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A GENERALIZATION OF SYMMETRIC RING PROPERTY

Hong Kee Kim, Tai Keun Kwak, Seung Ick Lee, Yang Lee, Sung Ju Ryu, Hyo Jin Sung, and Sang Jo Yun

Abstract. This note focuses on a ring property in which upper and lower nilradicals coincide, as a generalizations of symmetric rings. The concept of symmetric ideal and ring in the noncommutative ring theory was initially introduced by Lambek, as an extension of the usual commutative ideal theory. The investigation of symmetric rings provided many useful results to the study in the noncommutative ring theory. So the results obtained from this study may be applicable to observing the structure of zero divisors in various kinds of algebraic systems containing matrix rings and polynomial rings.

1. Introduction

Throughout this note all rings are associative with identity unless otherwise stated. Lambek introduced the concept of *symmetric* in the noncommutative ring theory as an extension of the usual commutative ideal theory, unifying the sheaf representation of commutative rings and reduced rings, in [16]. Lambek called a right ideal I of a ring R *symmetric* if $rst \in I$ implies $rts \in I$ for all $r, s, t \in R$, and if the zero ideal of R is symmetric, then R is usually called *symmetric*, while Anderson-Camillo [2] used the term ZC_3 for this concept. It is proved by Lambek that a ring R is symmetric if and only if $r_1r_2 \cdots r_n = 0$ implies $r_{\sigma(1)}r_{\sigma(2)} \cdots r_{\sigma(n)} = 0$ for any permutation σ of the set $\{1, 2, \ldots, n\}$, where $n \geq 1$ and $r_i \in R$ for all i (see [16, Proposition 1]). Anderson-Camillo also obtained this result independently in [2, Theorem I.1]. A ring is usually called *reduced* if it has no nonzero nilpotent elements. Commutative rings are clearly symmetric. Reduced rings are symmetric by [2, Theorem I.3]. There exist many non-reduced commutative rings, and many noncommutative reduced rings. A ring R is called *IFP* [5] if $ab = 0$ implies $aRb = 0$ for $a, b \in R$. Shin [22] used the term *SI* for the IFP, while Narbonne [19] used *semicommutative* in place of the IFP. In this note, we choose the term "semicommutative", so as to cohere with references. A ring is usually called *abelian* if every idempotent is central.

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Symmetric rings are semicommutative, and semicommutative rings are abelian, but not conversely for each case.

For a ring R, $N_*(R)$, $N^*(R)$, and $N(R)$ denote the lower nilradical (i.e., the prime radical), the upper nilradical (i.e., the sum of all nil ideals), and the set of all nilpotent elements in R, respectively. It is well-known that $N_*(R) \subseteq N^*(R) \subseteq N(R)$.

Recall that a ring R is called *nil-semicommutative* [18, Definition 2.1] if $ab = 0$ implies $aRb = 0$ for $a, b \in N(R)$. Clearly semicommutative rings are nil-semicommutative. Nil-semicommutative rings need not be abelian by [18, Example 2.2]. Recently, Chakraborty and Das called a ring R *right* (resp., *left*) *nil-symmetric* [6, Definition 1] if $abc = 0$ (resp., $cab = 0$) implies $acb = 0$ for $a, b \in N(R)$ and $c \in R$, and the ring R is *nil-symmetric* if it is both right and left nil-symmetric. It is proved that every right (left) nil-symmetric ring is nil-semicommutative but not conversely by [6, Proposition 7 and Example 11], respectively.

For a ring R, $\text{Mat}_n(R)$ and $U_n(R)$ denote the n by n full matrix ring and the upper triangular matrix ring over R, respectively. We note that

$$N(U_n(R)) = \begin{pmatrix} N(R) & R & R & \cdots & R \\ 0 & N(R) & R & \cdots & R \\ 0 & 0 & N(R) & \cdots & R \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & N(R) \end{pmatrix}.$$

Let $D_n(R) = \{(a_{ij}) \in U_n(R) \mid a_{11} = \cdots = a_{nn}\}$ and use E_{ij} for the matrix with (i, j)-entry 1 and elsewhere 0. We denote $\mathbb{Z}_n$ by the ring of integers modulo n.

2. Definition

We consider the following condition which is a generalization of nil-symmetric rings for a given ring R:

$$(*) \qquad abc = 0 \text{ implies } acb = 0 \text{ for all } a, b, c \in N(R).$$

Every right nil-symmetric ring does clearly satisfy the condition $(*)$, but the converse does not hold by next example.

Example 2.1. Consider the ring $R = U_3(A)$ over a reduced ring A. Then R obviously satisfies the condition $(*)$, since

$$N(R) = \begin{pmatrix} 0 & R & R \\ 0 & 0 & R \\ 0 & 0 & 0 \end{pmatrix}.$$

But R is not right nil-symmetric by [6, Example 11].

Hence we have a new class of rings which is a generalization of (nil-) symmetric rings as follows.

Definition 2.2. A ring is called *weak right* (resp., *left*) *nil-symmetric* if it satisfies the condition $(*)$ (resp., the left version of the condition $(*)$), and the ring R is called *weak nil-symmetric* if it is both weak left and right nil-symmetric.

The weak nil-symmetric ring property is not left-right symmetric by the following.

Example 2.3. Let K be a field and $A = K\langle a, b\rangle$ be the free algebra with noncommuting indeterminates a, b over K.

