Bull. Korean Math. Soc. **53** (2016), No. 5, pp. 1395–1399
http://dx.doi.org/10.4134/BKMS.b150719
pISSN: 1015-8634 / eISSN: 2234-3016

ON FINITE BARELY NON-ABELIAN p-GROUPS

Dong Seung Kang

Abstract. We will classify the finite barely non-abelian p-groups.

1. Introduction

It is an important theme to determine the structure of a group by using its subgroup in the group theory. Let p be prime and let G be a finite p-group. An old group-theoretic result of Rèdei [3] proved that if every proper subgroup of G is abelian then either abelian or minimal non-abelian. Blackburn [1] showed that if every proper subgroup of G is generated by two elements then G is either metacyclic or a 3-group of maximal class with a few exceptions. Minimal non-abelian p-groups have been investigated in recent years; see [6], [7] and [8].

We will say that a p-group G is *barely non-abelian* if it satisfies the following conditions: (1) every proper subgroup of G is abelian and (2) if $H_0 \subsetneq H \subset G$ are subgroups, where H is cyclic and H_0 is normal in G, then G/H_0 is abelian. For $p = 2$, this class of groups naturally came up in [2]. The main result of [2] relies on the classification of barely non-abelian 2-groups; see [2, Proposition 4.6]. The proof of [2, Proposition 4.6] depends, in turn, on a result of Rèdei; see [3]. The purpose of this paper is to classify barely non-abelian p-groups for every prime p. Our main result is as follows.

Theorem 1.1. *A non-abelian p-group G is barely non-abelian if and only if $|G| = p^3$ or G is isomorphic to $M(p^k)$, where $k \geq 4$.*

The remainder of this paper will be devoted to proving this theorem. Our proof will be entirely elementary; we will not appeal to Rèdei's theorem. In particular, for $p = 2$ we will give a new elementary proof of [2, Proposition 4.6].

2. Barely non-abelian p-groups

In this section, we introduce a barely non-abelian p-group G and investigate the properties of G.

Received September 3, 2015; Revised November 18, 2015.
2010 *Mathematics Subject Classification.* 20D15, 11E04.
Key words and phrases. minimal non-abelian groups, metacyclic, barely non-abelian groups, p-groups.

Definition 2.1. We call a finite non-abelian p-group G *barely non-abelian p-group* if it satisfies the following conditions:
(1) every proper subgroup of G is abelian,
(2) if $H_0 \subsetneq H \subset G$ are subgroups, where H is cyclic and H_0 is normal in G, then G/H_0 is abelian.

Example 2.2. Let $p = 2$. Q_8, D_8 and $M(2n)$ ($n \geq 4$ a power of 2) are barely non-abelian 2-groups. We define the group $M(2n)$ as the semidirect product of $\mathbb{Z}/n\mathbb{Z} \rtimes \mathbb{Z}/2\mathbb{Z}$, where the nontrivial element of $\mathbb{Z}/2\mathbb{Z}$ acts on $\mathbb{Z}/n\mathbb{Z}$ by sending 1 to $\frac{n}{2} + 1$. Equivalently,

$$M(2n) = \{r, s \mid r^n = s^2 = 1,\ sr = r^{n/2+1}s\}. \tag{2.1}$$

Note that $M(8)$ is the dihedral group D_8 ; see [2, Proposition 4.6].

Let p be an odd prime. There are two barely non-abelian p-groups of order p^3.

$$G_1 = \langle r, s \mid r^{p^2} = s^p = 1,\ sr = r^{p+1}s\rangle,$$
$$G_2 = \langle r, s \mid r^p = s^p = c^p = 1,\ rc = cr,\ sc = cs,\ sr = crs\rangle$$

We know that the barely non-abelian p-groups of order p^3 is isomorphic to the semidirect product of $\mathbb{Z}/p^2\mathbb{Z}$ and $\mathbb{Z}/p\mathbb{Z}$ for G_1 and $\mathbb{Z}/p\mathbb{Z} \times \mathbb{Z}/p\mathbb{Z}$ and $\mathbb{Z}/p\mathbb{Z}$ for G_2, respectively.

Now, we define the group $M(p^n)$ as the semidirect product of $\mathbb{Z}/p^{n-1}\mathbb{Z} \rtimes \mathbb{Z}/p\mathbb{Z}$, that is,

$$M(p^n) = \{r, s \mid r^{p^{n-1}} = s^p = 1,\ sr = r^{p^{n-2}+1}s\}, \tag{2.2}$$

where $n \geq 4$. Note that $M(p^3) = G_1$.

