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COFINITENESS OF GENERAL LOCAL COHOMOLOGY MODULES FOR SMALL DIMENSIONS

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Abstract. Let R be a commutative Noetherian ring, Φ a system of ideals of R and $I \in \Phi$. In this paper among other things we prove that if M is finitely generated and $t \in \mathbb{N}$ such that the R-module $\mathrm{H}^i_\Phi(M)$ is $\mathrm{FD}_{\leq 1}$ (or weakly Laskerian) for all $i < t$, then $\mathrm{H}^i_\Phi(M)$ is Φ-cofinite for all $i < t$ and for any $\mathrm{FD}_{\leq 0}$ (or minimax) submodule N of $\mathrm{H}^t_\Phi(M)$, the R-modules $\mathrm{Hom}_R(R/I, \mathrm{H}^t_\Phi(M)/N)$ and $\mathrm{Ext}^1_R(R/I, \mathrm{H}^t_\Phi(M)/N)$ are finitely generated. Also it is shown that if $\mathrm{cd}\, I = 1$ or $\dim M/IM \leq 1$ (e.g., $\dim R/I \leq 1$) for all $I \in \Phi$, then the local cohomology module $\mathrm{H}^i_\Phi(M)$ is Φ-cofinite for all $i \geq 0$. These generalize the main results of Aghapournahr and Bahmanpour [2], Bahmanpour and Naghipour [6, 7]. Also we study cominimaxness and weakly cofiniteness of local cohomology modules with respect to a system of ideals.

1. Introduction

Throughout this paper R is a commutative Noetherian ring with non-zero identity. For an R-module M, the i^{th} local cohomology module M with respect to an ideal I is defined as

$$\mathrm{H}^i_I(M) \cong \varinjlim_n \mathrm{Ext}^i_R(R/I^n, M).$$

Grothendieck in [21] made the following conjecture:

Conjecture 1.1. *Let M be a finite R–module and I be an ideal of R. Then $\mathrm{Hom}_R(R/I, \mathrm{H}^i_I(M))$ is finite for all $i \geq 0$.*

Hartshorne provides a counterexample to this conjecture in [22]. So it is false in general but there are some effort to show that under some conditions, for some number t, the module $\mathrm{Hom}_R(R/I, H^t_I(M))$ is finite, see [4, Theorem 3.3], [19, Theorem 6.3.9], [18, Theorem 2.1], [7, Theorem 2.6], [6, Theorem 2.3] and [2, Theorem 3.4]. He defined a module M to be *I–cofinite* if $\mathrm{Supp}_R(M) \subseteq \mathrm{V}(I)$

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and $\mathrm{Ext}^i_R(R/I, M)$ is finitely generated for all $i \geq 0$. He also asked the following question:

Question 1.2. Let M be a finite R–module and I be an ideal of R. When are $\mathrm{H}^i_I(M)$ I–cofinite for all $i \geq 0$?

The answer is negative in general, see [17] for a counterexample, but it is true in the following cases:

(a) $\mathrm{cd}(I) = 1$ or $\dim R \leq 2$. See [31].
(b) $\dim R/I \leq 1$. See [7].

Note that $\mathrm{cd}(I)$ (the cohomological dimension of I in R) is the smallest integer n such that the local cohomology modules $\mathrm{H}^i_I(S) = 0$ for all R–modules S and for all $i > n$.

There are some generalizations of the theory of ordinary local cohomology modules. The following is given by Bijan-Zadeh in [12].

Let Φ be a non-empty set of ideals of R. We call Φ a *system of ideals* of R if, whenever $I_1, I_2 \in \Phi$, then there is an ideal $J \in \Phi$ such that $J \subseteq I_1 I_2$. For such a system, for every R-module M, one can define

$$\Gamma_\Phi(M) = \{\ x \in M \mid Ix = 0 \ \text{ for some } I \in \Phi\}.$$

Then $\Gamma_\Phi(-)$ is a functor from $\mathscr{C}(R)$ to itself (where $\mathscr{C}(R)$ denotes the category of all R-modules and all R-homomorphisms). The functor $\Gamma_\Phi(-)$ is additive, covariant, R-linear and left exact. In [11], $\Gamma_\Phi(-)$ is denoted by $L_\Phi(-)$ and is called the "general local cohomology functor with respect to Φ". For each $i \geq 0$, the i-th right derived functor of $\Gamma_\Phi(-)$ is denoted by $\mathrm{H}^i_\Phi(-)$. The functor $\mathrm{H}^i_\Phi(-)$ and $\varinjlim_{I \in \Phi} \mathrm{H}^i_I(-)$ (from $\mathscr{C}(R)$ to itself) are naturally equivalent (see [12]). For an ideal I of R, if $\Phi = \{I^n | n \in \mathbb{N}_0\}$, then the functor $\mathrm{H}^i_\Phi(-)$ coincides with the ordinary local cohomology functor $\mathrm{H}^i_I(-)$. It is shown that, the study of torsion theory over R is equivalent to study the general local cohomology theory (see [11]).

