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CRITERIA OF NORMALITY CONCERNING THE SEQUENCE OF OMITTED FUNCTIONS

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Abstract. In this paper, we research the normality of sequences of meromorphic functions concerning the sequence of omitted functions. The main result is listed below. Let $\{f_n(z)\}$ be a sequence of functions meromorphic in D, the multiplicities of whose poles and zeros are no less than $k+2,\ k \in \mathbb{N}$. Let $\{b_n(z)\}$ be a sequence of functions meromorphic in D, the multiplicities of whose poles are no less than $k+1$, such that $b_n(z) \overset{\chi}{\Rightarrow} b(z)$, where $b(z)(\neq 0)$ is meromorphic in D. If $f_n^{(k)}(z) \neq b_n(z)$, then $\{f_n(z)\}$ is normal in D. And we give some examples to indicate that there are essential differences between the normal family concerning the sequence of omitted functions and the normal family concerning the omitted function. Moreover, the conditions in our paper are best possible.

1. Introduction and main results

Throughout this paper, unless otherwise specified, we use the standard notions and notation of Nevanlinna theory ([4], [14]). For example, D is a domain in $\mathbb{C}$. For $z_0 \in \mathbb{C},\ r > 0,\ \Delta'(z_0, r) = \{z : 0 < |z - z_0| < r\}$, $\Delta(z_0, r) = \{z : |z - z_0| < r\}$ and $\Delta'_r = \Delta'(0, r)$, $\Delta_r = \Delta(0, r)$, $\partial\Delta_r = \{z : |z| = r\}$. The unit disc will be marked as $\Delta = \Delta(0, 1)$. $f_n(z) \overset{\chi}{\Rightarrow} f(z)$ in D shows that the sequence $\{f_n(z)\}$ converges to $f(z)$ in the spherical metric uniformly in compact subsets of D and $f_n(z) \Rightarrow f(z)$ in D if the convergence is in the Euclidean metric.

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$n(r, f = 0)$ is the number of zeros of meromorphic function $f(z)$ in Δ_r (counting multiplicities), $\bar{n}(r, f = 0)$ is the number of distinct zeros of meromorphic function $f(z)$ in Δ_r (without counting multiplicities).

Recall that a family $\mathcal{F}$ of meromorphic functions defined in D is said to be normal (quasinormal of order ν) in D, if each sequence $\{f_n(z)\} \subset \mathcal{F}$ contains a subsequence which converges spherically locally in D (minus a set that contains at most ν points). The subtracted set may depend on the subsequence. See ([4], [9], [10], [14]).

It is well-known that most of the existing results on normal criteria of a family of meromorphic functions are about the omitted function, while little is known on criteria of normality concerning the sequence of omitted functions which is the main topic of the present paper.

In 2005, Pang and Zalcman ([7]) researched the normality of sequences of meromorphic functions concerning the omitted function and proved the following result.

Theorem A ([7, Theorem 1.1]). *Let $\{f_n(z)\}$ be a sequence of functions meromorphic in D, all of whose poles and zeros are multiple. Let $b(z)(\neq 0)$ be meromorphic in D. If $f'_n(z) \neq b(z)$, then $\mathcal{F}$ is normal in D.*

In ([13]), it was generalized to higher derivatives.

Theorem B ([13, Theorem 1]). *Let $\{f_n(z)\}$ be a sequence of functions meromorphic in D, all of whose poles are multiple and the multiplicities of whose zeros are no less than $k+1$, $k \in \mathbb{N}$. Let $b(z)(\neq 0)$ be meromorphic in D. If $f_n^{(k)}(z) \neq b(z)$, then $\{f_n(z)\}$ is normal in D.*

This paper bring our study of normal families of meromorphic functions concerning the sequence of omitted functions, begun in ([3]), to a certain completion. In that paper, we proved the following result.

Theorem C ([3, Theorem 1.1]). *Let $\{f_n(z)\}$ be a sequence of functions meromorphic in D, the multiplicities of whose poles and zeros are no less than 3. Let $\{b_n(z)\}$ be a sequence of functions meromorphic in D, the multiplicities of whose poles are no less than 2, such that $b_n(z) \stackrel{\chi}{\Rightarrow} b(z)$, where $b(z)(\neq 0)$ is meromorphic in D. If $f'_n(z) \neq b_n(z)$, then $\{f_n(z)\}$ is normal in D.*

A natural question is whether Theorem C can be generalized to higher derivatives. We give a positive answer as follows.

Theorem 1.1. *Let $\{f_n(z)\}$ be a sequence of functions meromorphic in D, the multiplicities of whose poles and zeros are no less than $k+2$, $k \in \mathbb{N}$. Let $\{b_n(z)\}$ be a sequence of functions meromorphic in D, the multiplicities of whose poles are no less than $k+1$, such that $b_n(z) \stackrel{\chi}{\Rightarrow} b(z)$, where $b(z)(\neq 0)$ is meromorphic in D. If $f_n^{(k)}(z) \neq b_n(z)$, then $\{f_n(z)\}$ is normal in D.*

A further consequence of normal criteria about the sequence of omitted functions can be obtained under certain conditions.