Lemma 2.3. (a) *Every proper subgroup of $M(p^n)$ is abelian.*
(b) *Every proper quotient of $M(p^n)$ is abelian.*
(c) *$M(p^n)$ is barely non-abelian for any $n \geq 4$.*

Proof. (a) Let S be a proper subgroup of $M(p^n)$. If S contains the index p subgroup $\langle r\rangle$, then $S = \langle r\rangle$ and hence S is abelian. If not, let $S_0 = S \cap \langle r\rangle$. Then $S_0 \subset \langle r^p\rangle$ is central in $M(p^n)$. Hence

$$S/S_0 \subset M(p^n)/\langle r\rangle \simeq \mathbb{Z}/p\mathbb{Z},$$

that is, S/S_0 is cyclic. Thus S is abelian, as desired.

(b) Assume $M(p^n)/N$ is not abelian for some non-trivial normal subgroup N of $M(p^n)$. Then N cannot contain $r^{p^{n-2}}$. Otherwise,

$$(sN)(rN) = srN = r^{p^{n-2}+1}sN = (rN)(sN).$$

Hence we have

$$N \cap \langle r\rangle = \{1\}.$$

Since $\langle r\rangle$ has an index p in $M(p^n)$, this implies that $|N| = p$. Moreover, N and $\langle r\rangle$ are complementary normal subgroups in $M(p^n)$. Thus $M(p^n) \simeq N \times \langle r\rangle$ is abelian. It is a contradiction.

(c) follows from (a) and (b). □

Theorem 2.4. *Suppose G be a barely non-abelian p-group of order $\geq p^4$.*

(a) *The center $Z(G)$ has index p^2 in G.*
(b) *If S is a proper subgroup of G, then $[S : (S \cap Z(G))] \leq p$.*
(c) *$x^p \in Z(G)$ for every $x \in G$.*

Let G' be the commutator subgroup of G.

(d) *$G' \subset Z(G)$.*
(e) *$|G'| = p$. In the sequel we shall denote the non-identity element of G' by c, that is, $G' = \langle c \rangle$ of order p.*
(f) *If $x \in G$ is an element of order $n \geq p^2$ then $x^{n/p} \in G'$.*
(g) *G is generated by two elements r and s such that $rs = csr$.*

Proof. (a) Let H be a subgroup of index p in G; see, e.g., [4, 5.3.1(ii)]. Choose $g \in G \backslash H$; applying [4, 5.3.1(ii)] once again, we can find a subgroup $H' \subset G$ such that $g \in H'$ and $[G : H'] = p$. Since G is a barely non-abelian group, both H and H' are abelian. Thus every $x \in H \cap H'$ commutes with g and with every element of H. Since H and g generate G, we conclude that $x \in Z(G)$, i.e.,

$$H \cap H' \subset Z(G). \tag{2.3}$$

Since G is non-abelian,

$$[G : Z(G)] \geq p^2\,; \tag{2.4}$$

see, e.g., [5, 6.3.4]. On the other hand, since $[G : H] = [G : H'] = p$, it is easy to see that

$$[G : (H \cap H')] = p^2. \tag{2.5}$$

Part (a) now follows from (2.3-2.5). For future reference we remark that our argument also shows that

$$H \cap H' = Z(G). \tag{2.6}$$

(b) By [4, 5.3.1(ii)], S is contained in a subgroup H of index p. By (2.6), $Z(G) = H \cap H'$, where H' is another subgroup of G of index p. Then $S \cap Z(G) = S \cap H'$, and the latter clearly has index $\leq p$ in S.

(c) Apply part (b) to the cyclic group $S = \langle x \rangle$.

(d) Follows from the fact that the factor group $G/Z(G)$ has order p^2 and, hence, is abelian.

(e) Since G is a non-abelian p-group, it has an element r of order $n \geq p^2$. Let $H = \langle r \rangle$ and $H_0 = \langle r^{n/p} \rangle$ be cyclic subgroups of G of orders n and p respectively. By part (c), $r^{n/p} = \left(r^{n/p^2}\right)^p \in Z(G)$ and hence $H_0 = \langle r^{n/p} \rangle \subseteq Z(G)$. Then H_0 is normal in G. Since G is a barely non-abelian group, G/H_0 is abelian. In other words,

$$G' \subset H_0. \tag{2.7}$$

Thus $|G'| \leq |H_0| = p$. On the other hand, since G is non-abelian, $|G'| \neq 1$. Thus G' has exactly p elements, as claimed.

(f) By (2.7), $x^{n/p} \in G'$.

