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GENERALIZED KKM-TYPE THEOREMS FOR BEST PROXIMITY POINTS

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Abstract. This paper is concerned with best proximity points for multimaps in normed spaces and in hyperconvex metric spaces. Using the generalized KKM theorem, we deduce new best proximity pair theorems for a family of multimaps with unionly open fibers in normed spaces. And we prove a new best proximity point theorem for quasi-lower semicontinuous multimaps in hyperconvex metric spaces.

1. Introduction and preliminaries

A *multimap* (or *map*) $F : X \multimap Y$ is a function from a set X into the power set 2^Y of Y; that is, a function with the *values* $F(x) \subset Y$ for $x \in X$. For $A \subset X$, let $F(A) = \bigcup\{F(x) : x \in A\}$. Let $\overline{}$ denote the closure of F.

Throughout this paper, we assume that multimaps have nonempty values otherwise explicitly stated or obvious from the context.

Let (M, d) be a metric space and let A and B be nonempty subsets of M.

For a multimap $F : A \multimap M$, a point $x_0 \in A$ is called a *best proximity point* of F if $d(x_0, F(x_0)) = d(A, B)$. In this case, $(x_0, F(x_0))$ is called a *best proximity pair* for F. Note that if $d(A, B) = 0$ and F is a single valued map, then the best proximity point is a fixed point of F.

The following notations are used in the sequel.

$$A_0 = \{x \in A : d(x, y) = d(A, B) \text{ for some } y \in B\},$$

$$B_0 = \{y \in B : d(x, y) = d(A, B) \text{ for some } x \in A\}.$$

The pair (A, B) is said to be a *proximal pair* if, for each $(x, y) \in A \times B$, there exists $(\tilde{x}, \tilde{y}) \in A \times B$ such that $d(x, \tilde{y}) = d(\tilde{x}, y) =d(A, B)$. A pair (A, B) is a proximal pair if and only if $A = A_0$ and $B = B_0$.

Sanka Raj and Somasundaram introduced R-KKM maps and proved an extended version of the Fan-Browder multivalued fixed point theorem with best proximity points setting;

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Theorem 1.1 ([14, Theorem 3.3]). *Let (A, B) be a nonempty compact convex proximal pair in a normed space M. Let $F : A \multimap B$ be a multimap such that*
(1) *$F(x)$ is a convex subset of B for each $x \in A$; and*
(2) *$F^{-1}(y)$ is open for each $y \in B$.*
Then there is a $w \in A$ such that $d(w, F(w)) = d(A, B)$.

Recently the author [7] proved the generalized KKM theorem in abstract convex spaces. The concept of abstract convex spaces is a far-reaching generalization of convex structures and it was introduced by Park [13].

In Section 2, using the generalized KKM theorem in [7] and the fact that R-KKM maps are generalized KKM maps, we shall deduce new best proximity pair theorems for a family of multimaps with unionly open fibers in normed spaces. These Theorems generalize Theorem 1.1.

A metric space (M, d) is said to be *hyperconvex* if

$$\bigcap_{\alpha} B(x_\alpha, \gamma_\alpha) \neq \emptyset$$

for any collection $\{B(x_\alpha, \gamma_\alpha)\}$ of closed balls in M for which $d(x_\alpha, x_\beta) \leq \gamma_\alpha + \gamma_\beta$.

In Section 3, we obtain a new best proximity points theorem in hyperconvex metric spaces which generalizes Kirk et al. ([9, Theorem 2.11]).

2. Best proximity points theorems in normed spaces

Let $\langle X \rangle$ denote the set of all nonempty finite subsets of X.

An *abstract convex space* $(X, D; \Gamma)$ consists of a topological space X, a nonempty set D, and a multimap $\Gamma : \langle D \rangle \multimap X$ with nonempty values $\Gamma_A := \Gamma(A)$ for $A \in \langle D \rangle$. When in case $X = D$, let $(X; \Gamma) := (X, X; \Gamma)$.

Let $(X, D; \Gamma)$ be an abstract convex space. If a map $F : D \multimap X$ satisfies $\Gamma_A \subset F(A)$ for all $A \in \langle D \rangle$, then F is called a *KKM* map.

The *partial KKM principle* for an abstract convex space $(X, D; \Gamma)$ is the statement that, for any closed-valued KKM map $F : D \multimap X$, the family $\{F(z)\}_{z \in D}$ has the finite intersection property.