(1) Let I be the ideal of A generated by

$$a^3, ab \text{ and } b^2.$$

Set $R = A/I$ and let a, b coincide with their images in R for simplicity. Then

$$N(R) = \{k_0a + k_1a^2 + k_2b + k_3ba + k_4ba^2 \mid k_0, k_1, k_2, k_3, k_4 \in K\}.$$

We next show that R is weak right nil-symmetric. Take $\alpha, \beta, \gamma \in N(R)$. Then

$$\alpha = h_0a + h_1a^2 + h_2b + h_3ba + h_4ba^2,$$
$$\beta = k_0a + k_1a^2 + k_2b + k_3ba + k_4ba^2,$$
$$\gamma = l_0a + l_1a^2 + l_2b + l_3ba + l_4ba^2$$

for some $h_i, k_i, l_i \in K$. Letting $\alpha\beta\gamma = 0$, we have

$$\begin{aligned}&(h_0a + h_1a^2 + h_2b + h_3ba + h_4ba^2)(k_0a + k_1a^2 + k_2b + k_3ba + k_4ba^2)\\&(l_0a + l_1a^2 + l_2b + l_3ba + l_4ba^2)\\&= (h_0k_0a^2 + h_2k_0ba + h_2k_1ba^2 + h_3k_0ba^2)(l_0a + l_1a^2 + l_2b + l_3ba + l_4ba^2)\\&= h_2k_0l_0ba^2 = 0.\end{aligned}$$

This implies $h_2k_0l_0 = 0$, entailing $\alpha\gamma\beta = h_2l_0k_0ba^2 = h_2k_0l_0ba^2 = 0$. Thus R is weak right nil-symmetric.

However, $a(ba) = 0$ but $(ba)a = ba^2 \neq 0$ for $a, b \in N(R)$. Thus R is not weak left nil-symmetric.

(2) Let J be the ideal of A generated by

$$a^3, ba \text{ and } b^2.$$

Let $R = A/J$. Then

$$N(R) = \{k_0a + k_1a^2 + k_2b + k_3ab + k_4a^2b \mid k_0, k_1, k_2, k_3, k_4 \in K\}.$$

Then R can be shown to weak left nil-symmetric, but not weak right nil-symmetric, through a similar computation to (1).

Due to Levitzki, an element a of a ring R is called *strongly nilpotent* if every sequence $a_0, a_1, a_2, \ldots$, such that $a_0 = a$ and $a_{n+1} \in a_nRa_n$ for all $n \geq 0$, is eventually zero. It is well-known that the prime radical is the set of all strongly nilpotent elements.

Theorem 2.4. *If R is a weak right (left) nil-symmetric ring, then $N_*(R) = N^*(R)$.*

Proof. Let R be a weak right nil-symmetric ring and $a \in N^*(R)$. We will show that a is strongly nilpotent in R, i.e., $a \in N_*(R)$. Let $a_0 = a$ and consider a sequence $a_0, a_1 = a_0r_0a_0, a_2 = a_1r_1a_1, \ldots, a_{n+1} = a_nr_na_n$, where r_i's are taken arbitrarily in R and $n \geq 0$.

Suppose $a^2 = 0$. Then $a_0^2 = 0$ implies $0 = (a_0a_0)(r_0a_0r_1a_0r_0)$. Note $r_0a_0r_1a_0r_0 \in N^*(R)$. Since R is weak right nil-symmetric, we have

$$0 = (a_0a_0)(r_0a_0r_1a_0r_0) = a_0(r_0a_0r_1a_0r_0)a_0 = a_1r_1a_1 = a_2.$$

Suppose $a^3 = 0$. Then $a_0^3 = 0$ implies $0 = (a_0a_0^2)(r_0a_0r_1a_0r_0)$. Note $r_0a_0r_1a_0r_0 \in N^*(R)$. Since R is weak right nil-symmetric, we have

$$0 = (a_0a_0^2)(r_0a_0r_1a_0r_0) = a_0(r_0a_0r_1a_0r_0)a_0a_0.$$

Next note that $r_2a_0r_0a_0r_1a_0r_0 \in N^*(R)$. Since R is weak right nil-symmetric, we have

$$\begin{aligned} 0 &= (a_0r_0a_0r_1a_0r_0a_0)(a_0)(r_2a_0r_0a_0r_1a_0r_0) \\ &= (a_0r_0a_0r_1a_0r_0a_0)(r_2a_0r_0a_0r_1a_0r_0)(a_0) \\ &= a_2r_2a_2 = a_3. \end{aligned}$$

We will extend this method to the general case. Let $a^n = 0$ for $n \geq 4$ and we will use freely the assumption that R is weak right nil-symmetric. Note that $a_i \in N^*(R)$ for all $i \geq 0$. Since $0 = a_0^{n-1}a_0(a_0r_0)$, we have

$$0 = a_0^{n-1}(a_0r_0)a_0 = a_0^{n-1}a_1.$$

Since $0 = a_0^{n-2}a_0a_1(r_0a_0r_1)$, we have

$$0 = a_0^{n-2}a_0a_1(r_0a_0r_1) = a_0^{n-2}a_0(r_0a_0r_1)a_1 = a_0^{n-2}a_1r_1a_1 = a_0^{n-2}a_2.$$

Since $0 = a_0^{n-3}a_0a_2(r_0a_0r_1a_1r_2)$, we have

$$\begin{aligned} 0 &= a_0^{n-3}a_0a_2(r_0a_0r_1a_1r_2) = a_0^{n-3}a_0(r_0a_0r_1a_1r_2)a_2 \\ &= a_0^{n-3}a_1r_1a_1r_2a_2 = a_0^{n-3}a_3. \end{aligned}$$

Now assume that $a_0^{n-k}a_k = 0$ for $k \leq n$. Since

$$0 = a_0^{n-(k+1)}a_0a_k(r_0a_0r_1a_1r_2 \cdots r_{k-1}a_{k-1}r_k),$$

we have

$$\begin{aligned} 0 &= a_0^{n-(k+1)}a_0a_k(r_0a_0r_1a_1r_2 \cdots r_{k-1}a_{k-1}r_k) \\ &= a_0^{n-(k+1)}a_0(r_0a_0r_1a_1r_2 \cdots r_{k-1}a_{k-1}r_k)a_k \\ &= a_0^{n-(k+1)}a_kr_ka_k = a_0^{n-(k+1)}a_{k+1}. \end{aligned}$$

We inductively obtain $a_0a_{n-1} = 0$ by the preceding result. Since

$$0 = a_0a_{n-1}(r_0a_0r_1a_1r_2 \cdots r_{n-2}a_{n-2}r_{n-1}),$$

we finally have

$$\begin{aligned}0 &= a_0a_{n-1}(r_0a_0r_1a_1r_2\cdots r_{n-2}a_{n-2}r_{n-1})\\ &= a_0(r_0a_0r_1a_1r_2\cdots r_{n-2}a_{n-2}r_{n-1})a_{n-1}\\ &= a_1r_1a_1(r_2a_2r_3\cdots r_{n-2}a_{n-2}r_{n-1})a_{n-1}\\ &= a_2r_2a_2(r_3a_3r_4\cdots r_{n-2}a_{n-2}r_{n-1})a_{n-1}\\ &= a_{n-2}(r_{n-2}a_{n-2}r_{n-1})a_{n-1} = a_{n-1}r_{n-1}a_{n-1} = a_n.\end{aligned}$$

This implies that a is strongly nilpotent, proving that $N_*(R) = N^*(R)$.

