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COMBINATORIAL ENUMERATION OF THE REGIONS OF SOME LINEAR ARRANGEMENTS

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Abstract. Richard Stanley suggested the problem of finding combinatorial proofs of formulas for counting regions of certain hyperplane arrangements defined by hyperplanes of the form $x_i = 0$, $x_i = x_j$, and $x_i = 2x_j$ that were found using the finite field method. We give such proofs, using embroidered permutations and linear extensions of posets.

1. Introduction

Much work has been devoted in recent years to studying hyperplane arrangements, especially finding their characteristic polynomials and number of regions. Several authors have worked on computing the number of regions of specific hyperplane arrangements. See for example [1, 2, 7].

Stanley's paper [10] on hyperplane arrangements contains classic and more recent results in this field, together with numerous problems. We consider the following problems [10, p. 469, problem 9] of counting the regions of certain linear arrangements.

(α) Let α_n be the arrangement in $\mathbb{R}^n$ with hyperplanes $x_i = 0$ for all i, $x_i = x_j$ for all $i < j$, and $x_i = 2x_j$ for all $i \neq j$. Show that the number of regions $r(\alpha_n)$ of α_n is given by $r(\alpha_n) = 2(2n+1)!/(n+2)!$.

(β) Let β_n be the arrangement in $\mathbb{R}^n$ with hyperplanes $x_i = x_j$ for all $i < j$, and $x_i = 2x_j$ for all $i \neq j$. Show that $r(\beta_n) = 6n^2(2n-1)!/(n+2)!$.

(γ) Let γ_n be the arrangement in $\mathbb{R}^n$ with hyperplanes $x_i = 0$ for all i, $x_i = x_j$ for all $i < j$, and $x_i = 2x_j$ for all $i < j$. Find $r(\gamma_n)$.

(δ) Let δ_n be the arrangement in $\mathbb{R}^n$ with hyperplanes $x_i = x_j$ for all $i < j$, and $x_i = 2x_j$ for all $i < j$. Find $r(\delta_n)$.

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Note that problems (α), (β), (γ) are actually in Stanley's paper, while problem (δ) is not. But the problem comes along naturally, so we add (δ) to the problem list.

By finite field methods (see [2, 3, 10]), one can solve these problems without much difficulty. But Stanley asked for combinatorial proofs, i.e., bijections between the regions of the hyperplane arrangements and objects that are counted by the corresponding formulas.

The present paper aims to solve these 4 problems in bijective way. In Section 2, we introduce the basic notations of hyperplane arrangements and partially ordered sets. In Section 3, we describe bijections from regions of certain hyperplane arrangement to linear extensions of posets (for problem (α) and (β)) or permutations (problems (γ) and (δ)). In Section 4, we discuss the characteristic polynomial of the four kinds of hyperplane arrangements.

2. Preliminaries

We recall some of the basic concepts of hyperplane arrangements and partially ordered sets. For a more thorough introduction, see [6, 10] for hyperplane arrangements and [9, Ch. 3] for partially ordered sets.

Given a field K and a positive integer n, a hyperplane arrangement of dimension n over K is a finite set of affine hyperplanes in K^n. We will refer to hyperplane arrangements simply as arrangements. We will say that an arrangement $\mathcal{A}$ is linear if every hyperplane H in $\mathcal{A}$ is an $(n-1)$-dimensional subspace of K^n, i.e.,

$$H = \{\, v \in K^n \mid \xi \cdot v = 0 \,\},$$

for some nonzero vector ξ in K^n, where $\xi \cdot v$ is the usual inner product.

Now, let $K = \mathbb{R}$. A region of an arrangement $\mathcal{A}$ is a connected component of the complement X of the hyperplanes:

$$X = \mathbb{R}^n - \bigcup_{H \in \mathcal{A}} H.$$

Let $\mathcal{R}(\mathcal{A})$ denote the set of regions of $\mathcal{A}$, and let $r(\mathcal{A})$ be the number of regions.

