mathbb{H}$ given by $\Phi_q(f) = f(q)$ is a bounded right-linear functional on $\mathcal{A}(\Omega_I)$ for every $I \in \mathbb{S}^2$. By Riesz representation theorem, there exists the unique function $K_q(\cdot) \in \mathcal{A}(\Omega_I)$ such that $\Phi_q(f) = \langle K_q(\cdot), f \rangle_{\mathcal{A}(\Omega_I)}$ for all $f \in \mathcal{A}(\Omega_I)$. Define the Bergman kernel for Ω_I as $K_{\Omega_I}(q, \cdot) := \overline{K_q(\cdot)}$. The Bergman kernel K_{Ω_I} is a quaternion-valued function, but it satisfies Cauchy-Riemann equation on each slice $\Omega_I = \Omega \cap L_I$. Thus we have

Proposition 2.3 ([4]). *The Bergman kernel K_{Ω_I} for Ω_I coincides with the complex Bergman kernel for Ω_I for any $I \in \mathbb{S}^2$.*

By using Representation Formula (Proposition 2.2), we can define the slice Bergman kernel for the whole domain Ω .

Definition 2.4. Let $K_{\Omega_I}(\cdot, \cdot) : \Omega_I \times \Omega_I \rightarrow \mathbb{H}$ be the Bergman kernel for Ω_I . The slice Bergman kernel K_{Ω} for Ω is defined by a function $K_{\Omega} : \Omega \times \Omega \rightarrow \mathbb{H}$ satisfying

$$K_{\Omega}(x + yI, r) := \frac{1}{2}(1 - IJ)K_{\Omega_J}(x + yJ, r) + \frac{1}{2}(1 + IJ)K_{\Omega_J}(x - yJ, r).$$

Definition 2.5. (i) Define the operator $Q_I : \mathcal{SR}(\Omega) \rightarrow \text{Hol}(\Omega_I) + \text{Hol}(\Omega_I)J$ by

$$Q_I(f) = f|_{\Omega_I} = F + JG,$$

which is well-defined for any chosen $I, J \in \mathbb{S}^2$ with $I \perp J$.

(ii) Define the operator $P_I : Hol(\Omega_I) + Hol(\Omega_I)J \rightarrow \mathcal{SR}(\Omega)$ by

$$P_I[f](q) = P_I[f](x + yI_q) = \frac{1}{2}[(1 + I_q I)f(x - yI) + (1 - I_q I)f(x + yI)].$$

The operator P_I extends a function $F + GJ$ defined on Ω_I to the smallest axially symmetric domain containing Ω_I . The operator P_I is well-defined by the Representation Formula.

Remark 2.6. Note that $Q_I \circ P_I$ and $P_I \circ Q_I$ are identities.

Let $f, g \in \mathcal{SR}(\Omega)$ and $I \in \mathbb{S}^2$. Then by splitting lemma (Proposition 2.1), we have

$$Q_I[f] = f_1 + f_2 J, \quad Q_I[g] = g_1 + g_2 J,$$

where $J \in \mathbb{S}^2$ is orthogonal to I . The regular product is defined as

$$Q_I[f] * Q_I[g](z) := [f_1(z)g_1(z) - f_2(z)\overline{g_2(\bar{z})}] + [f_1(z)g_2(z) + f_2(z)\overline{g_1(\bar{z})}]J.$$

Then the $*$ -product $f * g$ is defined to be the slice regular extension of $Q_I[f] * Q_I[g]$ as

$$f * g := P_I[Q_I[f] * Q_I[g]].$$

Proposition 2.7 ([5]). *Let Ω^1 and Ω^2 be axially symmetric slice domains and there exists a slice biregular function $\phi \in \mathcal{N}(\Omega^1)$ with $\phi(\Omega^1) = \Omega^2$. Then*

$$K_{\Omega_I^1}(q, r) = K_{\Omega_I^2}(\phi(q), \phi(r)) * \phi'(q)\overline{\phi'(r)},$$

for all $q \in \Omega^1$ and all $r \in \Omega_I^1$.