Any convex subset X of a topological vector space is an abstract convex space satisfying the partial KKM principle (X, Γ) by putting $\Gamma_A = \mathrm{co}A$, the convex hull of A. Other examples of an abstract convex space satisfying the partial KKM principle are any convex space, any pseudo-convex space, any homeomorphic image of a convex space, any contractible space, any C-space, any generalized convex space, and so on. See [13].

Let $(X, D; \Gamma)$ be an abstract convex space and Z be a nonempty set. A map $F : Z \multimap X$ is called a *generalized KKM map* provided that for each $N \in \langle Z \rangle$, there exists a function $\sigma : N \to D$ such that $\Gamma_{\sigma(M)} \subset F(M)$ for each $M \in \langle N \rangle$. If σ is an identity function on D, then F is a KKM map. For details, see [7].

When A is a nonempty subset of a normed space M and $y \in M$, let $P_A(y) = \{x \in A : ||x - y|| = d(y, A)\}$.

We extend the notion of *R-KKM* maps defined by Sankar Raj and Somasundaram [14] as follows:

Let $I = \{1, \dots, n\}$, A be a nonempty convex subset of a normed space M, and B_i be a nonempty subset of M for each $i \in I$ such that $\bigcap_{i \in I} P_A(y_i) \neq \emptyset$ for each $(y_1, \dots, y_n) \in B := \prod_I B_i$. The map $F : B \multimap A$ is said to be an R-KKM map if for any $\{y^1 = (y_1^1, \dots, y_n^1), \dots, y^m = (y_1^m, \dots, y_n^m)\} \in \langle B \rangle$, there exists an $x_j \in \bigcap_{i \in I} P_A(y_i^j)$ for each $j = 1, \dots, m$ such that $\mathrm{co}\{x_1, \dots, x_m\} \subset \bigcup_{j=1,\dots,m} F(y^j)$.

Note that R-KKM maps are generalized KKM maps.

Consider the following related three conditions for $F : Z \multimap X$;

(a) $\bigcap_{z \in Z} \overline{F(z)} = \overline{\bigcap_{z \in Z} F(z)}$ (F is *intersectionally closed-valued* [11]).

(b) $\bigcap_{z \in Z} \overline{F(z)} = \bigcap_{z \in Z} F(z)$ (F is *transfer closed-valued*).

(c) F is closed-valued.

Luc et al. [11] noted that (c) $\Longrightarrow$ (b) $\Longrightarrow$ (a).

A multimap $F : Z \multimap X$ is said to be *unionly open-valued* (resp., *transfer open-valued*) on Z if and only if the multimap $G : Z \multimap X$, defined by $G(z) = X \setminus F(z)$ for every $z \in Z$, is intersectionally closed-valued (resp., transfer closed-valued) on Z. See [11] and Tian [15].

The following KKM type theorem is due to the author [7];

Theorem 2.1. *Let B be a nonempty set, $(X, D; \Gamma)$ be an abstract convex space satisfying the partial KKM principle, and $F : B \multimap X$ be a multimap satisfying*

(1) *$\overline{F}$ is a generalized KKM map.*

Then $\{\overline{F(z)}\}_{z \in B}$ has the finite intersection property.
Further, if

(2) *F is intersectionally closed-valued; and*

(3) *there exists a nonempty compact subset K of X such that $\bigcap_{z \in N} \overline{F(z)} \subset K$ for some $N \in \langle B \rangle$.*

Then $\bigcap_{z \in B} F(z) \neq \emptyset$.

Since an R-KKM map is a generalized KKM map, we obtain the following theorem which generalizes Theorem 3.2 in [14];

Theorem 2.2. *Let A be a nonempty convex subset of a normed space M, and B_i be a nonempty subset of M for each $i \in I := \{1, \dots, n\}$ such that $\bigcap_{i \in I} P_A(y_i) \neq \emptyset$ for each $(y_1, \dots, y_n) \in B := \prod_I B_i$. Let $F : B \multimap A$ be a multimap satisfying*

(1) *$\overline{F}$ is an R-KKM map.*

Then $\{\overline{F(z)}\}_{z \in B}$ has the finite intersection property.
Further, if

(2) *F is intersectionally closed-valued; and*

(3) *there exists a nonempty compact subset K of A such that $\bigcap_{z \in N} \overline{F(z)} \subset K$ for some $N \in \langle B \rangle$.*

Then $\bigcap_{z\in B} F(z) \neq \emptyset$.