Theorem 1.2. *Let $\{f_n(z)\}$ be a sequence of functions meromorphic in D, each of which has a multiple pole at most, and the multiplicities of whose zeros are no less than $k+2,\ k \in \mathbb{N}$. Let $\{b_n(z)\}$ be a sequence of functions meromorphic in D, the multiplicities of whose poles are no less than k, such that $b_n(z) \overset{\chi}{\Rightarrow} b(z)$, where $b(z)(\neq 0)$ is meromorphic in D. If $f_n^{(k)}(z) \neq b_n(z)$, then $\{f_n(z)\}$ is normal in D.*

The following example shows that there are essential differences between the normal family concerning the sequence of omitted functions and the normal family concerning the omitted function when $k=1$.

Example 1.3. Let $\{f_n(z)\}$ and $\{b_n(z)\}$ be sequences of functions meromorphic in $\Delta = \{z : |z| < 1\}$, such that

$$f_n(z) = \frac{n^2(z-1/n^2)^2}{3z^2},\ b_n(z) = \frac{2}{3(z-e^{i2\pi/3}/n^2)(z-e^{i4\pi/3}/n^2)}.$$

Because the poles of $f_n(z)$ are different from the poles of $b_n(z)$ and

$$f_n'(z) - b_n(z) = \frac{-\frac{2}{n^6}}{3z^3(z-e^{i2\pi/3}/n^2)(z-e^{i4\pi/3}/n^2)} \neq 0,$$

it is clear that $f_n'(z) \neq b_n(z)$. Furthermore, $b_n(z) \overset{\chi}{\Rightarrow} b(z) = \frac{2}{3z^2} \neq 0$. But, $\{f_n(z)\}$ is not normal at $z=0$.

Moreover, Example 1.3 also shows that the the condition on the multiplicity of zeros of functions in $\{f_n(z)\}$ cannot be weakened when $k=1$, especially in Theorem 1.2.

One cannot omit the requirement that $b(z) \neq 0$ in D, as is shown in the following example.

Example 1.4. Let $\{f_n(z)\}$ and $\{b_n(z)\}$ be sequences of functions meromorphic in $\Delta = \{z : |z| < 1\}$, such that

$$f_n(z) = \frac{(z^2-1/n^6)^3}{4z^2},\ b_n(z) = z^3 - \frac{3}{2n^6}z.$$

Then each $f_n(z)$ has two zeros of multiplicity 3 and a single pole of multiplicity 2. $b_n(z) \Rightarrow b(z) = z^3$, where $b(z)$ has a zero $z=0$ in Δ. And

$$f_n'(z) = z^3 - \frac{3}{2n^6}z + \frac{1}{2n^{18}}\frac{1}{z^3} \neq b_n(z).$$

But, $\{f_n(z)\}$ is not normal at $z=0$.

2. Preliminary results

Lemma 2.1 ([16, Theorem 1])**.** *Let $f(z)$ be a function transcendental meromorphic in $\mathbb{C}$, and let $Q(z)$ be rational, $Q(z) \not\equiv 0$. Suppose that, with at most many exceptions, the multiplicities of all zeros of $f(z)$ are no less than $k+1$. Then $f^{(k)}(z) - Q(z)$ has infinitely many zeros.*

Lemma 2.1 generalizes the main results of ([6]), where the case $k = 1$ was proved.

Lemma 2.2 ([13, Lemma 6]). *Let $Q(z)$ be rational in $\mathbb{C}$. The multiplicities of all zeros of $Q(z)$ are no less than k. If $Q^{(k)}(z) \neq z^{-l}$, where $k, l \in \mathbb{N}$, then $Q(z)$ is constant.*

Lemma 2.3 ([3, Lemma 3.1]). *Each polynomial of degree k takes two distinct finite values at least $k+1$ distinct points in $\mathbb{C}$, where $k \in \mathbb{N}$.*

Lemma 2.4 ([5, Lemma 6]). *Let $\{f_n(z)\}$ be a sequence of functions meromorphic in D, all of whose poles are multiple and the multiplicities of whose zeros are no less than $k+1$, where $k \in \mathbb{N}$. And let $\{\varphi_n(z)\}$ be a sequence of functions holomorphic in D, such that $\varphi_n(z) \Rightarrow \varphi(z)$ in D, where $\varphi(z) \neq 0, \infty$ in D. If $f_n^{(k)}(z) \neq \varphi_n(z)$, then $\mathcal{F}$ is normal in D.*

Lemma 2.5 ([2, Lemma 2.3]). *Let $\{f_n(z)\}$ and $\{\psi_n(z)\}$ be sequences of functions meromorphic in D. Let $f(z)$ and $\psi(z)$ be meromorphic in D. If* (i) *$f_n(z) \overset{\chi}{\Rightarrow} f(z)$, $\psi_n(z) \overset{\chi}{\Rightarrow} \psi(z)$ and* (ii) *$f_n^{(k)}(z) \neq \psi_n(z)$, then either $f^{(k)}(z) \equiv \psi(z)$ or $f^{(k)}(z) \neq \psi(z)$.*