A linear extension of a poset P is an extension of P to a total order, that is, a total order $<_T$ on the underlying set of P such that if $x <_P y$, then $x <_T y$. It is convenient to represent the total order $<_T$ as a listing of the elements of P, so a linear extension of P is a listing of the elements of P in which x comes before y whenever $x <_P y$. Let $E(P)$ denote the set of linear extensions of P. From now on we will write $\{1, 2, \ldots, n\}$ as $[n]$, and $\{m, m+1, \ldots, n\}$ as $[m, n]$.

3. Bijective proofs

3.1. Problem (α)

Given a positive integer n, let α_n be the arrangement in $\mathbb{R}^n$ with hyperplanes

$$x_i = 0 \;\text{ for all } i, \;\; x_i = x_j \;\text{ for all } i < j, \;\; x_i = 2x_j \;\text{ for all } i \neq j.$$

Let $\mathcal{R}_0(\alpha_n)$ be the set of regions of α_n contained in $x_1 < x_2 < \cdots < x_n$. Note that $|\mathcal{R}(\alpha_n)| = n!\,|\mathcal{R}_0(\alpha_n)|$.

For positive integers l and m, let $P_{l,m}$ be the poset on the set $\{l, l+1, \ldots, m\} \cup \{l', (l+1)', \ldots, m'\}$ such that $l <_P l+1 <_P \cdots <_P m$ and $l' <_P (l+1)' <_P \cdots <_P m'$, where any two elements $i \in \{l, l+1, \ldots, m\}$ and $j' \in \{l', (l+1)', \ldots, m'\}$ are not comparable. Let $P^-_{l,m}$ be the poset obtained from $P_{l,m}$ by adding $i' <_P i$ for all $i \in [l, m]$. Similarly, let $P^+_{l,m}$ be obtained from $P_{l,m}$ by adding $i <_P i'$ for all $i \in [l, m]$. If l is greater than m, then we take $P_{l,m} = P^-_{l,m} = P^+_{l,m}$ to be a poset with one element (and therefore one linear extension).

Recall that $E(P)$ is the set of linear extensions of a finite poset P. For a nonnegative integer $0 \le k \le n$, let U_k be the set $E(P^-_{1,k}) \times E(P^+_{k+1,n})$.

Lemma 3.1. *For $n \ge 1$, there is a bijection from $\mathcal{R}_0(\alpha_n)$ to $\bigcup_{k=0}^n U_k$.*

Proof. Let R_0 be a region in $\mathcal{R}_0(\alpha_n)$ and let $(x_1, \ldots, x_n)$ be a point in R_0. Let k be the largest number such that $x_1, \ldots, x_k$ are negative; if x_1 is positive, take $k = 0$. Arrange the numbers $x_1, \ldots, x_k, 2x_1, \ldots, 2x_k$ in increasing order and then replace each x_i with i and each $2x_i$ with i', and take the resulting list to be a total order A of $\{1, \ldots, k\} \cup \{1', \ldots, k'\}$. Since $x_1 < \cdots < x_k$ and $2x_i < x_i$ for all i, this total order is an element of $E(P^-_{1,k})$.

Similarly, the positive coordinates $x_k, \ldots, x_n$ give an element B of $E(P^+_{k+1,n})$.

It is easy to show that points in the same region give the same pair (A, B) in $E(P^-_{1,k}) \times E(P^+_{k+1,n})$, that points in different regions give different pairs, and that every element of $E(P^-_{1,k}) \times E(P^+_{k+1,n})$ comes from some region. Thus we have a desired bijection.

We show that for any A in $E(P^-_{1,k})$, there is a corresponding point $(x_1, \ldots, x_k)$. We proceed by induction on k. (A very similar argument works for $E(P^+_{k+1,n})$.) The cases $k = 0$ and 1 are trivial. Suppose that $k > 1$. Let A be a linear of extension of $P^-_{1,k}$, represented as a list. Removing k and k' from this list gives an element $\tilde{A}$ of $E(P^-_{1,k-1})$. By induction, there is some point $(x_1, x_2, \ldots, x_{k-1})$, with negative coordinates, corresponding to $\tilde{A}$. Then A may be obtained from $\tilde{A}$ by inserting k' anywhere to the right of $(k-1)'$ and then inserting k at the right end of the list. We now need to choose x_k with $x_{k-1} < x_k < 0$ so that $a < 2x_k < b$ for certain numbers a and b, where $x_{k-1} \le a < b \le 0$, and it is clear that we can do this. □