The following is an existence theorem for the best proximity pairs;

Theorem 2.3. *Let* $I = \{1, \ldots, n\}$ *and for each* $i \in I$, (A, B_i) *be a nonempty convex proximal pair in a normed space* M *such that* $\bigcap_{i\in I} P_A(y_i) \neq \emptyset$ *for each* $(y_1, \ldots, y_n) \in B = \prod B_i$. *Let* A *be compact. For each* $i \in I$, *let* $F_i : A \multimap B_i$ *be a multimap such that*

(1) $F_i(x)$ *is convex for each* $x \in A$; *and*

(2) $F_i^{-1}(y_i)$ *is unionly open for each* $y_i \in B_i$.

Then there is a $w \in A$ *such that* $d(w, F_i(w)) = d(A, B_i)$ *for each* $i \in I$.

Proof. Define $F : A \multimap B$ and $G : B \multimap A$ by

$$F(x) = \prod F_i(x) \text{ and } G(y_1, \ldots, y_n) = A \backslash F^{-1}(y_1, \ldots, y_n).$$

Then $G(y_1, \ldots, y_n) = \{x \in A : y_i \notin F_i(x) \text{ for some } i \in I\} = \bigcup_I (A\backslash F_i^{-1}(y_i))$.

Suppose $G(y_1, \ldots, y_n) = \emptyset$ for some $(y_1, \ldots, y_n) \in B$. Then $F^{-1}(y_1, \ldots, y_n) = A$, that is, $(y_1, \ldots, y_n) \in F(x)$ for all $x \in A$. Since $\bigcap_{i\in I} P_A(y_i) \neq \emptyset$ and $y_i \in B_i = B_{i0}$, there exists a $w \in A$ such that $||y_i - w|| = d(y_i, A) = d(A, B_i)$ for all $i \in I$. Since $(y_1, \ldots, y_n) \in F(w)$, $d(w, F_i(w)) = d(A, B_i)$ for each $i \in I$.

Assume that $G(y_1, \ldots, y_n)$ is nonempty for each $(y_1, \ldots, y_n) \in B$. Then

$$\begin{aligned}
\overline{\bigcap_B G(y_1, \ldots, y_n)} &= \overline{\bigcap_B \bigcup_I (A\backslash F_i^{-1}(y_i))} = \overline{\bigcup_I \bigcap_{y_i\in B_i} (A\backslash F_i^{-1}(y_i))} \\
&= \bigcup_I \overline{\bigcap_{y_i\in B_i} (A\backslash F_i^{-1}(y_i))} = \bigcup_I \bigcap_{y_i\in B_i} \overline{(A\backslash F_i^{-1}(y_i))} \\
&= \bigcap_B \bigcup_I \overline{(A\backslash F_i^{-1}(y_i))} = \bigcap_B \overline{\bigcup_I (A\backslash F_i^{-1}(y_i))} \\
&= \bigcap_B \overline{G(y_1, \ldots, y_n)},
\end{aligned}$$

by (2). That is, G is intersectionally closed-valued. And

$$\bigcap_B G(y_1, \ldots, y_n) = \bigcup_I \bigcap_{B_i} (A\backslash F_i^{-1}(y_i)) = \bigcup_I (A\backslash \bigcup_{B_i} F_i^{-1}(y_i)) = \bigcup_I \emptyset = \emptyset.$$

By Theorem 2.2, $\overline{G}$ is not an R-KKM map. So there exist $\{y^j = (y_1^j, \ldots, y_n^j) \mid j = 1, \ldots, m\} \subset B$ and $x_j \in \bigcap_{i\in I} P_A(y_i^j)$ for $j = 1, \ldots, m$ such that $\mathrm{co}\{x_1, \ldots, x_m\} \not\subset \bigcup_{j=1,\ldots,m} \overline{G(y^j)}$. Choose $w = \sum_j \lambda_j x_j \in \mathrm{co}\{x_1, \ldots, x_m\} \backslash \bigcup_j \overline{G(y^j)}$, then $w \in \bigcap_{j=1}^m F^{-1}(y^j)$. Therefore $y_i^j \in F_i(w)$ for each $i \in I$ and $j = 1, \ldots, m$.