Lemma 2.6 ([2, Lemma 3.2]). *Let $\{f_n(z)\}$ be a sequence of functions meromorphic in D and let $\{b_n(z)\}$ be a sequence of functions meromorphic in D, $b_n(z) \overset{\chi}{\Rightarrow} b(z)$ in D, where $b(z) \not\equiv 0, \infty$. If $f_n(z) \neq 0$, $f_n^{(k)}(z) \neq b_n(z)$, then $\{f_n(z)\}$ is normal in D.*

By the proof method in [13, Theorem 1], [18, Theorem 1', P. 67] and combining Lemma 2.4 and Lemma 2.6, one can get the following Lemma (cf. [2, Lemma 2.10]; [15, Lemma 3.8]).

Lemma 2.7. *Let $\{f_n(z)\}$ be a sequence of functions meromorphic in D, all of whose poles are multiple and the multiplicities of whose zeros are no less than $k+1$, where $k \in \mathbb{N}$. Let $\{\varphi_n(z)\}$ be a sequence of functions meromorphic in D, such that $\varphi_n(z) \overset{\chi}{\Rightarrow} \varphi(z)$ in D, where $\varphi(z) (\neq 0)$ is meromorphic in D. Suppose that $\varphi_n(z)$ and $\varphi(z)$ have the same poles, and all with the same multiplicity. If $f_n^{(k)}(z) \neq \varphi_n(z)$, then $\{f_n(z)\}$ is normal in D.*

3. Auxiliary lemmas

Lemma 3.1. (i) *Let $\{f_n(z)\}$ be a sequence of functions meromorphic in D.*

(ii) *Let $\{\varphi_n(z)\}$ be a sequence of functions meromorphic in D, such that $\varphi_n(z) \overset{\chi}{\Rightarrow} \varphi(z)$ in D, where $\varphi(z) (\not\equiv 0)$ is meromorphic, $\varphi_n(z)$ and $\varphi(z)$ have the same poles and zeros in D, and all with the same multiplicity.*

(iii) *Let $\{F_n\} = \{F_n(z) \mid F_n(z) = f_n(z)/\varphi_n(z), z \in D$, where $f_n(z)$ and $\varphi_n(z)$ have no common zeros and poles$\}$.*

If $\{F_n(z)\}$ is normal in D, then $\{f_n(z)\}$ is normal in D.

Proof. Suppose that $z_0 \in D$ is an arbitrary point. Making standard normalizations, we may assume that $D = \Delta$, $z_0 = 0$, $\varphi(z) = z^l \phi(z)$, where $l \in \mathbb{N}$, $\phi(z) \neq 0, \infty$, $0 < m \leq |\phi(z)| \leq M$ for all $z \in \Delta$, $m, M \in \mathbb{R}$. On the basis of the condition (ii), we have $\varphi_n(z) = z^l \phi_n(z)$, $\phi_n(z) \Rightarrow \phi(z)$, $\phi_n(z) \neq 0, \infty$, $0 < m \leq |\phi_n(z)| \leq M$ for all $z \in \Delta$.

Because $\{F_n(z)\}$ is normal in Δ, reselecting and renumbering subsequences, it can assume that $F_n(z) \overset{\chi}{\Rightarrow} F(z)$ in Δ, where $F_n = f_n/\varphi_n$. In the following, our goal is to prove that $\{f_n(z)\}$ is normal at $z = 0$ and the discussion is divided into three cases.

Case 1. $l = 0$.

It yields $\varphi_n(z) = \phi_n(z)$, where $\phi_n(z) \Rightarrow \phi(z)$, $0 < m \leq |\phi_n(z)| \leq M$ for all $z \in \Delta$. Since $F_n(z) \overset{\chi}{\Rightarrow} F(z)$ and $f_n(z) = F_n(z)\varphi_n(z) = F_n(z)\phi_n(z)$ in Δ, it follows that $f_n(z) \overset{\chi}{\Rightarrow} F(z)\phi(z)$ in Δ. Thus $\{f_n(z)\}$ is normal at $z = 0$.

Case 2. $l > 0$.

Now $\varphi_n(z) = z^l \phi_n(z)$. Since $f_n(z)$ and $\varphi_n(z)$ have no common zeros, we get $f_n(0) \neq 0, F_n(0) = \infty$. According to the normality of $F_n(z)$, there exists $r > 0$, such that $|F_n(z)| \geq 1$ in $\Delta_{2r} \subset \Delta$ for large enough n. Then, it is clear that $f_n(z) \neq 0$ in Δ'_{2r}. Thus, $1/f_n$ is holomorphic in Δ_{2r}. Because

$$\left|\frac{1}{f_n(z)}\right| = \left|\frac{1}{F_n(z)} \cdot \frac{1}{z^l \phi_n(z)}\right| \leq \frac{1}{r^l m} \text{ in } \partial\Delta_r,$$

the above formula is all right in Δ_r on the basis of Maximum Modulus Principle. And then $\{f_n(z)\}$ is normal at $z = 0$.