Let C_n be the nth Catalan number $\frac{1}{n+1}\binom{2n}{n}$, with $C_{-1} = 0$. Let $\mathcal{D}_n$ be the set of Dyck words of length $2n$, i.e.,

$$\mathcal{D}_n := \left\{(d_1, \ldots, d_n) \in \{-1, 1\}^n \mid \textstyle\sum_{i=1}^j d_i \ge 0,\ \forall\, j \le n-1 \text{ and } \sum_{i=1}^n d_i = 0\right\}.$$

It is well known that $|\mathcal{D}_n| = C_n$.

Theorem 3.2. *For $n \ge 1$, we have $|\mathcal{R}_0(\alpha_n)| = C_{n+1}$.*

Proof. There is a well known bijection from $E(P^-_{1,k})$ to $\mathcal{D}_k$: We define ϕ : $E(P^-_{1,k}) \to \mathcal{D}_k$ by replacing i' with 1 and i with -1 for each $i \in [k]$ in a linear extension of $P^-_{1,k}$. Thus we have $|E(P^-_{1,k})| = C_k$. Similarly we have $|E(P^+_{k+1,n})| = C_{n-k}$. So $|U_k| = C_k\, C_{n-k}$. From Lemma 3.1 we obtain

$$\begin{aligned} |\mathcal{R}_0(\alpha_n)| &= \sum_{k=0}^{n} |U_k| = \sum_{k=0}^{n} C_k\, C_{n-k} \\ &= C_{n+1}\,, \end{aligned}$$

which completes the proof. □

Corollary 3.3. *The number of regions of α_n is given by*

$$|\mathcal{R}(\alpha_n)| = \frac{2(2n+1)!}{(n+2)!}\,.$$

Proof.

$$|\mathcal{R}(\alpha_n)| = n!\,|\mathcal{R}_0(\alpha_n)| = n!\,C_{n+1} = \frac{2(2n+1)!}{(n+2)!}\,.$$ □

3.2. Problem (β)

Fix a positive integer n. Let β_n be the arrangement in $\mathbb{R}^n$ with hyperplanes

$$x_i = x_j \text{ for all } i<j \quad\text{and}\quad x_i = 2x_j \text{ for all } i \neq j.$$

Let $\mathcal{R}(\beta_n)$ be the set of regions of β_n, and $\mathcal{R}_0(\beta_n)$ be the set of regions of β_n contained in $x_1 < x_2 < \cdots < x_n$. Note that $|\mathcal{R}(\beta_n)| = n!\,|\mathcal{R}_0(\beta_n)|$.

Let $P^\times_{l,m}$ be the refined poset of $P_{l,m}$ by adding relations $k < (k+1)'$ and $k' < (k+1)$ for all $k \in [l,m]$. For nonnegative integers i, j satisfying $i+j \le n$, let $V_{i,j}$ be the set of pairs (A,B) in $E(P^-_{1,i}) \times E(P^+_{n-j+1,n})$ such that A *does not* end with $i'i$ and B *does not* begin with $(n-j+1)(n-j+1)'$.

Lemma 3.4. *For $n \ge 1$, we have a bijection between the following two sets*:

$$\mathcal{R}_0(\beta_n) \overset{\text{bij}}{\longleftrightarrow} \bigsqcup_{i,j} V_{i,j}\,, \quad (\textit{disjoint union})$$

where the (disjoint) union is over all nonnegative integers i and j such that $i+j \le n$.