(g) Choose two non-commuting elements r and s in G. Since G is a barely non-abelian group, these elements generate G. By part (e), $rsr^{-1}s^{-1} \in G' = \langle c \rangle$. Without loss of generality, we may assume let $rs = csr$, as desired. □

We now proceed to give a complete list of barely non-abelian p-groups.

Theorem 2.5. *Let G be a barely non-abelian p-group. Then G is one of the following groups*:

(a) $|G| = p^3$.

(b) $M(p^k)$, *where* $k \geq 4$.

Proof. (a) follows from Example 2.2.

(b) Write $G = \langle r, s \rangle$, $G' = \langle c \rangle$, and $sr = crs$. Denote the orders of r and s by n and m respectively. We may assume without loss of generality that p is an odd prime, $|G| \geq p^4$ and $n \geq m$. Let $n = m = p$. Then G/G' is an abelian group of order $\leq p^2$. Hence $|G| \leq p^2|G'| = p^3$. Now let $n \geq m \geq p^2$. By Theorem 2.4(c), we have $r^{n/p} \in G'$, where the order of $r^{n/p}$ is p. We may assume that

$$G' = \langle r^{n/p} \rangle.$$

By Theorem 2.4(c) once again, $s^{m/p} \in G'$. Then there exists a positive integer t such that $s^{m/p} = \left(r^{n/p}\right)^t$. Let $\widetilde{s} = \left(r^{n/m}\right)^{-t} s$. We claim that

$$\widetilde{s}^{m/p} = 1. \tag{2.8}$$

We now consider two cases.

Case I: $m < n$.

$$\widetilde{s}^{m/p} = \left(\left(r^{n/m}\right)^{-t} s\right)^{m/p} = \left(r^{n/p}\right)^{-t} s^{m/p} = 1,$$

where $r^{n/m} \in Z(G)$, as claimed.

Case II: $m = n$. Then

$$\widetilde{s}^p = \left(\left(r^{n/m}\right)^{-t} s\right)^p = \left(r^{-t} s\right)^p = c^{pt} s^p r^{-pt},$$

where s^p and $(r^{-t})^p$ are in $Z(G)$. Hence

$$\widetilde{s}^{m/p} = \left(c^{pt} s^p r^{-pt}\right)^{m/p^2} = c^{\frac{mt}{p}} s^{\frac{m}{p}} (r^{\frac{m}{p}})^{-t} = 1,$$

where $s^{m/p} = \left(r^{m/p}\right)^t$ because $n = m$. This proves the claim. Now observe that $G = \langle r, s \rangle = \langle r, \widetilde{s} \rangle$ and $rsr^{-1}s^{-1} = r\widetilde{s}r^{-1}\widetilde{s}^{-1} = c$, where $c = r^{n/p}$. Thus we may replace s by $\widetilde{s}$. By (2.8), $\widetilde{s}$ has order $\leq m/p$. After repeating this process a finite number of times, we may assume $m = p$.

Thus G is generated by elements r and s such that $r^n = s^p = 1$ and $sr = r^{n/p+1}s$, where $n \geq p^3$ is a power p. This completes the proof of Theorem 2.5. $\square$

Acknowledgement. We would like to thank referee for helpful comments and bringing Theorem 2.5 to our attention.

References

[1] N. Blackburn, *On a special class of p-groups*, Acta Math. **100** (1958), 45–92.
[2] D.-S. Kang and Z. Reichstein, *Trace forms of Galois field extensions in the presence of roots of unity*, J. Reine Angew. Math. **549** (2002), 79–89.
[3] L. Rédei, *Das schiefe Produkt in der Gruppentheorie*, Comment. Math. Helv. **20** (1947), 225–267.
[4] D. J. S. Robinson, *A Course in the Theory of Groups*, Second edition, Springer-Verlag, New York, 1996.
[5] W. R. Scott, *Group Theory*, Dover Publications, Inc., 1987.
[6] M. Xu, *A theorem on metabelian p-groups and some consequences*, Chinese Ann. Math. Ser. B **5** (1984), no. 1, 1–6.
[7] M.-Y. Xu and Q. Zhang, *A classification of metacyclic 2-groups*, Algebra Colloq. **13** (2006), no. 1, 25–34.
[8] Q. Zhang, X. Sun, L. An, and M. Xu, *Finite p-groups all of whose subgroups of index* p^2 *are abelian*, Algebra Colloq. **15** (2008), no. 1, 167–180.

Dong Seung Kang
Department of Mathematical Education
Dankook University
Yongin 448-701, Korea
E-mail address: dskang@dankook.ac.kr