It can be similarly obtained that $N_*(R) = N^*(R)$, in case of R is a weak left nil-symmetric ring. □

The converse of Theorem 2.4 need not hold as can be seen by $R = \mathrm{Mat}_n(A)$ for $n \geq 4$ over a simple ring A. Note $N^*(R) = N_*(R) = 0$. Since $E_{12}E_{34}E_{23} = 0 = E_{23}E_{12}E_{34}$ and $E_{12}E_{23}E_{34} = E_{14}$, R is neither weak left nil-symmetric nor weak right nil-symmetric.

Considering Theorem 2.4, one may ask whether R is a weak one-sided nil-symmetric ring when $N^*(R) = N_*(R) = N(R)$. But the answer is negative by Example 2.3. The ring R in Example 2.3(1) (resp., Example 2.3(2)) satisfies

$$N^*(R) = N_*(R) = N(R) = RaR + RbR,$$

but it is not weak left (resp., right) nil-symmetric.

As noted earlier, every right (left) nil-symmetric ring is both nil-semicommutative and weak right (left) nil-symmetric, but not conversely. Moreover, the class of weak right nil-symmetric rings and the class of nil-semicommutative rings do not imply each other by the following example.

Example 2.5. (1) We apply [3, Example 4.8]. Let K be a field and $A = K\langle a, b\rangle$ be the free algebra with noncommuting indeterminates a, b over K. Let I be the ideal of A generated by a^2. Set $R = A/I$ and let a, b coincide with their images in R for simplicity. It is easily checked that

$$N(R) = \{ka + afa \mid k \in K, f \in A\}.$$

So R is weak right nil-symmetric. But R is not nil-semicommutative by [18, Theorem 2.5] as can be seen by $a^2 = 0$ and $aba \neq 0$.

(2) We use the ring in [13, Example 2]. Let

$$A = \mathbb{Z}_2\langle a_0, a_1, a_2, b_0, b_1, b_2, c\rangle$$

be the free algebra generated by noncommuting indeterminates $a_0, a_1, a_2, b_0, b_1, b_2, c$ over $\mathbb{Z}_2$. Let I be the ideal of A generated by

$$a_0b_0, a_1b_2 + a_2b_1, a_0b_1 + a_1b_0, a_0b_2 + a_1b_1 + a_2b_0, a_2b_2,$$

$$a_0rb_0, a_2rb_2, (a_0 + a_1 + a_2)r(b_0 + b_1 + b_2) \text{ with } r \in A$$

and

$$r_1r_2r_3r_4 \text{ with } r_1, r_2, r_3, r_4 \in A_0,$$

where A_0 is the subset of all elements in A of zero constants. Set $R = A/I$. Then R is semicommutative (hence nil-semicommutative) by the argument in [13, Example 2]. But R is not weak right nil-symmetric as can be seen by $\bar{c}\bar{a}_0\bar{b}_0 = 0$ and $\bar{c}\bar{b}_0\bar{a}_0 \neq 0$, noting $\bar{a}_0, \bar{b}_0, \bar{c} \in N(R)$.

Proposition 2.6. *If R is a weak right nil-symmetric ring, then $IJK = 0$ implies $IKJ = 0$ for all nil ideals I, J and K of R. The converse holds for a nil-semicommutative ring R.*

Proof. Suppose that R is weak right nil-symmetric and let $IJK = 0$ for nil ideals I, J and K of R. Then, for all $a \in I$, $b \in J$, and $c \in K$, we have $abc = 0$. Since $a, b, c \in N(R)$, $acb = 0$ and this yields $IKJ = 0$.

Conversely, let R be a nil-semicommutative ring and assume that $IJK = 0$ implies $IKJ = 0$ for all nil ideals I, J, K of R. Let $abc = 0$ for $a, b, c \in N(R)$. Then $RaRbRcR = 0$ since R is nil-semicommutative. Here RaR, RbR, RcR are nil ideals of R by [18, Theorem 2.5]. Thus $(RaR)(RcR)(RbR) = 0$ by assumption, entailing $acb = 0$. Thus R is weak right nil-symmetric. □

The condition "R is a nil-semicommutative ring" in Proposition 2.6 cannot be dropped by the following example.

Example 2.7. Consider the ring $R = \mathrm{Mat}_3(A)$, over a reduced ring A. Then R is not weak right nil-symmetric by Example 3.6(1) to follow. Moreover R is not nil-semicommutative: Indeed, $E_{12}^2 = 0$ for $E_{12} \in N(R)$, but $0 \neq E_{12} = E_{12}E_{21}E_{12} \in E_{12}RE_{12}$.

Since R has no nonzero nil ideals, R always satisfies the condition that $IJK = 0$ implies $IKJ = 0$ for all nil ideals I, J and K of R.

3. Structure and properties

Following the literature, the *index* (of nilpotency) of a nilpotent element a in a ring R is the least positive integer n such that $a^n = 0$, write $i(a)$ for n, the *index* (of nilpotency) of a subset S of R is the supremum of the indices (of nilpotency) of all nilpotent elements in S, write $i(S)$, and if such a supremum is finite, then S is said to be *of bounded index* (of nilpotency).