3. Transformation formula of the Bergman kernel

In this section we prove our Main Theorem on the transformation formula of the slice Bergman kernels under slice biregular function. Let Ω^1 and Ω^2 be axially symmetric slice domains in \mathbb{H} and $\phi : \Omega^1 \rightarrow \Omega^2$ be a slice biregular function. And we assume that ϕ is a member of

$$\mathcal{N}(\Omega^1) = \{\phi \in \mathcal{SR}(\Omega^1) : \phi(\Omega^1 \cap L_I) \subset L_I \text{ for all } I \in \mathbb{S}^2\}.$$

For any $q, r \in \Omega^1$, we choose $I, J \in \mathbb{S}^2$ such that

$$q \in \Omega_I^1 = \Omega^1 \cap L_I, \quad r \in \Omega_J^1 = \Omega^1 \cap L_J.$$

For convenience, we write

$$q = x + yI, \quad q^* = x + yJ, \quad \overline{q^*} = x - yJ.$$

Lemma 3.1. *Let $\phi \in \mathcal{N}(\Omega)$. Then $\phi(q)^* = \phi(q^*)$ and $\overline{\phi(q)} = \phi(\overline{q})$.*

Proof. Since $\phi \in \mathcal{N}(\Omega^1)$, we have $\phi(q) = a(x, y) + b(x, y)I$ for some real-valued functions $a(x, y)$ and $b(x, y)$. From $I^2 = J^2 = -1$ and $L_I \cong L_J \cong \mathbb{C}$, it is easily obtained that $\phi(q)^* = \phi(q^*)$ and $\overline{\phi(q)} = \phi(\overline{q})$. \square

Now we prove our transformation formula.

Theorem 3.2 (Main Theorem). *Let Ω^1 and Ω^2 be axially symmetric slice domains and there exists a slice biregular function $\phi \in \mathcal{N}(\Omega^1)$ with $\phi(\Omega^1) = \Omega^2$. Then for any $q \in \Omega^1 \cap L_I$ and $r \in \Omega^1 \cap L_J$, it holds that*

$$\begin{aligned} & K_{\Omega^1}(q, r) \\ &= \frac{1}{4}((1 - IJ)\phi'(q^*)(1 - JI) + (1 + IJ)\phi'(\overline{q^*})(1 + JI))K_{\Omega^2}(\phi(q), \phi(r))\overline{\phi'(r)} \\ & \quad + \frac{1}{4}((1 - IJ)\phi'(q^*)(1 + JI) + (1 + IJ)\phi'(\overline{q^*})(1 - JI))K_{\Omega^2}(\phi(\overline{q}), \phi(r))\overline{\phi'(r)}. \end{aligned}$$

Note that in general, $JI \neq -IJ$ for $I, J \in \mathbb{S}^2$, since

$$IJ + JI = -\langle I, J \rangle + I \times J - \langle J, I \rangle + J \times I = -2\langle I, J \rangle.$$

Proof. By representation formula, we have

$$(3.1) \quad K_{\Omega^1}(q, r) = \frac{1}{2}(1 - IJ)K_{\Omega^1_j}(q^*, r) + \frac{1}{2}(1 + IJ)K_{\Omega^1_j}(\overline{q^*}, r).$$

Note that q^* and r lie in the same slice of Ω^1 , so that $q^*, r \in \Omega^1_j$. By Proposition 2.3, the kernel $K_{\Omega^1_j}$ ($k = 1, 2$) is identical to the complex Bergman kernel for Ω^k_j and it satisfies the transformation formula (1.1). Thus combining Lemma 3.1 and Proposition (1.1) we have

$$(3.2) \quad K_{\Omega^1_j}(q^*, r) = \phi'(q^*)K_{\Omega^2_j}(\phi(q^*), \phi(r))\overline{\phi'(r)}$$

and

$$(3.3) \quad K_{\Omega^1_j}(\overline{q^*}, r) = \phi'(\overline{q^*})K_{\Omega^2_j}(\phi(\overline{q^*}), \phi(r))\overline{\phi'(r)}.$$

On the other hand, by representation formula and Lemma 3.1 again, we have

$$K_{\Omega^2}(\phi(q), \phi(r)) = \frac{1}{2}(1 - IJ)K_{\Omega^2_j}(\phi(q^*), \phi(r)) + \frac{1}{2}(1 + IJ)K_{\Omega^2_j}(\phi(\overline{q^*}), \phi(r)).$$