Case 3. $l < 0$.

It is evident that $\varphi_n(z) = z^l \phi_n(z)$. Because $f_n(z)$ and $\varphi_n(z)$ have no common poles, it follows that $f_n(0) \neq \infty, F_n(0) = 0$. On the basis of the normality of $F_n(z)$, there exists $r > 0$, such that $|F_n(z)| \leq 1$ in $\Delta_{2r} \subset \Delta$ for large enough n. Obviously, $f_n(z) \neq \infty$ in Δ'_{2r}. Then, $f_n(z)$ is holomorphic in Δ_{2r}. Because

$$|f_n(z)| = \left|F_n(z) \cdot z^l \phi_n(z)\right| \leq r^l M \text{ in } \partial\Delta_r,$$

the above formula is all right in Δ_r on the basis of Maximum Modulus Principle. Thus $\{f_n(z)\}$ is normal at $z = 0$. □

Lemma 3.2. *In a sequence $\{\Delta_{r_n}\}$ of domains, where $r_n \to \infty$, let $\{f_n(z)\}$ be a sequence of meromorphic functions, the multiplicities of whose poles and zeros are all no less than $k+2$. And let $\{b_n(z)\}$ be a sequence of meromorphic functions in $\{\Delta_{r_n}\}$, all of whose poles are multiple, such that $b_n(z) \overset{\chi}{\Rightarrow} \frac{1}{c(z)}$ in $\mathbb{C}$, where $c(z)$ is a polynomial, the degree of which is a positive integer s. If* (i) *$f_n^{(k)}(z) \neq b_n(z)$ in Δ_{r_n}, and* (ii) *$f_n(z) \overset{\chi}{\Rightarrow} f(z)$ in $\mathbb{C}$, where $f(z)$ is a meromorphic function in $\mathbb{C}$, then either $f(z) \neq 0$ or $f(z) \equiv 0$.*

Proof. Suppose, to the contrary, that $f(z) \not\equiv 0$ and $f(z)$ has at least one zero, the multiplicity of which is no less than $k+2$. Lemma 2.5 implies that either $f^{(k)}(z) \neq \frac{1}{c(z)}$ or $f^{(k)}(z) \equiv \frac{1}{c(z)}$ in $\mathbb{C}$. The latter possibility contradicts the assumption. Then $f^{(k)}(z) \neq \frac{1}{c(z)}$. So, $f(z)$ is rational by Lemma 2.1. Since all zeros of $c(z)$ are the poles of $f^{(k)}(z) - \frac{1}{c(z)}$, it is clear that $f^{(k)}(z) - \frac{1}{c(z)}$ is a non-polynomial rational function.

Therefore, there exist a monic polynomial $d(z)$ and a complex constant $a \neq 0$ such that

$$f^{(k)}(z) = \frac{1}{c(z)} + \frac{a}{c(z)d(z)} = \frac{d(z)+a}{c(z)\cdot d(z)}. \tag{1}$$

By equation (1), $f(z) \equiv 0$ which contradicts our assumption, if $f(z) \neq \infty$. Suppose that

$$f(z) = \frac{a_1 \Pi_{i=1}^{m}(z-e_i)^{m_i}}{\Pi_{j=1}^{n}(z-g_j)^{n_j}}, \tag{2}$$

where $a_1 \neq 0$ is a complex constant, $m \geq 1, n \geq 1, m_i \geq k+2, n_j \geq k+2$ are all integers. Set $\Sigma_{i=1}^{m} m_i = M$, $\Sigma_{j=1}^{n} n_j = N$.

Thus

$$f^{(k)}(z) = \frac{\Pi_{i=1}^{m}(z-e_i)^{m_i-k}}{\Pi_{j=1}^{n}(z-g_j)^{n_j+k}}\phi(z), \tag{3}$$

where $\phi(z) = a_1(M-N)(M-1-N)\cdots[M-(k-1)-N]z^{k(m+n-1)} + \cdots + a_0$ is a polynomial, whose degree is not more than $k(m+n-1), a_0 \in \mathbb{C}$. It follows from equations (1) and (3) that

$$d(z) = \Pi_{j=1}^{n}(z-g_j)^{n_j+k}, \tag{4}$$

$$d(z) + a = c(z)\Pi_{i=1}^{m}(z-e_i)^{m_i-k}\phi(z), \tag{5}$$

$$d'(z) = (d(z)+a)' = \Pi_{j=1}^{n}(z-g_j)^{n_j+k-1}[\Sigma_{j=1}^{n}(n_j+k)\Pi_{j'\neq j}(z-g_{j'})]. \tag{6}$$

It can be asserted that either $N+w = M (0 \leq w \leq k-1)$ or $N-M = s-k$. In fact, if $N+w \neq M (0 \leq w \leq k-1)$, then $\deg(\phi) = k(m+n-1)$. By equation (5), $N-M = s-k$.