Proposition 3.1. *Let R be a ring of bounded index with $i(R) = 2$.*

(1) *If R is weak right nil-symmetric, then $N(R)$ forms a subring of R such that $ab = -ba$ for all $a, b \in N(R)$.*

(2) *If R is a weak right nil-symmetric ring of characteristic* 2*, then $N(R)$ forms a commutative subring of R.*

Proof. Let R be weak right nil-symmetric and $a, b \in N(R)$. Then $a^2 = 0$ and $b^2 = 0$ by hypothesis, and this yields $a(a-b)b = aab - abb = 0$. So $ba(a-b) \in N(R)$. Since R is weak right nil-symmetric, $b(ba(a-b))a = 0$ implies $ba(ba(a-b)) = 0$. It then follows that $babab = 0$ and $(ba)^3 = 0$, entailing $(ba)^2 = 0$ and $(ab)^2 = 0$ by hypothesis. Next consider $(a-b)^k$ for $k = 2, 3, \ldots$.

$(a-b)^2 = a^2-ab-ba+b^2 = -ab-ba$, $(a-b)^3 = (-ab-ba)(a-b) = -aba+bab$, and

$$(a-b)^4 = (-aba+bab)(a-b) = baba + abab = 0,$$

entailing $a-b \in N(R)$. Then $(a-b)^2 = 0$ by hypothesis, forcing $ab = -ba$.

(2) This is an immediate consequence of (1). □

Proposition 3.2. (1) *Let R be a ring. If $N(R)^3 = 0$, then R is weak nil-symmetric.*

(2) *The class of weak right nil-symmetric rings is closed under subrings.*

(3) *Let R be a ring such that the multiplicative group of units in R, $U(R)$ say, is an Abelian group. Then R is weak nil-symmetric.*

(4) *Let R_λ ($\lambda \in \Lambda$) be rings. Then R_λ is weak right nil-symmetric for all $\lambda \in \Lambda$ if an only if the direct product $\prod_{\lambda\in\Lambda} R_\lambda$ of R_λ is weak right nil-symmetric.*

(5) *Let R be a ring and e be a central idempotent in R. Then both eR and $(1-e)R$ are weak right nil-symmetric if and only if R is weak right nil-symmetric.*

Proof. (1) It is an immediate consequence of the definition of a weak nil-symmetric ring.

(2) This comes from the fact that $N(S) = S \cap N(R)$ for any ring R and any subring S of R.

(3) Let $a, b, c \in N(R)$. Then $1-a, 1-b, 1-c \in U(R)$. But $U(R)$ is Abelian, and so we have $(1-a)(1-b) = (1-b)(1-a)$ (resp., $(1-b)(1-c) = (1-c)(1-b)$). This yields $ab = ba$ (resp., $bc = cb$), entailing $bac = 0$ (resp., $acb = 0$) from $abc = 0$. Thus R is weak nil-symmetric.

(4) By (2), it suffices to establish necessity. Note that $N(\prod_{\lambda\in\Lambda} R_\lambda) \subseteq \prod_{\lambda\in\Lambda} N(R_\lambda)$. Suppose that R_λ is weak right nil-symmetric for all $\lambda \in \Lambda$. Let $(a_\lambda)(b_\lambda)(c_\lambda) = 0$ for $(a_\lambda), (b_\lambda), (c_\lambda) \in N(\prod_{\lambda\in\Lambda} R_\lambda)$. Then $a_\lambda, b_\lambda, c_\lambda \in N(R_\lambda)$ satisfying $a_\lambda b_\lambda c_\lambda = 0$ for all λ. Since R_λ is weak right nil-symmetric, $a_\lambda c_\lambda b_\lambda = 0$ for all $\lambda \in \Lambda$. This yields $(a_\lambda)(c_\lambda)(b_\lambda) = 0$, concluding that R is weak right nil-symmetric.

(5) It is shown by (2) and (4), since $R \cong eR \oplus (1-e)R$. □

The following argument shows that Proposition 3.2(1, 3) need not hold when the hypotheses do not hold. In fact, we recall the weak right nil-symmetric ring R in Example 2.3(1) (which is not weak left nil-symmetric). Let

$$\alpha = h_0a + h_1a^2 + h_2b + h_3ba + h_4ba^2, \beta = k_0a + k_1a^2 + k_2b + k_3ba + k_4ba^2,$$

$$\gamma = l_0a + l_1a^2 + l_2b + l_3ba + l_4ba^2 \text{ and } \delta = m_0a + m_1a^2 + m_2b + m_3ba + m_4ba^2$$

be in R. Then

$$\alpha\beta\gamma\delta = h_2k_0l_0ba^2(m_0a + m_1a^2 + m_2b + m_3ba + m_4ba^2) = 0,$$

entailing $N(R)^4 = 0$. But $N(R)^3 \neq 0$ since $baa \neq 0$. Moreover $U(R)$ is non-Abelian since $1-a-b = (1-a)(1-b) \neq (1-b)(1-a) = 1-a-b+ba$.

As a corollary of Proposition 3.2(1, 2), we obtain the following.

Corollary 3.3. *If R is a reduced ring, then both $U_n(R)$ and $D_n(R)$ are weak nil-symmetric for $n = 2, 3$.*

Based on Corollary 3.3, one may suspect that $U_n(R)$ over a reduced ring R may be weak nil-symmetric for $n \geq 4$. But the following example eliminates the possibility.

Example 3.4. For any ring A, consider $R = D_4(A)$. For

$$a = \begin{pmatrix} 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} = c \text{ and } b = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix} \in N(R),$$

we have $abc = 0$, but

$$acb = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \neq 0.$$

Thus R is not weak right nil-symmetric.

By Proposition 3.2(2), we conclude that both $U_n(A)$ and $D_n(A)$ for any ring A and $n \geq 4$ cannot be weak right nil-symmetric, noting that $D_4(A)$ is isomorphic to a subring of $U_n(A)$ $(n \geq 4)$ and $D_n(A)$ $(n \geq 5)$.

The following example also illuminates that $U_3(R)$ and $D_3(R)$ are not weak right nil-symmetric any more, if we take the weaker condition "R is symmetric" instead of "R is reduced" in Corollary 3.3.