In [4, Definition 1.2] the authors introduced the concept of Φ-*cofiniteness* of general local cohomology modules. The general local cohomology module $\mathrm{H}^j_\Phi(M)$ is defined to be Φ-cofinite if there exists an ideal $I \in \Phi$ such that $\mathrm{Ext}^i_R(R/I, \mathrm{H}^j_\Phi(M))$ is finitely generated, for all $i, j \geq 0$. As an special case of [32, Definition 2.1] and generalization of FSF modules (see [23, Definition 2.1]), in [2, Definition 2.1] the authors of this paper introduced the class of $\mathrm{FD}_{\leq n}$ modules. A module M is said to be $\mathrm{FD}_{\leq n}$ module, if there exist a finitely generated submodule N of M such that $\dim M/N \leq n$. For more details about properties of this class see [2, Lemma 2.3]. Recall that an R-module M is called *weakly Laskerian* if $\mathrm{Ass}_R(M/N)$ is a finite set for each submodule N of M. The class of weakly Laskerian modules is introduced in [20]. Bahmanpour in [5, Theorem 3.3] proved that over Noetherian rings, an R-module M is weakly Laskerian if and only if M is FSF module. The purpose of this note is to make a suitable generalization of Conjecture 1.1, Question 1.2 and the cases (a) and (b) in the context of general local cohomology modules. In this direction in Section 2, we generalize [2, Theorem 3.4 and Corollaries 3.5 and 3.6]. Note that the class of weakly Laskerian modules is contained in the class of $\mathrm{FD}_{\leq 1}$ modules. More precisely, we shall show that:

Theorem 1.3. *Let R be a Noetherian ring and I an ideal of R. Let M be a finitely generated R-module and $t \geq 1$ be a positive integer such that the R-modules $\mathrm{H}^i_\Phi(M)$ are $\mathrm{FD}_{\leq 1}$ (or weakly Laskerian) R-modules for all $i < t$. Then the following conditions hold*:

(i) *The R-modules $\mathrm{H}^i_\Phi(M)$ are Φ-cofinite for all $i < t$.*

(ii) *For all $\mathrm{FD}_{\leq 0}$ (or minimax) submodule N of $\mathrm{H}^t_\Phi(M)$, the R-modules*

$$\mathrm{Hom}_R(R/I, \mathrm{H}^t_\Phi(M)/N) \text{ and } \mathrm{Ext}^1_R(R/I, \mathrm{H}^t_\Phi(M)/N)$$

are finitely generated.

As consequences we generalize and answer the cases (a) and (b) in the following corollaries:

Corollary 1.4. *Let R be a Noetherian ring and $I \in \Phi$ an ideal of R . Let M be a finitely generated R-module such that $\dim M/IM \leq 1$ (e.g., $\dim R/I \leq 1$) for all $I \in \Phi$. Then*

(i) *the R-modules $\mathrm{H}^i_\Phi(M)$ are Φ-cofinite for all i;*

(ii) *for any $i \geq 0$ and for any $\mathrm{FD}_{\leq 0}$ (or minimax) submodule N of $\mathrm{H}^i_\Phi(M)$, the R-module $\mathrm{H}^i_\Phi(M)/N$ is Φ-cofinite.*

Corollary 1.5. *Let M be a finitely generated R-module and suppose one of the following cases holds*:

(a) $\mathrm{cd}(\Phi) = 1$.

(b) $\dim R/I \leq 1$ *for all $I \in \Phi$.*

Then $\mathrm{H}^i_\Phi(M)$ is Φ-cofinite for all $I \in \Phi$ and $i \geq 0$.

Recall that a module M is a *minimax* module if there is a finitely generated submodule N of M such that the quotient module M/N is Artinian. Minimax modules have been studied Zöschinger in [33]. In Section 3 we generalize R. Belshof, S. P. Slattery and C. Wickham results in [9, 10] about Matlis reflexive modules as a special case of minimax modules.

In Section 4, we generalize [7, Corollaries 3.2 and 3.3] by studding weakly cofiniteness of general local cohomology modules.