First of all, it is evident that the multiplicities of all zeros of $f(z)$ are no less than $k+2$ and the multiplicities of all zeros of $c(z)$ are multiple. It follows from equation (5) that all zeros of $f(z)$ and $c(z)$ are the multiple zeros of $d(z)+a$. Secondly, it is obvious that all zeros of $d(z)$ are distinct from zeros of $f(z)$ and $c(z)$. Clearly, $\bar{n}(r, f=0) = m$. Writing $\bar{n}(r, c=0) = s_1 < s$. Therefore, on the basis of equation (6)

$$m + s_1 \leq n - 1. \tag{7}$$

By equation (3), it yields

$$\bar{n}(r, f^{(k)} = 0,\ f \neq 0,\ c \neq 0) \leq k(m+n-1). \tag{8}$$

According to equations (4), (1) and inequalities (8), (7), Lemma 2.3 shows that

$$\begin{aligned} N + nk + 1 &\leq \bar{n}(r, d=0) + \bar{n}(r, d+a=0) \\ &\leq n + s_1 + m + k(n+m-1). \end{aligned}$$

Then, $N + k + 2 \leq 2n + km$. Because the multiplicities of all zeros and poles of $\{f_n(z)\}$ are no less than $k+2$, it follows that $(k+2)m \leq M$, $(k+2)n \leq N$. And hence $N+k+2 \leq 2 \cdot \frac{N}{k+2} + k \cdot \frac{M}{k+2}$. That is $N - M \leq -(k+2) \cdot \frac{k+2}{k}$, which contradicts the fact that either $N + w = M (0 \leq w \leq k-1)$ or $N - M = s - k$. And thus it finishes the proof. □

Lemma 3.3. *In a sequence $\{\Delta_{r_n}\}$ of domains, where $r_n \to \infty$, let $\{f_n(z)\}$ be a sequence of meromorphic functions, the multiplicities of whose zeros are all no less than $k+2$ and all of which have a pole at most. And let $\{b_n(z)\}$ be a sequence of meromorphic functions in $\{\Delta_{r_n}\}$, such that $b_n(z) \overset{\chi}{\Rightarrow} \frac{1}{c(z)}$ in $\mathbb{C}$, where $c(z)$ is a polynomial, the degree of which is a positive integer t. If* (i) *$f_n^{(k)}(z) \neq b_n(z)$ in Δ_{r_n}, and* (ii) *$f_n(z) \overset{\chi}{\Rightarrow} f(z)$ in $\mathbb{C}$, where $f(z)$ is a meromorphic function in $\mathbb{C}$, then either $f(z) \equiv 0$ or $f(z) \neq 0$.*

Proof. Suppose, to the contrary, that $f(z) \not\equiv 0$ and $f(z)$ has at least one zero, the multiplicity of which is no less than $k+2$. Lemma 2.5 implies that either $f^{(k)}(z) \neq \frac{1}{c(z)}$ or $f^{(k)}(z) \equiv \frac{1}{c(z)}$ in $\mathbb{C}$. The latter possibility contradicts the assumption. Then $f^{(k)}(z) \neq \frac{1}{c(z)}$. So, $f(z)$ is rational by Lemma 2.1.

Then, $f(z) \equiv 0$ which contradicts our assumption, if $f(z) \neq \infty$. It follows that $f(z)$ has poles. And we claim that $f(z)$ has a pole at most. Suppose, to the contrary, that $f(z)$ has two distinct poles at least. We may assume that z_1 and z_2 are the two distinct poles of $f(z)$. Then there exists $\delta > 0$, such that $\Delta(z_1, \delta) \cap \Delta(z_2, \delta) = \phi$, and $f(z) \neq \infty$ in $\Delta'(z_1, \delta)$ and $\Delta'(z_2, \delta)$ according to the isolated of poles and $f(z) \not\equiv \infty$. $f_n(z)$ and $f(z)$ have the same number of poles in $\Delta(z_1, \delta)$ and $\Delta(z_2, \delta)$, respectively, by Hurwitz Theorem, $f_n(z) \overset{\chi}{\Rightarrow} f(z)$ in $\mathbb{C}$ and $f(z) \not\equiv \infty$. Then $f_n(z)$ has at least a pole in $\Delta(z_1, \delta)$ and $\Delta(z_2, \delta)$, respectively. Suppose that $f_n(z_{n,1}) = \infty$, $z_{n,1} \in \Delta(z_1, \delta)$ and $f_n(z_{n,2}) = \infty$, $z_{n,2} \in \Delta(z_2, \delta)$. It is clear that $z_{n,1} \neq z_{n,2}$ as $\Delta(z_1, \delta) \cap \Delta(z_2, \delta) = \phi$. This contradicts the condition of the lemma that $f_n(z)$ has a pole at most.