Example 3.5. Let $R = \mathbb{Z}_4$. Then R is a non-reduced symmetric (hence weak nil-symmetric) ring. Take

$$a = \begin{pmatrix} 2 & 1 & 0 \\ 0 & 2 & 2 \\ 0 & 0 & 2 \end{pmatrix}, b = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, c = \begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{pmatrix} \in N(D_3(R)).$$

In fact, $a^3 = 0$, $b^2 = 0$ and $c^3 = 0$. Then $abc = 0$ but

$$0 \neq \begin{pmatrix} 0 & 0 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = acb.$$

This shows that $D_3(R)$ is not weak right nil-symmetric. Consequently, $U_3(R)$ is not weak right nil-symmetric either, by Proposition 3.2(2).

We next see that the n by n full matrix ring $\mathrm{Mat}_n(A)$ over any ring A for $n \geq 2$ cannot be weak right nil-symmetric by the following example.

Example 3.6. For any ring A, let $R = \mathrm{Mat}_2(A)$. Then for

$$a = \begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix} = b \text{ and } c = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \in N(R),$$

we get $abc = 0$. But

$$acb = \begin{pmatrix} -1 & -1 \\ 1 & 1 \end{pmatrix} \neq 0,$$

implying that R is not weak right nil-symmetric. So $\mathrm{Mat}_n(A)$ is not weak right nil-symmetric by Proposition 3.2(2) even for the case of $n \geq 3$.

The next example shows that the class of weak right nil-symmetric rings is not closed under homomorphic images, being compared with Proposition 3.2(2).

Example 3.7. Let R be the ring of quaternions with integer coefficients. Then R is a domain and thus weak right nil-symmetric. However, for any odd prime integer q, there exists a ring isomorphism $R/qR \cong \mathrm{Mat}_2(\mathbb{Z}_q)$ by the argument in [10, Exercise 2A]. But $\mathrm{Mat}_2(\mathbb{Z}_q)$ is not weak right nil-symmetric by Example 3.6, and thus R/qR cannot be weak right nil-symmetric. Therefore the class of weak right nil-symmetric rings is not closed under homomorphic images.

For a ring R and $n \geq 2$, let $V_n(R)$ be the ring of all matrices (a_{ij}) in $D_n(R)$ such that $a_{st} = a_{(s+1)(t+1)}$ for $s = 1, \ldots, n-2$ and $t = 2, \ldots, n-1$. Note that $V_n(R) \cong \frac{R[x]}{x^n R[x]}$, where $R[x]$ denotes the polynomial ring with an indeterminate x over R. We use $(a_1, a_2, \ldots, a_n) \in V_n(R)$ to denote

$$\begin{pmatrix} a_1 & a_2 & a_3 & \cdots & a_n \\ 0 & a_1 & a_2 & \cdots & a_{n-1} \\ 0 & 0 & a_1 & \cdots & a_{n-2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & a_1 \end{pmatrix}.$$

For a reduced ring R and $n \geq 2$, $V_n(R)$ is symmetric by [11, Theorem 2.3] and so it is weak nil-symmetric. Here it is natural to ask whether $V_n(R)$ is weak right nil-symmetric over a symmetric ring R. But the following answers negatively.

Example 3.8. We adopt the ring and the argument in [14, Example 2.1]. Let

$$A = \mathbb{Z}_2\langle a_0, a_1, a_2, b_0, b_1, b_2, c\rangle$$

be the free algebra generated by noncommuting indeterminates $a_0, a_1, a_2, b_0, b_1, b_2, c$ over $\mathbb{Z}_2$. Next let I be the ideal of A generated by

$a_0b_0, a_0b_1 + a_1b_0, a_0b_2 + a_1b_1 + a_2b_0, a_1b_2 + a_2b_1, a_2b_2, a_0rb_0, a_2rb_2,$

$b_0a_0, b_0a_1 + b_1a_0, b_0a_2 + b_1a_1 + b_2a_0, b_1a_2 + b_2a_1, b_2a_2, b_0ra_0, b_2ra_2,$

$(a_0+a_1 + a_2)r(b_0 + b_1 + b_2), (b_0 + b_1 + b_2)r(a_0 + a_1 + a_2)$, and $r_1r_2r_3r_4$, where the constant terms of $r, r_1, r_2, r_3, r_4 \in A$ are zero. Now set $R = A/I$. Then R is a symmetric ring by the argument in by [11, Example 3.1].

We identify $a_0, a_1, a_2, b_0, b_1, b_2, c$ with their images in R for simplicity. Consider the extension ring $D_2(R)$ of R and take

$$\alpha = (a_0, a_0, \dots, a_1), \beta = (b_0, b_0, \dots, b_1) \text{ and } \gamma = (c, c, \dots, c) \in N(V_n(R)).$$

It can be also easily checked that $\alpha\beta\gamma = 0$, by the construction of I. But

$$\alpha\gamma\beta = (a_0cb_0, a_0cb_0, \dots, a_0cb_1 + a_1cb_0) = (0, 0, \dots, a_0cb_1 + a_1cb_0) \neq 0$$

since $a_1cb_0 + a_0cb_1 \notin I$. Thus $V_n(R)$ is not weak right nil-symmetric for $n \geq 2$.

Given a ring R and an (R, R)-bimodule M, the *trivial extension* of R by M is the ring $T(R, M) = R \oplus M$ with the usual addition and the following multiplication: $(r_1, m_1)(r_2, m_2) = (r_1r_2, r_1m_2 + m_1r_2)$. This is isomorphic to the ring of all matrices $\left(\begin{smallmatrix} r & m \\ 0 & r \end{smallmatrix}\right)$, where $r \in R$ and $m \in M$ and the usual matrix operations are used.

Notice that Example 3.8 also illuminates that the trivial extension of a weak right nil-symmetric ring need not be weak right nil-symmetric, since $T(R, R) = V_2(R)$.