Throughout this paper, R will always be a commutative Noetherian ring with non-zero identity. We denote $\{\mathfrak{p} \in \operatorname{Spec} R : \mathfrak{p} \supseteq I\}$ by $V(I)$. For any unexplained notation and terminology we refer the reader to [15], [16] and [28].

2. Cofiniteness

As a generalization of definition of cofinite modules with respect to an ideal ([31, Definition 2.2]), it is introduced in [1, Definition 2.2] the following definition.

Definition 2.1. An R-module M (not necessary I-torsion) is called ETH-cofinite with respect to an ideal I of R or I-ETH-cofinite if $\operatorname{Ext}^i_R(R/I, M)$ is a finite R-module for all i.

Remark 2.2. Let I be an ideal of R.

(i) All finite R-modules and all cofinite R-modules with respect to ideal I are I-ETH-cofinite.

(ii) Suppose M is an I-torsion module, then M is I-ETH-cofinite if and only if it is I-cofinite module.

We claim that the class of ETH-cofinite modules with respect to an ideal is strictly larger than the class of cofinite modules with respect to the same ideal. To do this, see the following example.

Example 2.3. Let $(R, \mathfrak{m})$ be a Noetherian local ring of dimension $d > 0$. Let $M = R \oplus E(R/\mathfrak{m})$. It is easy to see that M is an $\mathfrak{m}$-ETH-cofinite R-module that is neither finitely generated nor $\mathfrak{m}$-cofinite.

The following lemma is well-known for I-cofinite modules.

Lemma 2.4. *If* $0 \longrightarrow N \longrightarrow L \longrightarrow T \longrightarrow 0$ *is exact and two of the modules in the sequence are* I-ETH*-cofinite, then so is the third one.*

Lemma 2.5. *Let* I *be an ideal of a Noetherian ring* R *and* M *be an* $\mathrm{FD}_{\leq 0}$ (*or minimax*) R*-module. Then the following statements are equivalent*:

(i) M *is* I-ETH*-cofinite.*

(ii) *The* R*-module* $\operatorname{Hom}_R(R/I, M)$ *is finitely generated.*

Proof. See [29, Theorem 5.3] and [31, Theorem 2.1]. □

The following lemma that generalizes [2, 8, Proposition 2.6] is needed to prove the next theorem.

Lemma 2.6. *Let* R *be a Noetherian ring and* I *be an ideal of* R (*not necessary dimension one*)*. Let* M *be a non-zero* R*-module* (*not necessary* I*-torsion*) *such that* $\dim M \leq 1$*. Then the following conditions are equivalent*:

(i) *The* R*-modules* $\operatorname{Ext}^i_R(R/I, M)$ *are finitely generated for all* i;

(ii) $\mathrm{H}^i_I(M)$ *are* I*-cofinite for all* i;

(iii) *The* R*-modules* $\operatorname{Hom}_R(R/I, M)$ *and* $\operatorname{Ext}^1_R(R/I, M)$ *are finitely generated.*

Proof. (i)$\Longrightarrow$(iii) It is clear.

(iii)$\Longrightarrow$(ii) By the exact sequence

$$0 \longrightarrow \Gamma_I(M) \longrightarrow M \longrightarrow M/\Gamma_I(M) \longrightarrow 0,$$

we get the exact sequence

$$0 \longrightarrow \operatorname{Hom}_R(R/I, \Gamma_I(M)) \longrightarrow \operatorname{Hom}_R(R/I, M) \longrightarrow \operatorname{Hom}_R(R/I, M/\Gamma_I(M))$$
$$\longrightarrow \operatorname{Ext}^1_R(R/I, \Gamma_I(M)) \longrightarrow \operatorname{Ext}^1_R(R/I, M).$$

Since $\operatorname{Hom}_R(R/I, M/\Gamma_I(M)) = 0$, it follows that the R-modules

$$\operatorname{Hom}_R(R/I, \Gamma_I(M)) \text{ and } \operatorname{Ext}^1_R(R/I, \Gamma_I(M))$$

are finitely generated, and so in view of [8, Proposition 2.6] the R-module $\Gamma_I(M)$ is I-cofinite. Now as the R-module $\operatorname{Ext}^1_R(R/I, M)$ is finitely generated, it follows from [3, Corollary 4.4] that the R-module $\operatorname{Hom}_R(R/I, \mathrm{H}^1_I(M))$ is finitely generated. If $\mathfrak{p} \in \operatorname{Supp}_R(\mathrm{H}^1_I(M)) \subseteq \operatorname{Supp}_R(M)$, then

$$\mathrm{H}^1_{IR_\mathfrak{p}}(M_\mathfrak{p}) \cong \mathrm{H}^1_I(M)_\mathfrak{p} \neq 0.$$

