Bull. Korean Math. Soc. **53** (2016), No. 5, pp. 1353–1362
http://dx.doi.org/10.4134/BKMS.b150636
pISSN: 1015-8634 / eISSN: 2234-3016

COMMUTING ELEMENTS WITH RESPECT TO THE OPERATOR $\wedge$ IN INFINITE GROUPS

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Abstract. Using the notion of complete nonabelian exterior square $G\widehat{\wedge}G$ of a pro-p-group G (p prime), we develop the theory of the exterior degree $\widehat{\mathrm{d}}(G)$ in the infinite case, focusing on its relations with the probability of commuting pairs $\mathrm{d}(G)$. Among the main results of this paper, we describe upper and lower bounds for $\widehat{\mathrm{d}}(G)$ with respect to $\mathrm{d}(G)$. Here the size of the second homology group $H_2(G,\mathbb{Z}_p)$ (over the p-adic integers $\mathbb{Z}_p$) plays a fundamental role. A further result of homological nature is placed at the end, in order to emphasize the influence of $H_2(G,\mathbb{Z}_p)$ both on G and $\widehat{\mathrm{d}}(G)$.

1. Statement of the main results

The probability of finding two commuting elements has motivated several authors to introduce the *commutativity degree* of a finite group E

$$\mathrm{d}(E)=\frac{|\{(x,y)\in E\times E\mid [x,y]=1\}|}{|E|^2}=\frac{1}{|E|^2}\sum_{x\in E}|C_E(x)|=\frac{k(E)}{|E|},$$

where $k(E)$ is the number of E-conjugacy classes $[x]_E=\{x^g\mid g\in E\}$ that constitute E and $C_E(x)$ the centralizer of x in E. Of course, the first equality gives the definition, while the others follow easily. There is a wide production on $\mathrm{d}(E)$, for instance [1, 7, 8, 9, 17, 18]. The *exterior degree* of E is a more recent notion, studied for similar purposes in [2, 6, 13, 14]. It is defined by

$$(*)\qquad \mathrm{d}^{\wedge}(E)=\frac{|\{(x,y)\in E\times E\mid x\wedge y=1_{E\wedge E}\}|}{|E|^2}=\frac{1}{|E|}\sum_{i=1}^{k(E)}\frac{|C^{\wedge}_E(x_i)|}{|C_E(x_i)|},$$

where $\wedge$ denotes the operator of *nonabelian exterior square* in [4, 15, 16, 19] and the last equality is proved in [13, Lemma 2.2]. For instance, $\mathrm{d}^{\wedge}(E)\le \mathrm{d}(E)$ (see [13]) so the commutativity and the exterior degrees are connected.

Received August 9, 2015.
2010 *Mathematics Subject Classification.* Primary 20J05, 20J06; Secondary 20E10, 20P05.
Key words and phrases. exterior degree, exterior center, exterior centralizers, complete nonabelian exterior square.

In order to generalize the above concepts to the infinite case, we proceed as in [7, 8, 18]. We refer to [10] and [11, Window 5, pp. 357–365] for feedback on pro-p-groups. Briefly, a pro-p-group G is an projective limit of finite p-groups and the direct product $G \times G$ is again a pro-p-group. The notion of *complete nonabelian exterior square* $G\widehat{\wedge}G$ is more delicate and will be illustrated in Section 2. We will see that $G\widehat{\wedge}G$ is a pro-p-group, when so is G (Corollary 2.2 below). Then it is reasonable to consider the continuous map

$$(**) \qquad \widehat{f} : (x, y) \in G \times G \longmapsto x\widehat{\wedge}y \in G\widehat{\wedge}G$$

between the pro-p-groups $G \times G$ and $G\widehat{\wedge}G$. Section 2 contains more details on the well definition of $\widehat{f}$, its continuity and on the meaning of the symbol $x\widehat{\wedge}y$. Since there exists a unique normalized Haar measure μ on a pro-p-group (see [11, §11.0 and 11.1]), this is true for $G \times G$ when we consider the product measure $\mu \times \mu$. Then the set

$$C = \widehat{f}^{-1}(1) = \{(x, y) \in G \times G \mid x\widehat{\wedge}y = 1\} \subseteq G \times G$$

is a closed subgroup of $G \times G$ and the *exterior degree of a pro-p-group G* is

$$(***) \qquad \widehat{\mathrm{d}}(G) = (\mu \times \mu)(C).$$

In case G is finite and μ is the counting measure, $(***)$ becomes $(*)$. Our first main result correlates commutativity and exterior degrees.