We may therefore assume that $f^{(k)}(z) - \frac{1}{c(z)} = \frac{a}{c(z)(z-b)^n}$, where $n \in \mathbb{N}$ and $a (\neq 0)$ is a complex constant. That is

$$f^{(k)}(z) = \frac{(z-b)^n + a}{c(z)(z-b)^n}.$$

Because the multiplicities of all zeros of $f(z)$ are no less than $k+2$, it is clear that $(z-b)^n + a$ has multiple zeros, which is a contradiction. And thus it finishes the proof. □

4. Proof of Theorem 1.1

Proof. According to Lemma 2.4, it is enough to prove that $\{f_n(z)\}$ is normal at the point z_0, where $b(z_0) = \infty$ in D. Suppose that $D = \Delta$, $b(z) = \frac{\phi(z)}{z^l}$, where $\phi(z) \neq 0, \infty$ in Δ and $\phi(0) = 1$, $l \in \mathbb{N}$. Since $b_n(z) \overset{\chi}{\Rightarrow} b(z)$, $b_n(z) = \frac{\phi_n(z)}{(z-z_{n,1})^{l_1}(z-z_{n,2})^{l_2} \cdots (z-z_{n,s})^{l_s}}$, where $\phi_n \neq 0, \infty$ in Δ, and $\phi_n \overset{\chi}{\Rightarrow} \phi$, $z_{n,j}$ are

s different points satisfying $z_{n,j} \to 0$, $l_j \in \mathbb{N}$ $(1 \le j \le s)$ and $\Sigma_{j=1}^{s} l_j = l$. On the basis of the condition, $l_j \ge k+1, l \ge k+1$. Assume that $b_n(z) = \frac{\phi_n(z)}{z^{l_1}(z-z_{n,2})^{l_2}\cdots(z-z_{n,s})^{l_s}}$, as the normality of $\{f_n(z)\}$ is the same as $\{f_n(z+z_{n,1})\}$.

Using the principle of mathematical induction on l which is the multiplicity of $z=0$, that $\{f_n(z)\}$ is normal at $z=0$ is later proved to be true as follows.

Firstly, when $l=k+1$, $s=1$, the conclusion holds immediately according to Lemma 2.7.

Secondly, under the hypothesis that $\{f_n(z)\}$ is normal when $l < t$ $(t \in \mathbb{N},\ t > k+1)$, that $\{f_n(z)\}$ is normal at $z=0$ is later proved to be all right when $l=t$ on the basis of the mathematical induction principle.

Set $a_n = z_{n,l}$ such that $|a_n| \ge |z_{n,j}|$, where $2 \le j \le l$, $a_n \to 0$. Assign $g_n(z) = a_n^{t-k} f_n(a_n z)$ and $u_n(z) = a_n^t b_n(a_n z), z \in \Delta_{r_n}, r_n \to \infty$. Clearly, $u_n(z) = \frac{\phi_n(a_n z)}{z^{l_1}(z-\frac{z_{n,2}}{a_n})^{l_2}\cdots(z-\frac{z_{n,s-1}}{a_n})^{l_{s-1}}(z-1)^{l_s}} \overset{\chi}{\Rightarrow} u(z)$ in $\mathbb{C}$, where the points $z=0$ and $z=1$ are the poles of $u(z)$. Therefore, the multiplicities of all poles of $u(z)$ are less than t.

For $f_n^{(k)}(z) \ne b_n(z)$, so $g_n^{(k)}(z) \ne u_n(z)$. And Lemma 2.4 and the induction hypothesis imply that $\{g_n(z)\}$ is normal in $\mathbb{C}$. Reselecting and renumbering subsequences, it is clear that $g_n(z) \overset{\chi}{\Rightarrow} g(z)$ in $\mathbb{C}$. It follows from Lemma 3.2 that either $g(z) \ne 0$ or $g(z) \equiv 0$ in $\mathbb{C}$.

Assume that $\{f_n(z)\}$ is not normal at $z=0$. It follows from Lemma 2.6 that there exists a subsequence of $\{f_n(z)\}$ (still marked as $\{f_n(z)\}$), $\zeta_n \to 0$, satisfying $f_n(\zeta_n) = 0$. Suppose that ζ_n is the zero of $f_n(z)$ with the smallest modulus.

In the following, the discussion is divided into two cases.

Case 1. $g(z) \ne 0$.

Since $f_n(\zeta_n) = 0$, we get $g_n(\zeta_n/a_n) = a_n^{t-k} f_n(\zeta_n) = 0$. However, $g_n(z) = a_n^{t-k} f_n(a_n z) \overset{\chi}{\Rightarrow} g(z) \ne 0$ in $\mathbb{C}$. Then $\zeta_n/a_n \to \infty$. Writing $v_n(z) = \zeta_n^{t-k} f_n(\zeta_n z)$, $z \in \Delta_{r_n}, r_n \to \infty$. We have $v_n^{(k)}(z) \ne \frac{\phi_n(\zeta_n z)}{z^{l_1}(z-\frac{z_{n,2}}{\zeta_n})^{l_2}\cdots(z-\frac{z_{n,s}}{\zeta_n})^{l_s}} \overset{\chi}{\Rightarrow} \frac{1}{z^t}$ in $\mathbb{C}$. $\{v_n\}$ is normal in $\mathbb{C} \setminus \{0\}$ according to Lemma 2.4. And Lemma 2.6 implies that $\{v_n(z)\}$ is normal in Δ. Hence, $\{v_n(z)\}$ is normal in $\mathbb{C}$. Suppose that $v_n(z) \overset{\chi}{\Rightarrow} v(z)$ in $\mathbb{C}$. It follows from Lemma 3.2 that either $v(z) \ne 0$ or $v(z) \equiv 0$ in $\mathbb{C}$. For $v_n(1) = 0$, $v(z) \equiv 0$ which contradicts the fact that $v_n(0) = (\frac{\zeta_n}{a_n})^{t-k} g_n(0) \not\to 0$, $t-k > 0$. Thus case 1 can be ruled out.