If we substitute \overline{q} for q in the above identity, then since $(\overline{q})^* = \overline{q^*}$, we have

$$K_{\Omega^2}(\phi(\overline{q}), \phi(r)) = \frac{1}{2}(1 - IJ)K_{\Omega^2_j}(\phi(\overline{q^*}), \phi(r)) + \frac{1}{2}(1 + IJ)K_{\Omega^2_j}(\phi(q^*), \phi(r)).$$

From the above two identities, we obtain

$$K_{\Omega^2}(\phi(q), \phi(r)) + K_{\Omega^2}(\phi(\overline{q}), \phi(r)) = K_{\Omega^2_j}(\phi(q^*), \phi(r)) + K_{\Omega^2_j}(\phi(\overline{q^*}), \phi(r))$$

and

$$\begin{aligned} & K_{\Omega^2}(\phi(q), \phi(r)) - K_{\Omega^2}(\phi(\overline{q}), \phi(r)) \\ &= -IJK_{\Omega^2_j}(\phi(q^*), \phi(r)) + IJK_{\Omega^2_j}(\phi(\overline{q^*}), \phi(r)). \end{aligned}$$

Since $(IJ)^{-1} = J^{-1}I^{-1} = (-J)(-I) = JI$, it follows that

$$(3.4) \quad K_{\Omega^2_j}(\phi(q^*), \phi(r)) = \frac{1}{2}(1 - JI)K_{\Omega^2}(\phi(q), \phi(r)) + \frac{1}{2}(1 + JI)K_{\Omega^2}(\phi(\overline{q}), \phi(r))$$

and

(3.5)

$$K_{\Omega_j^2}(\phi(\bar{q}^*), \phi(r)) = \frac{1}{2}(1 + JI)K_{\Omega^2}(\phi(q), \phi(r)) + \frac{1}{2}(1 - JI)K_{\Omega^2}(\phi(\bar{q}), \phi(r)).$$

Combining the identities (3.1) ~ (3.4), we obtain

$$\begin{aligned} K_{\Omega_j^1}(q^*, r) &= \frac{1}{2}\phi'(q^*)(1 - JI)K_{\Omega^2}(\phi(q), \phi(r))\overline{\phi'(r)} \\ &\quad + \frac{1}{2}\phi'(q^*)(1 + JI)K_{\Omega^2}(\phi(\bar{q}), \phi(r))\overline{\phi'(r)} \end{aligned}$$

and

$$\begin{aligned} K_{\Omega_j^1}(\bar{q}^*, r) &= \frac{1}{2}\phi'(\bar{q}^*)(1 - JI)K_{\Omega^2}(\phi(q), \phi(r))\overline{\phi'(r)} \\ &\quad + \frac{1}{2}\phi'(\bar{q}^*)(1 + JI)K_{\Omega^2}(\phi(\bar{q}), \phi(r))\overline{\phi'(r)}. \end{aligned}$$

Thus by the representation formula (3.1), we prove

$$\begin{aligned} K_{\Omega^1}(q, r) &= \frac{1}{2}(1 - IJ)K_{\Omega_j^1}(q^*, r) + \frac{1}{2}(1 + IJ)K_{\Omega_j^1}(\bar{q}^*, r) \\ &= \frac{1}{4}(1 - IJ)\phi'(q^*)(1 - JI)K_{\Omega^2}(\phi(q), \phi(r))\overline{\phi'(r)} \\ &\quad + \frac{1}{4}(1 - IJ)\phi'(q^*)(1 + JI)K_{\Omega^2}(\phi(\bar{q}), \phi(r))\overline{\phi'(r)} \\ &\quad + \frac{1}{4}(1 + IJ)\phi'(\bar{q}^*)(1 + JI)K_{\Omega^2}(\phi(q), \phi(r))\overline{\phi'(r)} \\ &\quad + \frac{1}{4}(1 + IJ)\phi'(\bar{q}^*)(1 - JI)K_{\Omega^2}(\phi(\bar{q}), \phi(r))\overline{\phi'(r)}, \end{aligned}$$

which is our desired formula. □

Remark 3.3. The transformation formula of the slice Bergman kernels is more complicated than the complex setting. But from Theorem 3.2, if q and r lie in the same slice, then we see that

$$K_{\Omega^1}(q, r) = \phi'(q)K_{\Omega^2}(\phi(q), \phi(r))\overline{\phi'(r)},$$

where $q, r \in \Omega^1$.