Theorem 1.1. *A pro-p-group G satisfies the following inequality*

$$\widehat{\mathrm{d}}(G) \leq \mathrm{d}(G) - \left(\frac{p-1}{p}\right)\left(\mu(Z(G)) - \mu(\widehat{Z}(G))\right).$$

Furthermore, if $H_2(G, \mathbb{Z}_p)$ is finite, then

$$\widehat{\mathrm{d}}(G) \geq \mu(\widehat{Z}(G)) + \frac{1}{|H_2(G, \mathbb{Z}_p)|}\left(\mathrm{d}(G) - \mu(\widehat{Z}(G))\right).$$

Here $\widehat{Z}(G)$ denotes the *complete exterior center* of G, that is, the set

$$\widehat{Z}(G) = \{g \in G \mid g\widehat{\wedge}y = 1, \ \forall y \in G\}.$$

Some elementary properties of $\widehat{Z}(G)$ will be discussed in this paper. The size of $\widehat{Z}(G)$ and of $H_2(G, \mathbb{Z}_p)$ is fundamental in Theorem 1.1, so we prove a further result.

Theorem 1.2. *Let G be a pro-p-group,* $\mathrm{rk}(G/\widehat{Z}(G)) = n$, $\mathrm{rk}(H_2(G, \mathbb{Z}_p)) = m$ *and* $\mathrm{tf}(H_2(G, \mathbb{Z}_p)) = l$.

(i) *If $H_2(G, \mathbb{Z}_p)$ is finite, then $|Z(G)/\widehat{Z}(G)|$ divides $|H_2(G, \mathbb{Z}_p)|^n$.*
(ii) *If $H_2(G, \mathbb{Z}_p)$ is infinite, then* $\mathrm{rk}(Z(G)/\widehat{Z}(G)) \leq m^n$. *In particular, if $H_2(G, \mathbb{Z}_p)$ is torsion-free, then* $\mathrm{tf}(Z(G)/\widehat{Z}(G)) \leq l^n$.

The notion of rank, involved in Theorem 1.2, is usual (see [10, 11]). More precisely, the *rank* of a pro-p-group G is the number $\mathrm{rk}(G) = \sup\{m(H) \mid H = \overline{H} \leq G\}$, where $m(H)$ is the minimal number of elements which generate topologically H. If G is torsion-free, $l = \mathrm{rk}(G) = \mathrm{tf}(G)$ is called *torsion-free rank*. [5, Theorem A] provides useful information on the size of $H_2(G, \mathbb{Z}_p)$.

After the preliminaries of Section 2, we show some fundamental properties of $(***)$ in Section 3. Then Theorems 1.1 and 1.2 are proved in Section 4. Finally, Section 5 contains corollaries and applications of the main results.

2. Preliminaries

This section collects a series of information and some technical results in [4, 10, 11, 16, 19]. We begin to recall that a pro-p-group $G = \lim_{N \in \mathcal{P}(G)} G/N$ may be written as the projective limit of finite p-groups, where

$$\mathcal{P}(G) = \{N = \overline{N} \triangleleft G \mid G/N \text{ is a finite } p\text{-group}\}$$

is a filter basis. The complete nonabelian tensor square $G \widehat{\otimes} G$ of G is the group topologically generated by the symbols $x \widehat{\otimes} y$, subject to the relations $xy \widehat{\otimes} z = (x^y \widehat{\otimes} z^y)(y \widehat{\otimes} z)$ and $y \widehat{\otimes} tz = (y \widehat{\otimes} z)(y^z \widehat{\otimes} t^z)$ for all $x, y, z, t \in G$, where $x^y = y^{-1}xy$ denotes the conjugate of x with respect to y. This construction can be found in [12, 19] and generalizes the nonabelian tensor square in [4], originally formulated by Brown, Johnson and Robertson for abstract groups. A series of interesting properties have been studied with respect to the formation of the nonabelian tensor square (see [16]), in particular if P is a finite p-group, then so is $P \otimes P$. This allows us to ask whether $G \widehat{\otimes} G$ has the structure of projective limit of the finite p-groups $G/N \otimes G/N$, being $N \in \mathcal{P}(G)$. The answer is positive:

Theorem 2.1 (See [19], Theorem 1.1)**.** *Given a pro-p-group*

$$G = \lim_{N \in \mathcal{P}(G)} G/N,$$

there exists an isomorphism such that $G \widehat{\otimes} G \simeq \lim_{N \in \mathcal{P}(G)} G/N \otimes G/N$. *In particular,* $G \widehat{\otimes} G$ *is a pro-p-group.*

Here $\widehat{\nabla}(G) = \overline{\langle x \otimes x \mid x \in G \rangle}$ is a closed central subgroup of $G \widehat{\otimes} G$ and

$$G \widehat{\otimes} G / \widehat{\nabla}(G) = G \widehat{\wedge} G$$

defines the *complete nonabelian exterior square* of G. Using Theorem 2.1, we get:

Corollary 2.2. *Given a pro-p-group* $G = \lim_{N \in \mathcal{P}(G)} G/N$, *there exists an isomorphism such that* $G \widehat{\wedge} G \simeq \lim_{N \in \mathcal{P}(G)} G/N \wedge G/N$. *In particular,* $G \widehat{\wedge} G$ *is a pro-p-group.*

Corollary 2.2 clarifies the meaning of $G\widehat{\wedge}G$ in $(**)$. For the rest of this paper, it is good to recall here that the maps

$$\widehat{\kappa} : x\widehat{\otimes}y \in G\widehat{\otimes}G \mapsto [x,y] \in \overline{[G,G]} \text{ and } \widehat{\kappa'} : x\widehat{\wedge}y \in G\widehat{\wedge}G \mapsto [x,y] \in \overline{[G,G]},$$

are epimorphisms of pro-p-groups such that

$$\ker \widehat{\kappa} \supseteq \widehat{\nabla}(G) \text{ and } \ker \widehat{\kappa'} \simeq H_2(G,\mathbb{Z}_p).$$

Moreover $H_2(G,\mathbb{Z}_p)$ is a closed central subgroup of $G\widehat{\wedge}G$ (see [12, 19]). Also

$$\widehat{\varepsilon} : x\widehat{\otimes}y \in G\widehat{\otimes}G \mapsto x\widehat{\wedge}y \in G\widehat{\wedge}G \text{ and } \widehat{\psi} : x\widehat{\otimes}x \in \widehat{\nabla}(G) \mapsto x\widehat{\wedge}x \in H_2(G,\mathbb{Z}_p)$$

are epimorphisms of pro-p-groups. The commutative diagram in [4] becomes the following, whose rows are central extensions of pro-p-groups (see also [12, 16, 19]):

$$(\dagger)\qquad \begin{array}{ccccccccc} 1 & \longrightarrow & \widehat{\nabla}(G) & \longrightarrow & G\widehat{\otimes}G & \xrightarrow{\widehat{\kappa}} & \overline{[G,G]} & \longrightarrow & 1 \\ & & \widehat{\varphi}\Big\downarrow & & \widehat{\varepsilon}\Big\downarrow & & \Big\| & & \\ 1 & \longrightarrow & H_2(G,\mathbb{Z}_p) & \longrightarrow & G\widehat{\wedge}G & \xrightarrow{\widehat{\kappa'}} & \overline{[G,G]} & \longrightarrow & 1. \end{array}$$

It is appropriate to recall here that the *complete exterior centralizer* of a pro-p-group G is the set

$$\widehat{C}_G(x) = \{a \in G \mid a\widehat{\wedge}x = 1\}$$

which turns out to be a closed normal subgroup of $C_G(x)$ by [19, Lemma 4.2]. Moreover, $C_G(x)/\widehat{C}_G(x)$ is isomorphic to a closed subgroup of the abelian pro-p-group $H_2(G,\mathbb{Z}_p)$ by [19, Theorem 1.2]. Of course, if G is finite, $\widehat{\wedge}$ becomes the nonabelian exterior square $\wedge$ in [4] and $\widehat{\otimes}$ the nonabelian tensor square in $\otimes$. In particular, $\widehat{C}_G(x) = C^{\wedge}_G(x)$ and $\widehat{Z}(G) = Z^{\wedge}(G)$ have been studied in [2, 6, 13, 15]. Summarizing, we have the following lemma.