Since $\dim M \leq 1$, it is easy to see that $\dim R/\mathfrak{p} = 0$ or $\dim R/\mathfrak{p} = 1$. If $\dim R/\mathfrak{p} = 1$, then $M_\mathfrak{p}$ is a zero dimensional $R_\mathfrak{p}$-module that implies $\mathrm{H}^1_{IR_\mathfrak{p}}(M_\mathfrak{p}) = 0$ by using Grothendieck vanishing theorem [15, Theorem 6.1.2] which is a contradiction. Thus $\dim R/\mathfrak{p} = 0$ and so $\mathfrak{p}$ is a maximal ideal. So we have the following inclusion

$$\operatorname{Supp}_R(\operatorname{Hom}_R(R/I, \mathrm{H}^1_I(M))) \subseteq \operatorname{Supp}_R(\mathrm{H}^1_I(M)) \subseteq \operatorname{Max} R.$$

It yields that the R-module $\operatorname{Hom}_R(R/I, \mathrm{H}^1_I(M))$ has finite length, and so by [31, Proposition 4.1] the R-module $\mathrm{H}^1_I(M)$ is Artinian and I-cofinite. Thus in view of Grothendieck vanishing theorem [15, Theorem 6.1.2] the R-module $\mathrm{H}^i_I(M)$ is I-cofinite for all $i \geq 0$.

(ii)$\Longrightarrow$(i) By [31, Corollary 3.10], it follows that $\operatorname{Ext}^i_R(R/I, M)$ are finite for all $i \geq 0$, as required. □

The following theorem is a generalization of [2, Theorem 3.1] that in what follows the next theorem plays an important role.

Theorem 2.7. *Let R be a Noetherian ring and I be an ideal of R. Let M be an $\mathrm{FD}_{\leq 1}$ R-module. Then M is I-ETH-cofinite if and only if $\operatorname{Hom}_R(R/I, M)$ and $\operatorname{Ext}^1_R(R/I, M)$ are finitely generated.*

Proof. I-ETH-cofiniteness of M clearly implies that $\operatorname{Hom}_R(R/I, M)$ and $\operatorname{Ext}^1_R(R/I, M)$ are finitely generated. For another direction, note that by definition there is a finitely generated submodule N of M such that $\dim(M/N) \leq 1$. Also, the exact sequence

$$(*) \qquad 0 \to N \to M \to M/N \to 0$$

induces the following exact sequence

$$0 \longrightarrow \operatorname{Hom}_R(R/I, N) \longrightarrow \operatorname{Hom}_R(R/I, M) \longrightarrow \operatorname{Hom}_R(R/I, M/N)$$
$$\longrightarrow \operatorname{Ext}^1_R(R/I, N) \longrightarrow \operatorname{Ext}^1_R(R/I, M) \longrightarrow \operatorname{Ext}^1_R(R/I, M/N) \longrightarrow \operatorname{Ext}^2_R(R/I, N).$$

Whence, it follows that the R-modules $\operatorname{Hom}_R(R/I, M/N)$ and $\operatorname{Ext}^1_R(R/I, M/N)$ are finitely generated. Therefore, in view of Proposition 2.6, the R-module M/N is I-ETH-cofinite. Now it follows from the exact sequence $(*)$ and Lemma 2.4 that M is I-ETH-cofinite. □

The following lemma is needed in the proof of first main result of this paper.

Lemma 2.8. *Let I be an ideal of a Noetherian ring R, M a non-zero R-module and $t \in \mathbb{N}_0$. Suppose that the R-module $\mathrm{H}^i_\Phi(M)$ is Φ-cofinite for all $i = 0, \dots, t-1$, and the R-modules $\mathrm{Ext}^t_R(R/I, M)$ and $\mathrm{Ext}^{t+1}_R(R/I, M)$ are finitely generated. Then the R-modules $\mathrm{Hom}_R(R/I, \mathrm{H}^t_\Phi(M))$ and $\mathrm{Ext}^1_R(R/I, \mathrm{H}^t_\Phi(M))$ are finitely generated.*

Proof. See [25, Theorem 2.2]. □

We are now ready to state and prove the following main results (Theorem 2.9 and the Corollaries 2.10, 2.11, 2.15) which are extension of [2, Theorem 3.4], Bahmanpour-Naghipour's results in [7, 6], Brodmann-Lashgari's result in [14], Khashyarmanesh-Salarian's result in [24], Asadollahi et al. results in [4], Hong Quy's result in [23] and Melkersson results in [30, 31].