Case 2. $g(z) \equiv 0$.

Set $\mu_n(z) = z^{t-k} f_n(z), z \in \Delta$. For $f_n(0) \ne \infty$, it follows that $\mu_n(0) = 0$. And we claim that $\{\mu_n(z)\}$ is not normal at $z=0$. Suppose, to the contrary, that $\{\mu_n(z)\}$ is normal at $z=0$, where $\mu_n(z) = \frac{f_n(z)}{1/z^{t-k}}$. Since $f_n(0) \ne \infty$, $f_n(z)$ and $1/z^{t-k}$ have no common poles in Δ. Lemma 3.1 implies that $\{f_n(z)\}$ is normal at $z=0$, which contradicts our assumption that no subsequence of $\{f_n(z)\}$ is normal at $z=0$.

Montel Theorem shows that $\{\mu_n(z)\}$ is not locally uniformly bounded in $\Delta_\varepsilon \subset \Delta$, $\forall \varepsilon > 0$. Then $\exists \xi_n \to 0$, such that $\mu_n(\xi_n) \to \infty$. We have points $\omega_n \to 0$, such that $|\mu_n(\omega_n)| = 1$, on the basis of the continuity of $|\mu_n(z)|$. Suppose that the modulus of ω_n is the smallest among points which make the equation $|\mu_n(z)| = 1$ come true.

Writing $M_n(z) = \omega_n^{t-k} f_n(\omega_n z), z \in \Delta_{r_n}, r_n \to \infty$. It can be concluded that $\omega_n/a_n \to \infty$. Otherwise, assume that $\omega_n/a_n \to a$, a finite number. It follows that $g_n(\omega_n/a_n) = (\frac{a_n}{\omega_n})^{t-k} \omega_n^{t-k} f_n(\omega_n) \to 0, t-k > 0$. For $|\omega_n^{t-k} f_n(\omega_n)| = 1$, we get $a_n/\omega_n \to 0$, $\omega_n/a_n \to \infty$ which contradicts to $\omega_n/a_n \to a$. Therefore, $\omega_n/a_n \to \infty$. Obviously, $M_n^{(k)}(z) \neq \frac{\phi_n(\omega_n z)}{z^{l_1} \cdots (z - \frac{z_{n,s}}{\omega_n})^{l_s}} \overset{\chi}{\Rightarrow} \frac{1}{z^t}$ in $\mathbb{C}$. $\{M_n(z)\}$ is normal in $\mathbb{C} \backslash \{0\}$ according to Lemma 2.4.

It can be asserted that (i) $\{M_n(z)\}$ is holomorphic in Δ, and (ii) $\{M_n(z)\}$ is normal at $z = 0$.

First of all , we prove the claim (i). Suppose, to the contrary, that there exist β_n, where $0 < |\beta_n| < 1$, satisfying $M_n(\beta_n) = \infty$. It follows that $\mu_n(\beta_n \omega_n) = \infty$. Therefore, there are $\gamma_n \in \Delta$ with $0 < |\gamma_n| < |\beta_n \omega_n| < |\omega_n|$, such that $|\mu_n(\gamma_n)| = 1$, which contradicts the choice of ω_n. So the claim (i) is proved.

Next, we prove the claim (ii). Otherwise, it follows from the claim (i) and the normality of $\{M_n(z)\}$ in $\mathbb{C} \backslash \{0\}$ that $M_n(z) \overset{\chi}{\Rightarrow} \infty$ in Δ'. And furthermore, $M_n(z) \overset{\chi}{\Rightarrow} M(z)$ in $\mathbb{C} \backslash \{0\}$. Thus $M_n(z) \overset{\chi}{\Rightarrow} \infty$ in $\mathbb{C} \backslash \{0\}$. But, since $|M_n(1)| = 1$, it follows that $|M(1)| = 1$ which contradicts to what we have shown. Thus the claim (ii) holds immediately.