Lemma 2.3. *Let N be a closed normal subgroup of a pro-p-group G and $x \in G$. Then*

(i) $\widehat{C}_G(x)$ *is a closed normal subgroup of* $C_G(x)$;
(ii) $\widehat{Z}(G)$ *is a closed subgroup of* $Z(G)$;
(iii) $1 \longrightarrow \widehat{C}_G(x) \cap N \longrightarrow \widehat{C}_G(x) \longrightarrow \widehat{C}_{G/N}(xN) \longrightarrow 1$ *is a short exact sequence of pro-p-groups*;
(iv) $1 \longrightarrow \widehat{Z}(G) \cap N \longrightarrow \widehat{Z}(G) \longrightarrow \widehat{Z}(G/N) \longrightarrow 1$ *is a short exact sequence of pro-p-groups*;

Proof. (i). See [19, Lemma 4.2].

(ii). Since $\widehat{Z}(G) = \bigcap\limits_{g\in G} \widehat{C}_G(g)$, we use (i) and conclude $\widehat{Z}(G) \subseteq Z(G) = \bigcap\limits_{g\in G} C_G(g)$. The intersection of closed is closed so $\widehat{Z}(G)$ is closed by (i).

(iii). The natural epimorphism of pro-p-groups $\pi : g \in G \mapsto gN \in G/N$ restricts to the epimorphism of pro-p-groups $\pi_| : g \in \widehat{C}_G(x) \mapsto gN \in \widehat{C}_{G/N}(xN)$

and $\ker \pi_| = \widehat{C}_G(x) \cap N$ is a closed normal subgroup of $\widehat{C}_G(x)$. Therefore $1 \longrightarrow \widehat{C}_G(x) \cap N \longrightarrow \widehat{C}_G(x) \longrightarrow \widehat{C}_{G/N}(xN) \longrightarrow 1$ is a short exact sequence.

(iv). It follows from (ii) and (iii). □

We end with a notion from [12, 19], specialized to the present context. Let G and K be pro-p-groups. A map $\widehat{\psi} : G \times G \to K$ is a *crossed pairing* of pro-p-groups, if it is continuous and satisfies (for all $a, b, g, h \in G$) the rules $\widehat{\psi}(ag, h) = \widehat{\psi}(g^a, h^g)\widehat{\psi}(g, h)$, $\widehat{\psi}(g, h)\widehat{\psi}(a, b) = \widehat{\psi}(a^{[g,h]}, b^{[g,h]})\widehat{\psi}(g, h)$, $\widehat{\psi}(g, bh) = \widehat{\psi}(g, h)\widehat{\psi}(g^h, h^b)$ and $\widehat{\psi}(a, b)\widehat{\psi}(g, h) = \widehat{\psi}(g, h)\widehat{\psi}(a^{[h,g]}, b^{[h,g]})$. Roughly speaking, we are generalizing bilinear maps and the (usual) abelian tensor square. A corresponding universal property is described by [19, Lemma 2.5]. In particular, we may choose $K = G\widehat{\otimes}G$ in the above definition and define $\widehat{f} = \widehat{\varepsilon} \circ \widehat{\psi}$. This map is defined in $(**)$ and of course is continuous, since a composition of two continuous maps. Consequently, the set C and the measure $\widehat{\mathrm{d}}(C)$ are well defined in $(***)$.

3. Properties of the exterior degree

We clarified the main algebraic and topological properties of the complete nonabelian exterior square, hence we may prove some fundamental properties of the exterior degree of pro-p-groups. The next lemmas show that the methods of [7, 8, 9, 18] can be adapted here.