Theorem 2.9. *Let R be a Noetherian ring and $I \in \Phi$ an ideal of R. Let M be a finitely generated R-module and $t \geq 1$ be a positive integer such that the R-modules $\mathrm{H}^i_\Phi(M)$ are $\mathrm{FD}_{\leq 1}$ (or weakly Laskerian) R-modules for all $i < t$. Then the following conditions hold:*

(i) *The R-modules $\mathrm{H}^i_\Phi(M)$ are I-ETH-cofinite (in particular Φ-cofinite) for all $i < t$.*

(ii) *For all $\mathrm{FD}_{\leq 0}$ (or minimax) submodule N of $\mathrm{H}^t_\Phi(M)$, the R-modules*

$$\mathrm{Hom}_R(R/I, \mathrm{H}^t_\Phi(M)/N) \text{ and } \mathrm{Ext}^1_R(R/I, \mathrm{H}^t_\Phi(M)/N)$$

are finitely generated. In particular the sets

$$\mathrm{Ass}_R(\mathrm{Hom}_R(R/I, \mathrm{H}^t_\Phi(M)/N)) \text{ and } \mathrm{Ass}_R(\mathrm{Ext}^1_R(R/I, \mathrm{H}^t_\Phi(M)/N))$$

are finite sets.

Proof. (i) We proceed by induction on t. By Lemma 2.8 the case $t = 1$ is obvious since $\mathrm{H}^0_\Phi(M)$ is finitely generated. So, let $t > 1$ and the result has been proved for smaller values of t. By the inductive assumption, $\mathrm{H}^i_\Phi(M)$ is I-ETH-cofinite for $i = 0, 1, \dots, t-2$. Hence by Lemma 2.8 and assumption, $\mathrm{Hom}_R(R/I, \mathrm{H}^{t-1}_\Phi(M))$ and $\mathrm{Ext}^1_R(R/I, \mathrm{H}^{t-1}_\Phi(M))$ are finitely generated. Therefore by Theorem 2.7, $\mathrm{H}^i_\Phi(M)$ is I-ETH-cofinite and therefore Φ-cofinite for all $i < t$. This completes the inductive step.

(ii) In view of (i) and Lemma 2.8, $\mathrm{Hom}_R(R/I, \mathrm{H}^t_\Phi(M))$ and $\mathrm{Ext}^1_R(R/I, \mathrm{H}^t_\Phi(M))$ are finitely generated. On the other hand, according to Lemma 2.5, N is I-ETH-cofinite. Now, the exact sequence

$$0 \longrightarrow N \longrightarrow \mathrm{H}^t_\Phi(M) \longrightarrow \mathrm{H}^t_\Phi(M)/N \longrightarrow 0$$

induces the following exact sequence,

$$\mathrm{Hom}_R(R/I, \mathrm{H}^t_\Phi(M)) \longrightarrow \mathrm{Hom}_R(R/I, \mathrm{H}^t_\Phi(M)/N) \longrightarrow \mathrm{Ext}^1_R(R/I, N)$$
$$\longrightarrow \mathrm{Ext}^1_R(R/I, \mathrm{H}^t_\Phi(M)) \longrightarrow \mathrm{Ext}^1_R(R/I, \mathrm{H}^t_\Phi(M)/N) \longrightarrow \mathrm{Ext}^2_R(R/I, N).$$

Consequently

$$\mathrm{Hom}_R(R/I, \mathrm{H}^t_\Phi(M)/N) \text{ and } \mathrm{Ext}^1_R(R/I, \mathrm{H}^t_\Phi(M)/N)$$

are finitely generated, as required. $\square$

The following corollaries answer to Hartshorne's question.

Corollary 2.10. *Let R be a Noetherian ring and $I \in \Phi$ an ideal of R. Let M be a finitely generated R-module such that the R-modules $\mathrm{H}^i_\Phi(M)$ are $\mathrm{FD}_{\leq 1}$ (or weakly Laskerian) R-modules for all i. Then*

(i) *the R-modules $\mathrm{H}^i_\Phi(M)$ are I-ETH-cofinite (in particular, Φ-cofinite) for all i.*

(ii) *for any $i \geq 0$ and for any $\mathrm{FD}_{\leq 0}$ (or minimax) submodule N of $\mathrm{H}^i_\Phi(M)$, the R-module $\mathrm{H}^i_\Phi(M)/N$ is I-ETH-cofinite (in particular, Φ-cofinite).*

Proof. (i) Clear.