For each $i \in I$, put $z_i := \sum_j \lambda_j y_i^j$. Since $F_i(w)$ is convex, $z_i \in F_i(w)$. For each $i \in I$, $d(A, B_i) \leq d(w, F_i(w)) \leq ||w - z_i|| = ||\sum_j \lambda_j x_j - \sum_j \lambda_j y_i^j|| \leq \sum_j \lambda_j ||x_j - y_i^j|| = \sum_j \lambda_j d(A, y_i^j) = d(A, B_i)$.

Therefore $d(w, F_i(w)) = d(A, B_i)$ for all $i \in I$. □

If (A, B) is a proximal pair, then $P_A(y) \neq \emptyset$ for each $y \in B$. So we obtain the following corollary.

Corollary 2.4. *Let (A, B) be a nonempty convex proximal pair in a normed space M. Let A be compact and $F : A \multimap B$ be a multimap such that*
(1) *$F(x)$ is convex for each $x \in A$; and*
(2) *$F^{-1}(y)$ is unionly open for each $y \in B$.*
Then there is a $w \in A$ such that $d(w, F(w)) = d(A, B)$.

Corollary 2.4 improves Theorem 1.1.

If A and B are non-empty subsets of a normed space M such that $d(A, B) > 0$, then A_0 and B_0 are contained in the boundaries of A and B respectively, see [2].

Proposition 2.5. *Let A and B be nonempty subsets of a normed space M. And let A_0 and B_0 be nonempty.*
(1) *If the sets A and B are convex, then A_0 and B_0 are convex.*
(2) *If the sets A and B are compact, then A_0 and B_0 are compact.*

Proposition 2.5 is in the proof of Theorem 1 in Kim and Lee [8].

Proposition 2.6. *For each $i \in I = \{1, \ldots, n\}$, let A and B_i be nonempty subsets of a normed space M such that $\prod B_{i0} \neq \emptyset$ and $\bigcap_{i\in I} P_A(y_i) \neq \emptyset$ for each $(y_1, \ldots, y_n) \in \prod B_{i0}$. Then $\bigcap A_{0i} \neq \emptyset$ where $A_{0i} := \{x \in A : ||x - y|| = d(A, B_i)$ for some $y \in B_i\}$.*

Proof. If $y_i \in B_{i0}$, then $P_A(y_i) \subset A_{0i}$. Therefore $\bigcap A_{0i} \neq \emptyset$. □

The following is an existence theorem for the pairs $\{(A, B_i)\}_{i=1,\ldots,n}$ which are not proximal;

Theorem 2.7. *For each $i \in I = \{1, \ldots, n\}$, let A and B_i be nonempty compact convex subsets of a normed space M satisfying $\prod B_{i0} \neq \emptyset$. Let $F_i : A \multimap B_i$ be a multimap such that*
(1) *$\bigcap_{i\in I} P_A(y_i) \neq \emptyset$ for each $(y_1, \ldots, y_n) \in \prod B_{i0}$;*
(2) *$F_i(x) \cap B_{i0} \neq \emptyset$ for each $x \in C := \bigcap_{i\in I} A_{0i}$;*
(3) *$F_i(x)$ is convex for each $x \in C$; and*
(4) *$F_i^{-1}(y_i)$ is unionly open for each $y_i \in B_{i0}$.*
Then there is a $w \in C$ such that $d(w, F_i(w)) = d(A, B_i)$ for each $i \in I$.

Proof. By Proposition 2.6, $C \neq \emptyset$. By Proposition 2.5, A_{0i} and B_{i0} are compact and convex, so C is also compact and convex.

For each $x \in C$, there exists a $y_i \in B_{i0}$ such that $||x - y_i|| = d(A, B_i)$ for each $i \in I$. Since $d(A, B_i) \leq d(C, B_{i0}) \leq ||x - y_i||$, $||x - y_i|| = d(C, B_{i0})$ and $d(C, B_{i0}) = d(A, B_i)$.