Lemma 3.1. *Let G be a pro-p-group and $x \in G$. Then*

$$\widehat{\mathrm{d}}(G) = \int_G \mu(\widehat{C}_G(x))d\mu(x),$$

where $\mu(\widehat{C}_G(x)) = \int_{G\times G} \chi_C(x, y)d\mu(y)$ and χ_C is the characteristic map of C.

Proof. Fubini-Tonelli's Theorem implies:

$$\begin{aligned}\widehat{\mathrm{d}}(G) = (\mu \times \mu)(C) &= \int_{G\times G} \chi_C(d\mu \times d\mu)\\ &= \int_G \left(\int_G \chi_C(x, y)d\mu(x)\right) d\mu(y) = \int_G \mu(\widehat{C}_G(x))d\mu(x).\end{aligned}$$

□

Lemma 3.2. *Let H be a closed subgroup of a pro-p-group G and $k \geq 1$. Then*

$$\mu(H) = \begin{cases} \frac{1}{p^k}, & \text{if} \quad |G : H| = p^k \\ 0, & \text{if} \quad |G : H| = \infty. \end{cases}$$

Proof. See [18, Lemma 2.4]. □

From Lemmas 3.1 and 3.2, we may characterize pro-p-groups with $\widehat{\mathrm{d}}(G) \in \{0, 1\}$.

Proposition 3.3. *A pro-p-group G has $0 < \widehat{\mathrm{d}}(G) \leq 1$. In particular,*

(i) $\widehat{\mathrm{d}}(G) = 0$ *if and only if* $|G : \widehat{C}_G(x)| = \infty$ *for all but finitely many* $x \in G$;

(ii) $\widehat{\mathrm{d}}(G) = 1$ *if and only if* $\widehat{Z}(G) = G$.

Proof. Since μ is monotone, positive and normalized, $0 = (\mu \times \mu)(\{(1,1)\}) < \widehat{\mathrm{d}}(G) = (\mu \times \mu)(C) \leq (\mu \times \mu)(G) = 1$. Now (i) follows easily from Lemmas 3.1 and 3.2. (ii) is clear from the definition of $\widehat{Z}(G)$ and $(***)$. □

Two observations may be useful here.

Remark 3.4. A finite group E is capable if and only if $Z^\wedge(E) = 1$ (see [3, 15]). Then Proposition 3.3 allows us to find criteria for the size of $\widehat{Z}(G)$, getting information on the capability of a pro-p-group G.

A fundamental difference with the finite case is the following.

Remark 3.5. The group $\mathbb{Z}_p$ is an infinite abelian pro-p-group topologically generated by 1 element (but not generated by 1 element in the abstract sense). Here $\widehat{Z}(\mathbb{Z}_p) = \mathbb{Z}_p$, $\widehat{\mathrm{d}}(\mathbb{Z}_p) = 1$ and $H_2(\mathbb{Z}_p, \mathbb{Z}_p)$ is trivial.

We describe the behaviour for quotients and direct products.

Proposition 3.6. *Let N be a closed normal subgroup of a pro-p-group G. Then* $\widehat{\mathrm{d}}(G) \leq \widehat{\mathrm{d}}(G/N)$ *and the equality holds if* $N \leq \widehat{Z}(G)$.

Proof. Assume that λ, μ and ν are corresponding Haar measure of N, G and G/N respectively. The Extended Weil's Formula (see [18, Equation $(*)$, p. 126]) and Lemma 3.1 imply

$$\begin{aligned}\widehat{\mathrm{d}}(G) &= \int_G \mu(\widehat{C}_G(x))d\mu(x) \leq \int_G \mu(\widehat{C}_G(x)N)d\mu(x) \\ &= \int_{\frac{G}{N}} \int_N \mu(\widehat{C}_G(xn)N)d\lambda(n)d\nu(xN).\end{aligned}$$

On the other hand, one can see without difficulties that

$$\mu(\widehat{C}_G(xn)N) = \nu(\widehat{C}_G(xn)N/N) \leq \nu(\widehat{C}_{G/N}(xN))$$

therefore,

$$\begin{aligned}\widehat{\mathrm{d}}(G) &\leq \int_{\frac{G}{N}} \int_N \nu(\widehat{C}_{G/N}(xN))d\lambda(n)d\nu(xN) \\ &= \int_{\frac{G}{N}} \nu(\widehat{C}_{G/N}(xN)) \left(\int_N d\lambda(n)\right) d\nu(xN) \\ &= \int_{\frac{G}{N}} \nu(\widehat{C}_{G/N}(xN))d\nu(xN) = \widehat{\mathrm{d}}(G/N).\end{aligned}$$