(ii) In view of (i) the R-module $\mathrm{H}^i_\Phi(M)$ is I-ETH-cofinite for all i. Hence the R-module $\mathrm{Hom}_R(R/I, N)$ is finitely generated, and so it follows from Lemma 2.5 that N is I-ETH-cofinite. Now, the exact sequence

$$0 \longrightarrow N \longrightarrow \mathrm{H}^i_\Phi(M) \longrightarrow \mathrm{H}^i_\Phi(M)/N \longrightarrow 0,$$

and Lemma 2.4 implies that the R-module $\mathrm{H}^i_\Phi(M)/N$ is I-ETH-cofinite. $\square$

The following corollary is a generalization of [7, Corollary 2.7].

Corollary 2.11. *Let R be a Noetherian ring and $I \in \Phi$ an ideal of R. Let M be a finitely generated R-module such that $\dim M/IM \leq 1$ (e.g., $\dim R/I \leq 1$) for all $I \in \Phi$. Then*

(i) *the R-modules $\mathrm{H}^i_\Phi(M)$ are I-ETH-cofinite (in particular, Φ-cofinite) for all i.*

(ii) *for any $i \geq 0$ and for any $\mathrm{FD}_{\leq 0}$ (or minimax) submodule N of $\mathrm{H}^i_\Phi(M)$, the R-module $\mathrm{H}^i_\Phi(M)/N$ is I-ETH-cofinite (in particular, Φ-cofinite).*

Proof. (i) Since by [12, Lemma 2.1],

$$\mathrm{H}^i_\Phi(M) \cong \varinjlim_{I \in \Phi} \mathrm{H}^i_I(M),$$

it is easy to see that $\mathrm{Supp}_R(\mathrm{H}^i_\Phi(M)) \subseteq \bigcup_{I\in\Phi} \mathrm{Supp}_R(\mathrm{H}^i_I(M))$ and therefore

$$\mathrm{dimSupp}\,\mathrm{H}^i_\Phi(M) \leq \sup\{\mathrm{dimSupp}\,\mathrm{H}^i_I(M) | I \in \Phi\} \leq 1,$$

thus $\mathrm{H}^i_\Phi(M)$ is $\mathrm{FD}_{\leq 1}$ R-module and the assertion follows by Corollary 2.10(i).

(ii) Proof is the same as Corollary 2.10(ii). $\square$

Lemma 2.12. *Let R be a Noetherian ring and $I \in \Phi$ an ideal of R. Let M be an I-ETH-cofinite R-module. If s is a number, such that $\mathrm{H}^i_\Phi(M)$ is I-ETH-cofinite for all $i \neq s$, then this is the case also when $i = s$.*

Proof. See [25, Proposition 2.7]. $\square$

Before bringing the following main result, recall that, for any proper ideal I of R, the *cohomological dimension of M with respect to I*, is defined as

$$\operatorname{cd}(I, M) = \sup\{i \in \mathbb{N}_0 \mid \mathrm{H}^i_I(M) \neq 0\},$$

if this supremum exists, otherwise, it is defined $-\infty$.

The above definition motivates the following definition:

Definition 2.13. For any system of ideals Φ of R and any R-module M, we denote the cohomological dimension of M with respect to Φ by $\operatorname{cd}(\Phi, M)$ and define as

$$\operatorname{cd}(\Phi, M) = \sup\{i \in \mathbb{N}_0 \mid \mathrm{H}^i_\Phi(M) \neq 0\},$$

if this supremum exists, otherwise, we define it $-\infty$.

Remark 2.14. Since $\mathrm{H}^i_\Phi(M) \cong \varinjlim_{I\in\Phi} \mathrm{H}^i_I(M)$, it is easy to see that $\operatorname{cd}(\Phi, M) \leq \sup\{\operatorname{cd}(I, M) \mid I \in \Phi\}$. We denote $\operatorname{cd}(\Phi, R)$ by $\operatorname{cd}\Phi$, therefore $\operatorname{cd}\Phi \leq \sup\{\operatorname{cd} I \mid I \in \Phi\}$.

Corollary 2.15. *Let M be a finitely generated R-module and suppose one of the following cases holds*:

(a) $\operatorname{cd} I \leq 1$ *for all* $I \in \Phi$;
(b) $\dim R/I \leq 1$ *for all* $I \in \Phi$;
(c) $(R, \mathfrak{m})$ *is local and* $\dim R \leq 2$.

Then $\mathrm{H}^i_\Phi(M)$ is I-ETH-cofinite (*in particular, Φ-cofinite*) *for all $I \in \Phi$ and $i \geq 0$.*

Proof. It follows by Corollary 2.11, Definition 2.13, Lemma 2.12 and [25, Corollary 2.8]. □

3. Cominimaxness

The definition of Φ-cofiniteness of $\mathrm{H}^i_\Phi(M)$ ([4, Definition 1.2]) motivates the following definition.