Proof. Fix a region R_0 in $\mathcal{R}_0(\beta_n)$. Let i be the largest number such that $x_1, \ldots, x_i$ are always negative in R_0. If there is no such i, take $i = 0$. Similarly, let j be the largest number such that $x_{n-j+1}, \ldots, x_n$ are always positive in R_0. Clearly we have $i+j \le n$. The region R_0 is determined by $x_l < 2x_m$, which gives a relation $l <_P m'$ on the set $P_{1,n}$. Then the negative coordinate part of R_0 corresponds to an element A of $E(P^-_{1,i})$ which doesn't end with $i'i$ – otherwise, x_i may have both signs. Similarly, the positive coordinate part of R_0 corresponds to an element B of $E(P^+_{n-j+1,n})$ which doesn't begin with $(n-j+1)(n-j+1)'$. The middle part of R_0 corresponds to the poset

$P^{\times}_{i+1,n-j}$, which is only one choice. It is easy to show that this correspondence R_0 to (A,B) is well-defined and invertible. □

Theorem 3.5. *For $n \geq 1$, we have*

$$|\mathcal{R}_0(\beta_n)| = C_{n+1} - C_n \,.$$

Proof. Recall that $\phi : E(P^-_{1,i}) \to \mathcal{D}_i$ is a bijection. Moreover, $A \in E(P^-_{1,i})$ ends with $i'i$ if and only if $\phi(A)$ ends with $(+1)(-1)$. Thus

$$\left|\{A \in E(P^-_{1,i}) \mid A \text{ does not end with } i'i\}\right| = C_i - C_{i-1}\,. \tag{1}$$

Similarly we also get

$$\left|\{B \in E(P^+_{n-j+1,n}) \mid B \text{ does not begin with } (n-j+1)(n-j+1)'\}\right| = C_j - C_{j-1}\,. \tag{2}$$

From (1) and (2), we have $|V_{i,j}| = (C_i - C_{i-1})\,(C_j - C_{j-1})$. By Lemma 3.4 we obtain

$$\begin{aligned}
|\mathcal{R}_0(\beta_n)| &= \sum_{i+j\leq n} |V_{i,j}| \\
&= \sum_{i+j\leq n} (C_i - C_{i-1})\,(C_j - C_{j-1}) \\
&= \sum_{i=0}^{n} (C_i - C_{i-1}) \sum_{j=0}^{n-i} (C_j - C_{j-1}) \\
&= \sum_{i=0}^{n} (C_i - C_{i-1})\, C_{n-i} \\
&= C_{n+1} - C_n\,,
\end{aligned}$$

where the last equality is due to the identity $\sum_{i=0}^{n} C_i\, C_{n-i} = C_{n+1}$. □

Corollary 3.6. *The number of regions of β_n is given by*

$$|\mathcal{R}(\beta_n)| = \frac{6n^2\,(2n-1)!}{(n+2)!}\,.$$

Proof.

$$|\mathcal{R}(\beta_n)| = n!\,|\mathcal{R}_0(\beta_n)| = n!\,(C_{n+1} - C_n) = \frac{6n^2\,(2n-1)!}{(n+2)!}\,.$$ □

3.3. Problem (γ)

Fix a positive integer n. Let γ_n be the arrangement in $\mathbb{R}^n$ with hyperplanes

$$x_i = 0 \;\text{ for all } i,\;\; x_i = x_j \;\text{ for all } i<j,\;\; x_i = 2x_j \;\text{ for all } i<j.$$

Let $\mathcal{R}(\gamma_n)$ be the set of regions of γ_n. Given a subset T of $[n]$, let $\mathcal{R}_T(\gamma_n)$ be the set of regions R of γ_n such that the set of negative coordinates of R is $\{x_t \mid t \in T\}$.

Given a subset T of $[n]$, an *embroidered permutation* of T consists of a permutation π of T together with a collection ε of arcs (i, j) such that:

(1) $1 \le i < j \le |T|$ for all $(i, j) \in \varepsilon$.
(2) If $i \le h \le k \le j$, then $(i, j) = (h, k)$. (nonnesting condition)
(3) If $(i, j) \in \varepsilon$, then $\pi_i < \pi_j$.

For a subset T of $[n]$, let $\mathcal{S}_T$ be the set of permutation on T, and $\mathcal{E}_T$ be the set of embroidered permutation on T. For example, given $T = \{1, 3\} \subset [3]$, we have

$$\mathcal{E}_T = \{(13, \emptyset), (13, \{(1, 2)\}), (31, \emptyset)\}.$$

By convention, set $\mathcal{S}_\emptyset := \{\emptyset\}$ and $\mathcal{E}_\emptyset := \{\emptyset\}$.