The property of weak right nil-symmetric and the abelian ring property do not follow each other by Corollary 3.3 and Example 3.4, noting that $U_3(A)$ is non-abelian and $D_4(A)$ is abelian by [12, Lemma 2] for any reduced ring A.

$J(R)$ denotes the Jacobson radical of a given ring R. Following the literature, R is *semilocal* if $R/J(R)$ is Artinian, and R is *semiperfect* if R is semilocal and idempotents can be lifted modulo $J(R)$. Local rings are abelian and semilocal obviously.

Proposition 3.9. *Let R be an abelian ring. Then R is weak right nil-symmetric and semiperfect if and only if R is a finite direct sum of local weak right nil-symmetric rings.*

Proof. Suppose that R is weak right nil-symmetric and semiperfect. Since R is semiperfect, R has a finite orthogonal set $\{e_1, e_2, \dots, e_n\}$ of local idempotents whose sum is 1 by [15, Proposition 3.7.2], say $R = \sum_{i=1}^{n} e_iR$ such that each e_iRe_i is a local ring. Since R is abelian and weak right nil-symmetric, $e_iR = e_iRe_i$ is weak right nil-symmetric by Proposition 3.2(3).

Conversely assume that R is a finite direct sum of local weak right nil-symmetric rings. Then R is semiperfect since local rings are semiperfect by [15, Corollary 3.7.1], and moreover R is weak right nil-symmetric by Proposition 3.2(2). □

By *minimal*, we mean "having smallest cardinality" of a given kind of a finite ring. $|R|$ means the order of a given ring R.

Proposition 3.10. *If R is a minimal noncommutative weak right nil-symmetric ring, then R is of order 8 and is isomorphic to $U_2(\mathbb{Z}_2)$.*

Proof. Let R be a minimal noncommutative weak right nil-symmetric ring. Eldridge proved that if the order of R has a cube free factorization, then R is a commutative ring in [8, Theorem]. This forces $|R| \geq 2^3$. If $|R| = 2^3$, then R is isomorphic to $U_2(\mathbb{Z}_2)$ by [8, Proposition]. Notice that $U_2(\mathbb{Z}_2)$ is a weak right nil-symmetric ring by Corollary 3.3. This yields that R is of order 8 and is isomorphic to $U_2(\mathbb{Z}_2)$. □

4. Extensions

In this section we study the structure of weak right nil-symmetric rings related to several sorts of ordinary ring extensions.

A ring R is called *directly finite* if $ab = 1$ implies $ba = 1$ for $a, b \in R$. Recall that weak right nil-symmetric rings and abelian rings are independent of each other, and abelian rings (e.g., symmetric rings) are clearly directly finite.

Proposition 4.1. *Every weak right nil-symmetric ring is directly finite.*

Proof. Let R be a weak right nil-symmetric ring and assume on the contrary that R is not directly finite. Then R contains an infinite set of matrix units, say

$$\{E_{11}, E_{12}, E_{13}, \ldots, E_{21}, E_{22}, E_{23}, \ldots\},$$

by [9, Proposition 5.5]. Consider $E_{21}, E_{12}, E_{23} \in N(R)$. Then $E_{21}E_{23}E_{12} = 0$ but $E_{21}E_{12}E_{23} = E_{23} \neq 0$, showing that R is not weak right nil-symmetric, a contradiction. Thus R is directly finite. □

Let A be an algebra over a commutative ring S. Due to Dorroh [7], the *Dorroh extension* of A by S is the Abelian group $A \oplus S$ with multiplication given by $(r_1, s_1)(r_2, s_2) = (r_1r_2 + s_1r_2 + s_2r_1, s_1s_2)$ for $r_i \in A$ and $s_i \in S$. We use $A \times S$ to denote the Dorroh extension of A by S.

Theorem 4.2. *Let R be an algebra with identity over a commutative reduced ing S. Then R is weak right nil-symmetric if and only if the Dorroh extension $D = R \times S$ is weak right nil-symmetric.*

Proof. It can be easily checked that $N(D) = (N(R), 0)$ since S is a commutative reduced ring. For any $(r_1, 0), (r_2, 0), (r_3, 0) \in N(D)$,

$$(r_1, 0)(r_2, 0)(r_3, 0) = (0, 0) \text{ if and only if } r_1r_2r_3 = 0.$$

This implies that R is weak right nil-symmetric if and only if the Dorroh extension D is weak right nil-symmetric. □

The symmetric ring property does not go up to polynomial rings by [11, Example 3.1]. So one may ask whether the polynomial rings over weak right nil-symmetric rings are weak right nil-symmetric. However the answer is negative by the following.

Example 4.3. We use the ring and apply the argument in [11, Example 3.1] and Example 3.8. Let R be the symmetric (hence weak nil-symmetric) ring in Example 3.8. Note that

$$N(R) = N_*(R) = \mathbb{Z}_2\langle a_0, a_1, a_2, b_0, b_1, b_2, c\rangle,$$

i.e., $R/N_*(R) \cong \mathbb{Z}_2$ is a reduced ring, where $\mathbb{Z}_2\langle a_0, a_1, a_2, b_0, b_1, b_2, c\rangle$ means the set of all elements in A of zero constants. It then follows from this that

$$N(R)[x] = \mathbb{Z}_2\langle a_0, a_1, a_2, b_0, b_1, b_2, c\rangle[x] = N(R[x]) = N^*(R[x])$$

and

$$\frac{R}{N^*(R)}[x] \cong \frac{R[x]}{N^*(R[x])} \cong \mathbb{Z}_2.$$

Now we take

$$f(x) = a_0 + a_1x + a_2x^2, g(x) = b_0c + b_1cx + b_2cx^2 \text{ and } h(x) = c$$

in $R[x]$. Then $f(x), g(x), h(x) \in N(R[x])$ and $f(x)g(x)h(x) = (a_0 + a_1x + a_2x^2)(b_0 + b_1x + b_2x^2)c = 0$. But

$$f(x)h(x)g(x) = (a_0 + a_1x + a_2x^2)c(b_0 + b_1x + b_2x^2) \neq 0,$$

since $a_0cb_1 + a_1cb_0 \notin I$. Thus $R[x]$ is not weak right nil-symmetric.