Definition 3.1. The general local cohomology module $\mathrm{H}^j_\Phi(M)$ is called Φ-cominimax if there exists an ideal $I \in \Phi$ such that the R-module $\operatorname{Ext}^i_R(R/I, \mathrm{H}^j_\Phi(M))$ is a minimax R-module, for all $i, j \geq 0$.

Lemma 3.2. *Suppose that for any finitely generated R-module N and for each $i \geq 0$, the local cohomology modules $\mathrm{H}^i_\Phi(N)$ are Φ-cofinite. Then for any minimax R-module M and for each $i \geq 0$, the local cohomology modules $\mathrm{H}^i_\Phi(M)$ are Φ-cominimax. In fact, for each $i \geq 1$, the local cohomology modules $\mathrm{H}^i_\Phi(M)$ are Φ-cofinite.*

Proof. Since M is minimax, there exists a short exact sequence

$$0 \to N \to M \to A \to 0,$$

where N is a finite module and A is an Artinian module. This induces the exact sequence

$$0 \to \Gamma_\Phi(N) \to \Gamma_\Phi(M) \to \Gamma_\Phi(A) \to \mathrm{H}^1_\Phi(N) \to \mathrm{H}^1_\Phi(M) \to 0,$$

and $\mathrm{H}^i_\Phi(N) \cong \mathrm{H}^i_\Phi(M)$ for all $i \geq 2$. Hence, for all $i \geq 2$, $\mathrm{H}^i_\Phi(M)$ is Φ-cofinite. Let K be the kernel of the map $\mathrm{H}^1_\Phi(N) \to \mathrm{H}^1_\Phi(M)$. Since A is Artinian and $(0 :_{\mathrm{H}^1_\Phi(N)} I)$ is finitely generated so K is Artinian and $(0 :_K I)$ has finite length. Thus, by Lemma 2.5 K is I-ETH-cofinite. From the exact sequence

$$0 \to K \to \mathrm{H}^1_\Phi(N) \to \mathrm{H}^1_\Phi(M) \to 0,$$

since K and $\mathrm{H}^1_\Phi(N)$ are I-ETH-cofinite so by Lemma 2.4, $\mathrm{H}^1_\Phi(M)$ is I-ETH-cofinite and therefore Φ-cofinite. Hence $\mathrm{H}^i_\Phi(M)$ is Φ-cofinite for all $i \geq 1$. □

Lemma 3.3. *Let M be a minimax R-module. Then, for any finite R-module N, $\mathrm{Ext}^i_R(N, \Gamma_\Phi(M))$ and $\mathrm{Tor}^R_i(N, \Gamma_\Phi(M))$ are minimax for all i. In particular, $\mathrm{Ext}^i_R(R/I, \Gamma_\Phi(M))$ and $\mathrm{Tor}^R_i(R/I, \Gamma_\Phi(M))$ are minimax for all i*

Proof. Since N is finitely generated it follows that N has a free resolution of finitely generated free modules. Note that by [31, Corollary 4.4] the category of minimax modules is a Serre subcategory of the category of R-modules i.e., it is closed under taking submodules, quotients and extensions. Now the assertion follows computing the modules $\mathrm{Ext}^i_R(N, \Gamma_\Phi(M))$ and $\mathrm{Tor}^R_i(N, \Gamma_\Phi(M))$, by this free resolution. □

The following theorem is a generalization of R. Belshof, S. P. Slattery and C. Wickham results in [9, 10] about Matlis reflexive modules.

Theorem 3.4. *Let M be a minimax R-module and suppose one of the following cases holds*:

(a) $\operatorname{cd} I \leq 1$ *for all* $I \in \Phi$;
(b) $\dim R/I \leq 1$ *for all* $I \in \Phi$;
(c) $(R, \mathfrak{m})$ *is local and* $\dim R \leq 2$.

Then $\mathrm{H}^i_\Phi(M)$ is Φ-cominimax. In fact, $\mathrm{H}^i_\Phi(M)$ is Φ-cofinite for all $i \geq 1$. In particular, for any $I \in \Phi$ and any finitely generated R-module N with $\mathrm{Supp}_R(N) \subseteq \mathrm{V}(I)$, $\mathrm{Ext}^j_R(N, \mathrm{H}^i_\Phi(M))$ and $\mathrm{Tor}^R_j(N, \mathrm{H}^i_\Phi(M))$ are minimax for all i and j. In fact $\mathrm{Ext}^j_R(N, \mathrm{H}^i_\Phi(M))$ and $\mathrm{Tor}^R_j(N, \mathrm{H}^i_\Phi(M))$ are finitely generated for all j when $i \geq 1$.