For $(y_1, \ldots, y_n) \in \prod B_{i0}$, there exists an $x \in \bigcap_{i\in I} P_A(y_i) \subset C$ such that $||x - y_i|| = d(A, y_i)$ for each $i \in I$, by condition (1). Since $d(A, y_i) = d(A, B_i)$ and $d(C, B_{i0}) = d(A, B_i)$, $||x - y_i|| = d(C, B_{i0})$.

Therefore $x \in \bigcap_{i\in I} P_C(y_i)$ for each $(y_1, \ldots, y_n) \in \prod B_{i0}$. And also (C, B_{i0}) is a proximal pair of M for each $i \in I$.

Define $S_i : C \multimap B_{i0}$ by $S_i(x) = F_i(x) \cap B_{i0}$ for each $x \in C$. Note that $S_i^{-1}(y_i)$ is unionly open for each $y_i \in B_{i0}$. Since S_i satisfies all the conditions of Theorem 2.3, there is a $w \in C$ such that $d(w, S_i(w)) = d(C, B_{i0})$. So $d(A, B_i) \leq d(w, F_i(w)) \leq d(w, S_i(w)) = d(C, B_{i0}) = d(A, B_i)$ for each $i \in I$. Therefore the conclusion holds. □

Corollary 2.8. *Let A and B be nonempty compact convex subsets of a normed space M, $A_0 \neq \emptyset$ and $F : A \multimap B$ be a multimap such that*
(1) *$F(x) \cap B_0 \neq \emptyset$ for each $x \in A_0$;*
(2) *$F(x)$ is convex for each $x \in A_0$; and*
(3) *$F^{-1}(y)$ is unionly open for each $y \in B_0$.*
Then there is a $w \in A_0$ such that $d(w, F(w)) = d(A, B)$.

3. Best proximity point theorems in hyperconvex metric spaces

Let (M, d) be a hyperconvex metric space.

The *admissible* subset of M is a set of the form $\bigcap_\alpha B(x_\alpha, \gamma_\alpha)$, i.e., a closed ball intersection in M.

A subset X of M is said to be *externally hyperconvex* (relative to M) if given any family $\{x_\alpha\}$ of points of M and any family of $\{\gamma_\alpha\}$ of real numbers satisfying for each α and β, $d(x_\alpha, x_\beta) \leq \gamma_\alpha + \gamma_\beta$ with $d(x_\alpha, X) \leq \gamma_\alpha$, it follows that $\bigcap_\alpha B(x_\alpha, \gamma_\alpha) \cap X \neq \emptyset$.

A subset X of M is said to be *weakly externally hyperconvex* (relative to M) if X is externally hyperconvex relative to $X \cup \{z\}$ for each $z \in M$, that is, given any family $\{x_\alpha\}$ of points of M all but at most one of which lies in X, and any family of $\{\gamma_\alpha\}$ of real numbers satisfying $d(x_\alpha, x_\beta) \leq \gamma_\alpha + \gamma_\beta$ with $d(x_\alpha, X) \leq \gamma_\alpha$ if $x_\alpha \notin X$, it follows that $\bigcap_\alpha B(x_\alpha, \gamma_\alpha) \cap X \neq \emptyset$.

For any $A \in \langle M \rangle$, let $\Gamma_A = \bigcap\{B : B$ is a closed ball containing $A\}$. A subset X of M is said to be *sub-admissible* if for each $N \in \langle X \rangle$, $\Gamma_N \subset X$.

Note that if A is a subset of a hyperconvex metric space M, then

A is admissible
$\Longrightarrow$ A is externally hyperconvex
$\Longrightarrow$ A is weakly externally hyperconvex
$\Longrightarrow$ A is hyperconvex
$\Longrightarrow$ A is sub-admissible.

For details, see [5], Theorem 6 in [1], and Theorem 3.10 in [3].

A subset A of M is called *proximinal* if $x \in M$, then there exists $a \in A$ such that $d(x, a) = d(x, A)$.

A subset A of M is called a *proximinal nonexpansive retract* of M if there exists a nonexpansive retraction r of M onto A for which $d(x, r(x)) = d(x, A)$ for each $x \in M$. Thus $d(r(x), r(y)) \leq d(x, y)$ for each $x, y \in M$.