In the complex setting, the property of the existence of zeros of the Bergman kernel is invariant under biholomorphic mappings by Proposition 1.1. In the quaternionic setting, the author of this paper proved in [9] that the slice Bergman kernels of the upper half plane

$$\mathbb{H}^+ = \{q = q_0 + q_1e_1 + q_2e_2 + q_3e_3 : q_0 > 0\}$$

and the unit disk

$$\mathbb{B} = \{q = q_0 + q_1e_1 + q_2e_2 + q_3e_3 : q_0^2 + q_1^2 + q_2^2 + q_3^2 < 1\}$$

have no zeros. Although the invariance is not known yet, we obtain the following result from Theorem 3.2.

Corollary 3.4. *Let Ω^1, Ω^2, ϕ be defined as in the hypothesis of Theorem 3.2. If there exist $q, r \in \Omega^1$ such that*

$$K_{\Omega^2}(\phi(q), \phi(r)) = K_{\Omega^2}(\phi(\bar{q}), \phi(r)) = 0,$$

then $K_{\Omega^1}(q, r) = 0$.

In fact, the Calyey transformation $\phi(q) = (1 - q)^{-1}(1 + q)$ is a slice regular function from \mathbb{B} onto \mathbb{H}^+ with the regular inverse function $\psi(q) = (q - 1)(q + 1)^{-1}$. Also note that ϕ and ϕ' belong to the class $\mathcal{N}(\Omega^1)$. If $\phi' \in \mathcal{N}(\Omega^1)$, then the transformation formula is reduced to the following simple form.

Corollary 3.5. *Let Ω^1, Ω^2, ϕ be defined as in the hypothesis of Theorem 3.2. If $\phi' \in \mathcal{N}(\Omega^1)$, then*

$$K_{\Omega^1}(q, r) = \phi'(q^*)K_{\Omega^2}(\phi(q), \phi(r))\overline{\phi'(r)}.$$

Proof. Assume that $\phi'(q^*) = a + bJ$ for some $a, b \in \mathbb{R}$. Then we have

$$\begin{aligned} & (1 - IJ)(a + bJ)(1 - JI) + (1 + IJ)(a - bJ)(1 + JI) \\ &= (a + aJI + bJ - bI - aIJ - a + bI + bIJI) \\ & \quad + (a - aJI - bJ - bI + aIJ - a + bI - bIJI) \\ &= 4(a + bI). \end{aligned}$$

Also note that

$$(1 - IJ)(a + bJ)(1 + JI) + (1 + IJ)(a - bJ)(1 - JI) = 0.$$

By Theorem 3.2, we obtain the desired formula. □

4. Slice monogenic functions in the Clifford algebra \mathbb{R}_n

The theory of slice regular functions has been studied for functions from an open set $\Omega \subset \mathbb{R}^{n+1}$ to the Clifford algebra \mathbb{R}_n generated over n imaginary units. A real differentiable function $f : \Omega \subset \mathbb{R}^{n+1} \rightarrow \mathbb{R}_n$ is called slice monogenic if it satisfies

$$\left(\frac{\partial}{\partial x} + I \frac{\partial}{\partial y} \right) f|_{\Omega_I}(x + yI) = 0, \quad \text{on } \Omega_I = \Omega \cap L_I$$

for any $I \in \mathbb{S}^{n-1} := \{(q_1, \dots, q_n) \in \mathbb{R}^n : q_1^2 + \dots + q_n^2 = 1\}$. Since the representation formula (Proposition 2.2) also holds for slice monogenic functions, the slice Bergman spaces and the slice Bergman kernels for Ω for slice monogenic functions were introduced in the same method explained in Section 2. Thus our Main Theorem also holds for slice monogenic functions, since we only use the fact that \mathbb{H} is a division algebra.

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