Proof. It follows by using Lemma 3.2, Corollary 2.15, [31, Theorem 2.1] and Lemma 3.3. □

4. Weakly cofiniteness

Concerning the category of weakly Laskerain modules, the definition of I-ETH-cofinite modules and Φ-cofiniteness of $\mathrm{H}^i_\Phi(M)$ ([4, Definition 1.2]) motivate the following definitions.

Definition 4.1. An R-module M (not necessary I-torsion) is called ETH-weakly cofinite with respect to an ideal I of R or I-ETH-weakly cofinite if $\mathrm{Ext}^i_R(R/I, M)$ is a weakly Laskerian R-module for all $i \geq 0$.

Definition 4.2. The general local cohomology module $\mathrm{H}^j_\Phi(M)$ is called Φ-weakly cofinite if there exists an ideal $I \in \Phi$ such that the R-module $\mathrm{Ext}^i_R(R/I, \mathrm{H}^j_\Phi(M))$ is a weakly Laskerin R-module, for all $i, j \geq 0$.

The following results are generalization of [7, Corollaries 3.2 and 3.3].

Theorem 4.3. *Let* $(R, \mathfrak{m})$ *be a Noetherian local ring and* $I \in \Phi$ *an ideal of* R *. Let* M *be a finitely generated* R*-module such that* $\dim M/IM \leq 2$ (*e.g.,* $\dim R/I \leq 2$) *for all* $I \in \Phi$*. Then the* R*–modules* $\mathrm{H}^i_\Phi(M)$ *are* I*-*ETH*-weakly cofinite* (*in particular,* Φ*-weakly cofinite*) *for all* $i \geq 0$.

Proof. Let $\Sigma = \{\mathrm{Ext}^j_R(R/I, \mathrm{H}^i_\Phi(M))) \,|\, i \geqslant 0, j \geqslant 0\}$. Let $T \in \Sigma$ and T' be a submodule of T. It is enough to show that $\mathrm{Ass}(T/T')$ is a finite set. By [13, Lemma 2.3(i)], [28, Exersize 7.7] and [26, Lemma 2.1(b)], without loss of generality, we may assume that R is complete. Suppose that $\mathrm{Ass}(T/T')$ is infinite. Hence, there exists a countably infinite subset $\{\mathfrak{p}_t\}_{t=1}^\infty$ of $\mathrm{Ass}(T/T')$, such that none of them is equal to $\mathfrak{m}$. Then by [27, Lemma 3.2], $\mathfrak{m} \not\subseteq \bigcup_{t=1}^\infty \mathfrak{p}_t$. Let S be the multiplicatively closed subset $R \setminus \bigcup_{t=1}^\infty \mathfrak{p}_t$. Then by Corollary 2.11 and [11, 2.6], it is easy to see that $S^{-1}T/S^{-1}T'$ is a finite $S^{-1}R$–module, and therefore $\mathrm{Ass}_{S^{-1}R}(S^{-1}T/S^{-1}T')$ is a finite set. But $S^{-1}\mathfrak{p}_t \in \mathrm{Ass}_{S^{-1}R}(S^{-1}T/S^{-1}T')$ for all $t = 1, 2, \ldots$ which is a contradiction. $\square$

Corollary 4.4. *Let* $(R, \mathfrak{m})$ *be a Noetherian local ring and* $I \in \Phi$ *an ideal of* R*. Let* M *be a finitely generated* R*-module such that* $\dim M/IM \leq 2$ (*e.g.,* $\dim R/I \leq 2$) *for all* $I \in \Phi$*. Then, for every weakly Laskerian submodule* N *of* $\mathrm{H}^i_\Phi(M)$*, the* R*–module* $\mathrm{H}^i_\Phi(M)/N$ *is* I*-*ETH*-weakly cofinite* (*in particular,* Φ*-weakly cofinite*) *for all* $i \geq 0$.

Proof. In view of Theorem 4.3, the R-module $\mathrm{H}^i_\Phi(M)$ is I-ETH-weakly cofinite for all $i \geq 0$. Now, the assertion follows from exact sequence

$$0 \longrightarrow N \longrightarrow \mathrm{H}^i_\Phi(M) \longrightarrow \mathrm{H}^i_\Phi(M)/N \longrightarrow 0,$$

and [20, Lemma 2.3]. $\square$

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