Now we study some conditions under which polynomial rings may be weak right nil-symmetric.

Recall that a ring R is called *Armendariz* [20] if whenever any polynomials $f(x) = \sum_{i=0}^{m} a_ix^i$, $g(x) = \sum_{j=0}^{n} b_jx^j \in R[x]$ satisfy $f(x)g(x) = 0$, $a_ib_j = 0$ for all i, j. Reduced rings are Armendariz by [4, Lemma 1]. Armendariz rings are abelian by the proof of [1, Theorem 6] or [13, Corollary 8]. Armendariz rings and weak right nil-symmetric rings are independent of each other by the following example.

Example 4.4. (1) Let K be a field and $A = K\langle a, b\rangle$ be the free algebra generated by the noncommuting indeterminates a, b over K. Let I be the ideal of A generated by a^4 and set $R = A/I$. Then R is Armendariz by [3, Example 4.8]. We have $\bar{a}, \bar{a}\bar{b}\bar{a}^3 \in N(R)$ and $\bar{a}(\bar{a}\bar{b}\bar{a}^3)\bar{a} = 0$, but $\bar{a}\bar{a}(\bar{a}\bar{b}\bar{a}^3) \neq 0$. So R is not weak right nil-symmetric.

(2) $U_2(R)$, over a reduced ring R, is a weak nil-symmetric ring by Corollary 3.3. But this is non-abelian, so not Armendariz.

A ring R is called (*von Neumann*) *regular* if for each $a \in R$ there exists $b \in R$ such that $a = aba$.

Proposition 4.5. *For a regular ring R the following conditions are equivalent*:

(1) *R is Armendariz.*
(2) *R is reduced.*
(3) *R is symmetric.*
(4) *R is semicommutative.*

(5) *R is abelian.*
(6) *R is right (left) nil-symmetric.*
(7) *R is nil-semicommutative.*
(8) *R is weak right (left) nil-symmetric.*

Proof. A regular ring R is Armendariz if and only if R is abelian if and only if R is reduced if and only if R is nil-semicommutative by help of [9, Theorem 3.2], [13, Corollary 8] and [18, Proposition 2.18]. (3)⇒(6) is clear and (6)⇒(2) comes from[18, Proposition 2.18]. So we show that (8)⇒(5). Let R be weak right nil-symmetric and assume on the contrary that there exist $e^2 = e, r \in R$ with $er - re \neq 0$. Then $er(1-e) \neq 0$ or $(1-e)re \neq 0$. Say $a = er(1-e) \neq 0$. Since R is regular, there exists $b \in R$ with $aba = a$. Note that $b = (1-e)be$, $a^2 = 0$ and $b^2 = 0$. Then $a^2b = 0$ but $aba = a \neq 0$ for $a, b \in N(R)$, contradicting that R is weak right nil-symmetric. The computation for the case of $a = (1-e)re \neq 0$ is similar. The proof of the weak left nil-symmetric case is also similar. □

On may raises two questions withe respect to Theorem 2.4, Proposition 4.1 and Proposition 4.5. But the regular ring $R = \mathrm{Mat}_n(F)$ for $n \geq 4$ over a field F with $N^*(R) = N_*(R)$ is directly finite but it is neither weak left nil-symmetric nor weak right nil-symmetric by Example 3.6.

An element u of a ring R is *right regular* if $ur = 0$ implies $r = 0$ for $r \in R$. Similarly, *left regular* elements can be defined. An element is *regular* if it is both left and right regular (i.e., not a zero divisor). Recall that $R \cong \frac{R[x]}{R[x](x-a)R[x]}$ for a central regular element a in R.

Proposition 4.6. (1) *If R is an Armendariz ring, then R is weak right nil-symmetric if and only if $R[x]$ is weak right nil-symmetric.*

(2) *For a ring R, assume that the center $C(R)$ of R contains infinitely many regular elements. Then R is weak right nil-symmetric if and only if $R[x]$ is weak right nil-symmetric.*

Proof. Each proof is enough to show that $R[x]$ is weak right nil-symmetric when R is weak right nil-symmetric, by Proposition 3.2(2).

(1) Assume that R is Armendariz and weak right nil-symmetric. Let

$$f(x)g(x) = 0 \text{ for } f(x) = \sum_{i=0}^{m} a_i x^i, g(x) = \sum_{j=0}^{n} a_j x^j, h(x) = \sum_{l=0}^{k} c_l x^l \in N(R[x]).$$

Then $f(x), g(x), h(x) \in N(R)[x]$ because $N(R[x]) = N(R)[x]$ by [3, Corollary 5.2]. Since R is Armendariz, $a_i b_j c_l = 0$ for all i, j and l. This implies that $a_i c_l b_j = 0$ for all i, j and l since R is weak right nil-symmetric. This yields $f(x)h(x)g(x) = 0$, proving that $R[x]$ is weak right nil-symmetric.

(2) Let R be weak right nil-symmetric. By assumption, $C(R)$ contains infinitely many regular elements, $\{a_i \mid i \in I\}$ say. Then $\bigcap_{i \in I} R[x](x - a_i)R[x] = 0$, entailing that $R[x]$ is a subdirect product of infinitely many copies of R. Thus $R[x]$ is weak right nil-symmetric by Proposition 3.2(4). □

The *Laurent* polynomial ring with an indeterminate x over a ring R consists of all formal sums $\sum_{i=k}^{n} m_i x^i$ with obvious addition and multiplication, where $m_i \in R$ and k, n are (possibly negative) integers, we denote it by $R[x; x^{-1}]$.

Proposition 4.7. *Let R be a ring. Then we have the following results.*

(1) *Let M be a multiplicatively closed subset of R consisting of central regular elements. Then R is weak right nil-symmetric if and only if $M^{-1}R$ is weak right nil-symmetric.*

(2) *$R[x]$ is weak right nil-symmetric if and only if $R[x; x^{-1}]$ is weak right nil-symmetric.*

Proof. (1) It comes from the fact of $N(M^{-1}R) = M^{-1}N(R)$.