Lemma 3.7. *For each subset T of $[n]$, we have a bijection between the following two sets*:

$$\mathcal{R}_T(\gamma_n) \overset{\text{bij}}{\longleftrightarrow} \mathcal{E}_T \times \mathcal{E}_{[n]\setminus T}\,.$$

Proof. Fix a region R in $\mathcal{R}_T(\gamma_n)$. Set $w = w_1 w_2 \cdots w_k \in \mathcal{S}_T$ and $w' = w'_1 w'_2 \cdots w'_{n-k} \in \mathcal{S}_{[n]\setminus T}$ defined by

$$x_{w_1} < x_{w_2} < \cdots < x_{w_k} < 0 < x_{w'_{n-k}} < \cdots < x_{w'_2} < x_{w'_1},$$

where $(x_1, x_2, \ldots, x_n) \in R$. Draw an arc (i, j) in w if $i < j$ and $x_i < 2x_j$, and remove redundant arcs (among nesting arcs, outer one). This defines an embroidered permutation A on T. Similarly, draw an arc (i', j') in w' if $i' < j'$ and $x_i > 2x_j$, and remove redundant arcs. This defines an embroidered permutation B on $[n] \setminus T$. □

Since the number of embroidered permutations on a k-set T is given by $(k+1)^{k-1}$ (see [10, p. 468]), we have

$$|\mathcal{R}_T(\gamma_n)| = |\mathcal{E}_T \times \mathcal{E}_{[n]\setminus T}| = (k+1)^{k-1}\,(n-k+1)^{n-k-1}\,. \tag{3}$$

Theorem 3.8. *For $n \ge 1$, we have*

$$|\mathcal{R}(\gamma_n)| = 2\,(n+2)^{n-1}\,.$$

Proof. Since $\{\, \mathcal{R}_T(\gamma_n) \mid T \subseteq [n] \,\}$ consists of a disjoint union of $\mathcal{R}(\gamma_n)$, we have

$$\begin{aligned} |\mathcal{R}(\gamma_n)| &= \sum_{T \subseteq [n]} |\mathcal{R}_T(\gamma_n)| \\ &= \sum_{k=0}^{n} \binom{n}{k} (k+1)^{k-1}\,(n-k+1)^{n-k-1} \qquad (4) \\ &= 2\,(n+2)^{n-1}\,. \end{aligned}$$

Since the number of $\mathbf{x}$-parking functions for $\mathbf{x} = (2, 1, \ldots, 1) \in \mathbb{P}^n$ is counted by (4), the last equation holds. □

Note that the last identity in (4) is the case $x = y = 1$ of the well-known Abel identity (see [8, p. 18])

$$\sum_{k=0}^{n} \binom{n}{k} x(x+k)^{k-1} y(y+n-k)^{n-k-1} = (x+y)(x+y+n)^{n-1}.$$

For more information of $\mathbf{x}$-parking functions, see [4, 5, 11].

3.4. Problem (δ)

Let δ_n be the arrangement in $\mathbb{R}^n$ with hyperplanes

$$x_i = x_j \ \text{ for all } i < j \ \text{ and } \ x_i = 2x_j \text{ for all } i < j.$$

Let $\mathcal{R}(\delta_n)$ be the set of regions of δ_n. Given disjoint subsets S, T of $[n]$, let $\mathcal{R}_{S,T}(\delta_n)$ be the set of regions R of δ_n such that the set of negative coordinates of R is S and the set of positive coordinates of R is T.

Suppose w be an embroidered permutation (π, ε), where $\pi = \pi_1\pi_2 \cdots \pi_l$ is an underlying permutation and ε is a set of nonnesting arcs. We call w *proper* if for some $i \in [1, l-1]$ there exists an arc $(i, l) \in \varepsilon$. Let $\mathcal{Q}_{S,T}$ be the set of pairs (A, B) in $\mathcal{E}_S \times \mathcal{E}_T$ such that both A and B are proper. Then the following lemma can be proved similarly to Lemma 3.7.