Now assume $N \subseteq \widehat{Z}(G)$. The canonical map $G \widehat{\wedge} G \to G/N \widehat{\wedge} G/N$ will be an isomorphism. Therefore Lemma 2.3(iii) implies $\widehat{C}_G(xn)N/N \simeq \widehat{C}_{G/N}(xN)$ for all $x \in G$ and so $\widehat{\mathrm{d}}(G) = \widehat{\mathrm{d}}(G/N)$. □

4. Proofs of the main theorems

The present section illustrates our main results.

Proof of Theorem 1.1. We begin with the upper bound. Lemma 3.1 implies

$$\begin{aligned}\widehat{\mathrm{d}}(G) &= \int_G \mu(\widehat{C}_G(x))d\mu(x)\\ &= \mu(\widehat{Z}(G)) + \int_{Z(G)-\widehat{Z}(G)} \mu(\widehat{C}_G(x))d\mu(x) + \int_{G-Z(G)} \mu(\widehat{C}_G(x))d\mu(x),\end{aligned}$$

where $x \notin \widehat{Z}(G)$. The monotonicity of μ implies $\mu(\widehat{C}_G(x)) \le \mu(C_G(x))$ and Lemma 3.2 implies $\mu(\widehat{C}_G(x)) = |G : \widehat{C}_G(x)|^{-1} \le \frac{1}{p}$, thus

$$\begin{aligned}&\le \mu(\widehat{Z}(G)) + \frac{1}{p}\Big(\mu(Z(G)) - \mu(\widehat{Z}(G))\Big) + \int_{G-Z(G)} \mu(C_G(x))d\mu(x)\\ &= \mu(\widehat{Z}(G)) + \frac{1}{p}\Big(\mu(Z(G)) - \mu(\widehat{Z}(G))\Big) + \mathrm{d}(G) - \mu(Z(G))\\ &= \mathrm{d}(G) - \left(\frac{p-1}{p}\right)\Big(\mu(Z(G)) - \mu(\widehat{Z}(G))\Big).\end{aligned}$$

For the lower bound, we apply Lemmas 3.1, 3.2 and $|C_G(x) : \widehat{C}_G(x)| \le |H_2(G, \mathbb{Z}_p)|$, which is shown in [19, Theorem 1.2]. Therefore

$$\begin{aligned}\widehat{\mathrm{d}}(G) &= \int_G \mu(\widehat{C}_G(x))d\mu(x) = \mu(\widehat{Z}(G)) + \int_{G-\widehat{Z}(G)} \frac{\mu(C_G(x))}{|C_G(x) : \widehat{C}_G(x)|} d\mu(x)\\ &\ge \mu(\widehat{Z}(G)) + \frac{1}{|H_2(G,\mathbb{Z}_p)|}\int_{G-\widehat{Z}(G)} \mu(C_G(x))d\mu(x)\\ &= \mu(\widehat{Z}(G)) + \frac{1}{|H_2(G,\mathbb{Z}_p)|}\left(\int_G \mu(C_G(x))d\mu(x) - \mu(\widehat{Z}(G))\right)\\ &= \mu(\widehat{Z}(G)) + \frac{1}{|H_2(G,\mathbb{Z}_p)|}\Big(\mathrm{d}(G) - \mu(\widehat{Z}(G))\Big).\end{aligned}$$

□

Now we show a bound for $H_2(G, \mathbb{Z}_p)$, in harmony with the results of [5, 12].