(2) Letting $M = \{1, x, x^2, \ldots\}$, M is clearly a multiplicatively closed subset of central regular elements in $R[x]$ such that $R[x; x^{-1}] = M^{-1}R[x]$. The proof is completed by (1). $\square$

A multiplicatively closed subset S of a ring R is said to satisfy the *right Ore condition* if for each $a \in R$ and $b \in S$, there exist $a_1 \in R$ and $b_1 \in S$ such that $ab_1 = ba_1$. It is shown by [17, Theorem 2.1.12] that S satisfies the right Ore condition and S consists of regular elements if and only if the right quotient ring of R with respect to S exists.

Theorem 4.8. *Let S be a multiplicatively closed subset of a ring R, and suppose that S satisfies the right Ore condition and S consists of regular elements. Assume that Q is a nil-semicommutative ring. Then R is weak right nil-symmetric if and only if so is the right quotient ring Q of R with respect to S.*

Proof. Let R be a weak right nil-symmetric ring. It suffices to show that the right quotient ring Q of R is weak right nil-symmetric by Proposition 3.2(2). We will freely use the assumption that Q is nil-semicommutative, and the fact that $N(Q) = N^*(Q)$ in [18, Theorem 2.5]. Suppose that $\alpha = au^{-1}, \beta = bv^{-1}, \gamma = cw^{-1} \in N(Q)$ and $\alpha\beta\gamma = 0$. Set I, J and K be the generated ideals of Q by α, β and γ, respectively. Then I, J and K are nil so that $a = \alpha u \in I$, $b = \beta v \in J$ and $c = \gamma w \in K$, entailing $a, b, c \in N(R)$. Since S satisfies the right Ore condition, $u^{-1}b = b_1 u_1^{-1}$ and $bu_1 = ub_1$ for some $b_1 \in R$ and $u_1 \in S$. Here $b_1 \in N(R)$ since $bu_1 = ub_1 \in J$ and $b_1 = u^{-1}(bu_1) \in J$. Then $0 = \alpha\beta\gamma = au^{-1}bv^{-1}cw^{-1} = ab_1u_1^{-1}v^{-1}cw^{-1}$. Similarly, there exist $c_1 \in R$ and $v_1 \in S$ such that $cv_1 = vc_1$ and $v^{-1}c = c_1 v_1^{-1}$. Here, $c_1 \in N(R)$ since $cv_1 = vc_1 \in K$ and $c_1 = v^{-1}(cv_1) \in K$. Then $0 = \alpha\beta\gamma = ab_1u_1^{-1}v^{-1}cw^{-1} = ab_1u_1^{-1}c_1v_1^{-1}w^{-1}$. Also, there exist $c_2 \in R$ and $u_2 \in S$ such that $c_1u_2 = u_1c_2$ and $u_1^{-1}c_1 = c_2u_2^{-1}$. Moreover, $c_2 \in N(R)$ since $c_1u_2 = u_1c_2 \in K$ and $c_2 = u_1^{-1}(u_1c_2) \in K$. Hence, $0 = \alpha\beta\gamma = ab_1u_1^{-1}c_1v_1^{-1}w^{-1} = ab_1c_2u_2^{-1}v_1^{-1}w^{-1}$ and so $ab_1c_2 = 0$. Then

$$0 = ab_1c_2 = aub_1c_2 = abu_1c_2 = abc_1u_2$$

and

$$0 = abc_1 = abvc_1 = abcv_1.$$

Hence we have $abc = 0$, and so $acb = 0$ since R is weak right nil-symmetric. There exist $c_3, b_3, b_4 \in R$ and $u_3, w_3, u_4 \in S$ such that $cu_3 = uc_3, bw_3 = wb_3$ and $b_3u_4 = u_3b_4$, where $c_3, b_3, b_4 \in N(R)$ by the same method as above. We will freely use R being weak right nil-symmetric. Then

$$0 = acb = acu_3b = a(uc_3)b = ab(uc_3) = ac_3(bu).$$

So $0 = ac_3b = ac_3bw_3 = a(c_3w)b_3$ since $bw_3 = wb_3$, and thus $0 = ab_3(c_3w)b_3$. Then $0 = ab_3c_3 = ac_3b_3$. By the similar computation to above,

$$0 = ac_3b_3 = ac_3(b_3u_4) = ac_3(u_3b_4)$$

and $0 = a(c_3u_3)b_4 = ab_4(c_3u_3)$. Thus we have $ab_4c_3 = 0$ and so $ac_3b_4 = 0$. Therefore

$$\begin{aligned}\alpha\gamma\beta &= au^{-1}cw^{-1}bv^{-1} = ac_3u_3^{-1}w^{-1}bv^{-1} \\ &= ac_3u_3^{-1}b_3w_3^{-1}v^{-1} = ac_3b_4u_4^{-1}w_3^{-1}v^{-1} = 0,\end{aligned}$$

showing that Q is weak right nil-symmetric. □

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Hong Kee Kim
Department of Mathematics and RINS
Gyeongsang National University
Jinju 52828, Korea
E-mail address: hkkim@gsnu.ac.kr

Tai Keun Kwak
Department of Mathematics
Daejin University
Pocheon 11159, Korea
E-mail address: tkkwak@daejin.ac.kr

Seung Ick Lee
Department of Mathematics
Pusan National University
Pusan 46241, Korea
E-mail address: maick@hanmail.net

Yang Lee
Department of Mathematics
Pusan National University
Pusan 46241, Korea
E-mail address: ylee@pusan.ac.kr

Sung Ju Ryu
Department of Mathematics
Pusan National University
Pusan 46241, Korea
E-mail address: sjryu@pusan.ac.kr

Hyo Jin Sung
Department of Mathematics
Pusan National University
Pusan 46241, Korea
E-mail address: hjsung@pusan.ac.kr

Sang Jo Yun
Department of Mathematics
Pusan National University
Pusan 46241, Korea
E-mail address: pitt0202@hanmail.net