Lemma 3.9. *For disjoint subsets S and T of $[n]$, we have a bijection between the following two sets*:

$$\mathcal{R}_{S,T}(\delta_n) \overset{\text{bij}}{\longleftrightarrow} \mathcal{Q}_{S,T} \times \mathcal{S}_{[n]\setminus(S\cup T)} .$$

Note that the number of proper embroidered permutations on a k-set T (say E_k) is given by

$$E_0 = 1, \quad \text{and} \quad E_k = (k+1)^{k-1} - k^{k-1} \ (k \geq 1) .$$

Thus if $|S| = k$ and $|T| = l$, then we have

$$|\mathcal{R}_{S,T}(\delta_n)| = |\mathcal{Q}_{S,T}| \cdot |\mathcal{S}_{[n]\setminus(S\cup T)}| = E_k\, E_l \cdot (n-k-l)! \,. \tag{5}$$

Theorem 3.10. *For $n \geq 1$, we have*

$$|\mathcal{R}(\delta_n)| = 2\,(n+2)^{n-1} - \, 2\,n\,(n+1)^{n-2} .$$

Proof. Let $\mathcal{B} = \{\, (S,T) \in 2^{[n]} \times 2^{[n]} \mid S \cap T = \emptyset \,\}$. Since $\{\, \mathcal{R}_{S,T}(\delta_n) \mid (S,T) \in \mathcal{B} \,\}$ is a disjoint union of $\mathcal{R}(\delta_n)$, from (5) we have

$$\begin{aligned}
|\mathcal{R}(\delta_n)| &= \sum_{(S,T)\in\mathcal{B}} |\mathcal{R}_{S,T}(\delta_n)| \\
&= \sum_{k+l+m=n} \binom{n}{k,l,m} E_k\, E_l\, m! \\
&= n! \sum_{k=0}^{n} \frac{E_k}{k!} \sum_{l=0}^{n-k} \frac{E_l}{l!}
\end{aligned}$$

$$
\begin{aligned}
&= n! \sum_{k=0}^{n} \frac{E_k}{k!} \left[1 + \sum_{l=1}^{n-k} \left(\frac{(l+1)^{l-1}}{l!} - \frac{l^{l-2}}{(l-1)!} \right) \right] \\
&= n! \sum_{k=0}^{n} \frac{E_k}{k!} \frac{(n-k+1)^{n-k-1}}{(n-k)!} \\
&= (n+1)^{n-1} + n! \sum_{k=1}^{n} \left(\frac{(k+1)^{k-1}}{k!} - \frac{k^{k-2}}{(k-1)!} \right) \frac{(n-k+1)^{n-k-1}}{(n-k)!} \\
&= \sum_{k=0}^{n} \binom{n}{k} (k+1)^{k-1} (n-k+1)^{n-k-1} - \sum_{k=1}^{n} \binom{n}{k} k^{k-3} (n-k+1)^{n-k-1} \\
&= 2\,(n+2)^{n-1} - 2\,n\,(n+1)^{n-2},
\end{aligned}
$$

where the last equality is by (4). □

4. Remarks

By the finite field method one can obtain the characteristic polynomials of hyperplane arrangements α_n and β_n, which are given by

$$
\begin{aligned}
\chi_{\alpha_n}(t) &= (t-1) \cdot (t-n-2)_{n-1}, \\
\chi_{\beta_n}(t) &= (t-1) \cdot (t^2 - (3n-1)t + 3n^2 - 3n) \cdot (t-n-2)_{n-3},
\end{aligned}
$$

where $(x)_m = x(x-1)\cdots(x-m+1)$. Note that $r(\mathcal{A})$ can be derived from its characteristic polynomial $\chi_{\mathcal{A}}(t)$ by Zaslavsky's formula [12]

$$r(\mathcal{A}) = (-1)^n \chi_{\mathcal{A}}(t).$$

Thus it would be interesting to find combinatorial proof for the characteristic polynomials of α_n and β_n, which will be a generalization of our results. Also, a combinatorial enumeration of the characteristic polynomials of γ_n and δ_n can be asked for.

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