According to the discussion as before, $\{M_n(z)\}$ is normal in$\mathbb{C}$. Suppose that $M_n(z) \overset{\chi}{\Rightarrow} M(z)$ in $\mathbb{C}$. Lemma 3.2 deduces that either $M(z) \equiv 0$ or $M(z) \neq 0$ in $\mathbb{C}$. The former possibility contradicts the fact that $|M(1)| = 1$. Thus $M(z) \neq 0$. Because $f_n(\zeta_n) = 0$, $M_n(\zeta_n/\omega_n) = \omega_n^{t-k} f_n(\zeta_n) = 0, \zeta_n/\omega_n \to \infty$. The following proof is similar to Case 1. Let $\tilde{g}_n(z) = \zeta_n^{t-k} f_n(\zeta_n z), z \in \Delta r_n, r_n \to \infty$. We also get a contradiction.

So $\{f_n(z)\}$ is normal at $z = 0$. According to the mathematical induction, Theorem 1.1 is proved. □

Remark 4.1. Finally, in order to complete the paper, Theorem 1.2 is proved by using the same argument as the proof of Theorem 1.1 and Lemma 3.3 as follows.

5. Proof of Theorem 1.2

Proof. According to Lemma 2.4, it is enough to prove that $\{f_n(z)\}$ is normal at the point z_0, where $b(z_0) = \infty$ in D. Suppose that $D = \Delta$, $b(z) = \frac{\phi(z)}{z^l}$, where $\phi(z) \neq 0, \infty$ in Δ and $\phi(0) = 1$, $l \in \mathbb{N}$. Since $b_n(z) \overset{\chi}{\Rightarrow} b(z)$, $b_n(z) = \frac{\phi_n(z)}{(z - z_{n,1})^{l_1} (z - z_{n,2})^{l_2} \cdots (z - z_{n,s})^{l_s}}$, where $\phi_n \neq 0, \infty$ in Δ, and $\phi_n \overset{\chi}{\Rightarrow} \phi$, $z_{n,j}$ are s different points satisfying $z_{n,j} \to 0$, $l_j \in \mathbb{N}$ $(1 \leq j \leq s)$ and $\Sigma_{j=1}^s l_j = l$. On the basis of the condition, $l_j \geq k, l \geq k$. Assume that $b_n(z) = \frac{\phi_n(z)}{z^{l_1}(z-z_{n,2})^{l_2}\cdots(z-z_{n,s})^{l_s}}$, as the normality of $\{f_n(z)\}$ is the same as $\{f_n(z+z_{n,1})\}$.

Using the principle of mathematical induction on l which is the multiplicity of $z=0$, that $\{f_n(z)\}$ is normal at $z=0$ is later proved to be true as follows.

Firstly, when $l=k$, $s=1$, the conclusion holds immediately according to Lemma 2.7.

Secondly, under the hypothesis that $\{f_n(z)\}$ is normal when $l<t$ ($t \in \mathbb{N}$, $t > k$), that $\{f_n(z)\}$ is normal at $z=0$ is later proved to be all right when $l=t$ on the basis of the mathematical induction principle.

Set $a_n = z_{n,l}$ such that $|a_n| \geq |z_{n,j}|$, where $2 \leq j \leq l$, $a_n \to 0$. Assign $g_n(z) = a_n^{t-k} f_n(a_n z)$ and $u_n(z) = a_n^t b_n(a_n z), z \in \Delta_{r_n}, r_n \to \infty$. Clearly, $u_n(z) = \frac{\phi_n(a_n z)}{z^{l_1}(z-\frac{z_{n,2}}{a_n})^{l_2}\cdots(z-\frac{z_{n,s-1}}{a_n})^{l_{s-1}}(z-1)^{l_s}} \stackrel{\chi}{\Rightarrow} u(z)$ in $\mathbb{C}$, where the points $z=0$ and $z=1$ are the poles of $u(z)$. Therefore, the multiplicities of all poles of $u(z)$ are less than t.

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Assume that $\{f_n(z)\}$ is not normal at $z=0$. It follows from Lemma 2.6 that there exists a subsequence of $\{f_n(z)\}$ (still marked as $\{f_n(z)\}$), $\zeta_n \to 0$, satisfying $f_n(\zeta_n)=0$. Suppose that ζ_n is the zero of $f_n(z)$ with the smallest modulus.

In the following, the discussion is divided into two cases.

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It can be asserted that (i) $\{M_n(z)\}$ is holomorphic in Δ, and (ii) $\{M_n(z)\}$ is normal at $z = 0$.

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Next, we prove the claim (ii). Otherwise, it follows from the claim (i) and the normality of $\{M_n(z)\}$ in $\mathbb{C}\backslash\{0\}$ that $M_n(z) \overset{\chi}{\Rightarrow} \infty$ in Δ'. And furthermore, $M_n(z) \overset{\chi}{\Rightarrow} M(z)$ in $\mathbb{C}\backslash\{0\}$. Thus $M_n(z) \overset{\chi}{\Rightarrow} \infty$ in $\mathbb{C}\backslash\{0\}$. But, since $|M_n(1)| = 1$, it follows that $|M(1)| = 1$ which contradicts to what we have shown. Thus the claim (ii) holds immediately.

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So $\{f_n(z)\}$ is normal at $z = 0$. According to the mathematical induction, Theorem 1.2 is proved. □

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