Proof of Theorem 1.2. First of all, we note that $H_2(G, \mathbb{Z}_p)$ is a closed central pro-p-subgroup of $G\widehat{\wedge}G$. This follows from the definitions, from the commutativity of (†) and from Corollary 2.2. Now we proceed to prove (i) and (ii). Assume that $G/\widehat{Z}(G) = \overline{\langle \bar{x}_1, \ldots, \bar{x}_n\rangle}$ for some elements $\bar{x}_1 = x_1\widehat{Z}(G), \ldots, \bar{x}_n = x_n\widehat{Z}(G)$ of $G/\widehat{Z}(G)$. Define from $Z(G)$ to the direct product of n copies of $H_2(G, \mathbb{Z}_p)$:

$$\widehat{\xi} : x \in Z(G) \longmapsto (x\,\widehat{\wedge}x_1, \ldots, x\widehat{\wedge}x_n) \in H_2(G, \mathbb{Z}_p)^n.$$

Here $\widehat{\xi}$ is a homomorphism of pro-p-groups, because for all $x, y \in Z(G)$ we have

$$xy\widehat{\wedge}x_i = (x\widehat{\wedge}x_i)\ (y\widehat{\wedge}x_i)^x = (x\widehat{\wedge}x_i)\ (y\widehat{\wedge}x_i)$$

for every $i \in \{1, \ldots, n\}$. We claim that $\ker \widehat{\xi} = \widehat{Z}(G)$. It is easy to check that $\widehat{Z}(G) \subseteq \ker \widehat{\xi}$. On the other hand, if $x \in \ker \widehat{\xi}$, then $x \widehat{\wedge} x_i = 1$ for every $i \in \{1, \ldots, n\}$. It is enough to show that $x \widehat{\wedge} y = 1$ for every $y \in G$ in order to finish our proof. If $y \in G \setminus \widehat{Z}(G)$, then we may always write $y = x_1^{\alpha_1} \cdots x_n^{\alpha_n}$, where $\alpha_1, \ldots, \alpha_n$ are suitable integers. Thus

$$x \widehat{\wedge} y = x \widehat{\wedge} (x_1^{\alpha_1} \cdots x_n^{\alpha_n}) = (x \widehat{\wedge} x_1^{\alpha_1}) \cdots (x \widehat{\wedge} x_n^{\alpha_n}) = (x \widehat{\wedge} x_1)^{\alpha_1} \cdots (x \widehat{\wedge} x_n)^{\alpha_n} = 1$$

and $\widehat{Z}(G) \supseteq \ker \widehat{\xi}$. Then $\widehat{Z}(G) = \ker \widehat{\xi}$ and First Isomorphism Theorem of pro-p-groups (see [10, 11]) implies that $Z(G)/\widehat{Z}(G) = Z(G)/\ker \widehat{\xi} \simeq \mathrm{Im}\ \widehat{\xi}$ is isomorphic to a closed subgroup of $H_2(G, \mathbb{Z}_p)^n$. In case (i), $H_2(G, \mathbb{Z}_p)$ is finite, hence $|Z(G)/\widehat{Z}(G)|$ must divide $|H_2(G, \mathbb{Z}_p)|^n$. In case (ii), $H_2(G, \mathbb{Z}_p)$ is infinite abelian of finite rank, and the bound of (ii) is true. The result follows. □

5. Some applications

Theorem 1.1 allows us to find an interesting bound.

Corollary 5.1. *A pro-p-group G satisfies $\widehat{\mathrm{d}}(G) \leq \mathrm{d}(G)$. In particular, if $\widehat{\mathrm{d}}(G) = \mathrm{d}(G)$, then $\widehat{Z}(G) = Z(G)$.*

For instance, $\widehat{\mathrm{d}}(G) = \mathrm{d}(G)$, whenever $H_2(G, \mathbb{Z}_p)$ is trivial. This happens for the infinite pro-2-group (with $r \geq 1$ arbitrary)

$$D = \langle a, t \mid a^{2^r} = 1, a^{-1}ta = t^{-1} \rangle = \mathbb{Z}_2 \rtimes C_{2^r},$$

described also in [5, §1]. Its exterior degree is computed below.