Lemma 3.1 ([4, Theorem 4.2]). *A compact subset X of a hyperconvex metric space M is weakly externally hyperconvex if and only if it is a proximinal nonexpansive retract of M.*

Let X be a topological space and (M, d) be a metric space. A multimap $F : X \multimap M$ is called;

(1) *lower semicontinuous* at $x \in X$, if for each open set W with $W \cap F(x) \neq \emptyset$, there is a neighborhood $U(x)$ of x such that $F(z) \cap W \neq \emptyset$ for all $z \in U(x)$.

(2) *quasi-lower semicontinuous* at $x \in X$, if for each $\epsilon > 0$, there are $y \in F(x)$ and a neighborhood $U(x)$ of x such that $F(z) \cap B(y, \epsilon) \neq \emptyset$ for all $z \in U(x)$.

If F is lower semicontinuous (quasi-lower semicontinuous, respectively) at each $x \in X$, F is called *lower semicontinuous* (*quasi-lower semicontinuous*, respectively). Note that (1) $\Longrightarrow$ (2).

Lemma 3.2 ([12, Theorem 1]). *Let X be a paracompact topological space, M be a hyperconvex metric space and $F : X \multimap M$ be a quasi-lower semicontinuous map with closed sub-admissible values. Then F has a continuous selection; i.e., there is a continuous function $f : X \to M$ such that $f(x) \in F(x)$ for each $x \in X$.*

Lemma 3.3 ([6], [10]). *Any continuous self map of a compact admissible subset of a hyperconvex metric space has a fixed point.*

Theorem 3.4. *Let (M, d) be a hyperconvex metric space, A be a compact weakly externally hyperconvex subset of M, B be a hyperconvex subset of M and A_0 be a nonempty admissible subset of M. Let $F : A \multimap B$ be a quasi-lower semicontinuous map such that*

(1) *for each $x \in A$, $F(x)$ is closed sub-admissible; and*

(2) *$F(A_0) \subset B_0$.*

Then there exists a best proximity point for F.

Proof. By Lemma 3.2, F has a continuous selection which we denote by f. Let $x \in A_0$, then by (2), $f(x) \in B_0$. So there exists an $a \in A_0$ such that $d(f(x), a) = d(B, A)$. Since A is a compact weakly externally hyperconvex, it is a proximinal nonexpansive retract of M. Let r be a proximinal nonexpansive retraction of M onto A. Then $d(f(x), r \circ f(x)) = d(f(x), A) \leq d(f(x), a)$. Therefore $d(f(x), r \circ f(x)) = d(B, A)$ and $r \circ f : A_0 \to A_0$.

Since A_0 is an admissible subset of a compact set A, A_0 is compact. By Lemma 3.3, there exists an $x_0 \in A_0$ such that $r \circ f(x_0) = x_0$. So $d(x_0, f(x_0)) = d(x_0, F(x_0)) = d(A, B)$, that is, x_0 is a best proximity point for F. □

For a subset A of M, $N_\epsilon(A) = \{x \in M : d(x, A) \leq \epsilon\}$.

The *Hausdorff metric* d_H on nonempty bounded closed subsets A, B of M is given by

$$d_H(A, B) = \inf\{\epsilon > 0 : A \subset N_\epsilon(B) \text{ and } B \subset N_\epsilon(A)\}.$$

From Theorem 3.4, we obtain the following corollary ([9, Theorem 2.11]);

Corollary 3.5. *Let A and B be two convex subsets of a hyperconvex metric space (M, d). Suppose that A is compact and weakly externally hyperconvex and that A_0 and B are admissible. Let $F : (A, d) \multimap (B, d_H)$ be a continuous map such that*

(1) *for each $x \in A$, $F(x)$ is an externally hyperconvex subset of B; and*
(2) *$F(A_0) \subset B_0$.*

Then there exists a best proximity point of F.

Proof. Note that if $F : (A, d) \multimap (B, d_H)$ is continuous, then $F : (A, d) \multimap (B, d)$ is lower semicontinuous.

It is shown in ([1, Theorem 7]) and ([3, Lemma 3.8]) that the externally hyperconvex subsets of M are proximinal in M. Therefore $F(x)$ is closed for each $x \in X$. □

In the proof of Corollary 3.5, Kirk et al. [9] showed that A_0 is hyperconvex and claimed that $r \circ F : A_0 \multimap A_0$ has a fixed point by Lemma 3.3, where r is a proximinal nonexpansive retraction of M onto A. So the condition "A_0 is admissible" must be added in order to use Lemma 3.3.

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