Example 5.2. It is clear that $Z(D) = \widehat{Z}(D) = 1$. For $i = 0$, $\mu(\widehat{C}_D(t^i)) = 1$; for all $i \neq 0$, $\mu(\widehat{C}_D(t^i)) = 1/2^r$; for all i and $1 \leq j \leq 2^r - 1$, $\mu(\widehat{C}_D(a^j t^i)) = 0$. By Lemma 3.1 we find that

$$\begin{aligned}\widehat{\mathrm{d}}(D) &= \mu(\widehat{Z}(D)) + \int_{\langle t \rangle - \widehat{Z}(D)} \mu(\widehat{C}_D(x)) d\mu(x) + \int_{D - \langle t \rangle} \mu(\widehat{C}_D(x)) d\mu(x) \\ &= \frac{1}{2^r} \mu(\langle t \rangle - \{1\}) = \frac{1}{2^r} \mu(\langle t \rangle) = \frac{1}{4^r}.\end{aligned}$$

The importance of the condition $\widehat{\mathrm{d}}(G) = \mathrm{d}(G) = \mathrm{d}^{\wedge}(G)$ is illustrated in [13] in finite case. The following result is an application of Theorem 1.1 and, at the same time, is a generalization of the corresponding results of [13] to the infinite case.

Corollary 5.3. *Assume that G is a pro-p-group.*

(i) *If G is abelian of $\mathrm{rk}(G) > 1$, then $\widehat{\mathrm{d}}(G) \leq \frac{p^2+p-1}{p^3}$ and the equality holds if and only if $G/\widehat{Z}(G)$ is p-elementary abelian of rank two.*

(ii) *If G is nonabelian and $\widehat{Z}(G) \neq Z(G)$, then $\widehat{\mathrm{d}}(G) \leq \frac{p^3+p-1}{p^4}$.*

Proof. (i) Abelian pro-p-groups of $\mathrm{rk}(G) = 1$ are procyclic pro-p-groups and they are either isomorphic to a cyclic p-group or to $\mathbb{Z}_p$. Both of them have $\widehat{Z}(G) = Z(G) = G$ by Proposition 3.3. Let $\widehat{Z}(G) \neq Z(G) = G$ and $\mathrm{d}(G) = 1$. For all $x \notin \widehat{Z}(G)$, arguing as in Theorem 1.1, we find that $|G : \widehat{C}_G(x)| \geq p$, $|\widehat{C}_G(x) : \widehat{Z}(G)| \geq p$ and $\mu(\widehat{Z}(G)) \leq 1/p^2$. From Theorem 1.1, we have

$$\widehat{\mathrm{d}}(G) \leq 1 - \left(\frac{p-1}{p}\right)\left(1 - \frac{1}{p^2}\right) = \frac{p^2+p-1}{p^3}.$$

Now assume $G/\widehat{Z}(G)$ is p-elementary abelian of rank two. From Lemma 3.1,

$$\begin{aligned}\widehat{\mathrm{d}}(G) &= \int_G \mu(\widehat{C}_G(x))d\mu(x) = \mu(\widehat{Z}(G)) + \int_{G-\widehat{Z}(G)} \mu(\widehat{C}_G(x))d\mu(x)\\ &= \frac{1}{p} + \mu(\widehat{Z}(G))\Big(1 - \frac{1}{p}\Big) = \frac{p^2+p-1}{p^3}.\end{aligned}$$

Conversely,

$$\frac{p^2+p-1}{p^3} = \widehat{\mathrm{d}}(G) = \int_G \mu(\widehat{C}_G(x))d\mu(x) \leq \frac{1}{p} + \mu(\widehat{Z}(G))\Big(1 - \frac{1}{p}\Big)$$

implies $\mu(\widehat{Z}(G)) \geq 1/p^2$ and then $\mu(\widehat{Z}(G)) = 1/p^2$. Hence $G/\widehat{Z}(G)$ is p-elementary abelian of rank two.

(ii) Assume that G is nonabelian and $\widehat{Z}(G) \neq Z(G)$. Then we may argue as (i) above, getting $\mathrm{d}(G) \leq (p^2+p-1)/p^3$, $\mu(Z(G)) \leq 1/p^2$ and $\mu(\widehat{Z}(G)) \leq 1/p^3$. Again an application of Theorem 1.1 allows us to conclude the proof. □

Acknowledgments. The second author was supported in part by Grant CSUR 93652 and in part by Grant CPRR 14071175245 of NRF (South Africa) for this research. We also thank the referees and the editors for their excellent job.

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