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What Distinguishes Mathematical Experience from Other Kinds of Experience?[1]

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Investigating students' lived mathematical experiences presents dual challenges for the researcher. On the one hand, we must respect that students' experiences are not directly accessible to us and are likely very different from our own experiences. On the other hand, we might not want to rely upon the students' own characterizations of what constitutes mathematics because these characterizations could be limited to the formal products students learn in school. I suggest a characterization of mathematics as objectified action, which would lead the researcher to focus on students' operations—mental actions organized as objects within structures so that they can be acted upon. Teachers' and researchers' models of these operations and structures can be used as a launching point for understanding students' experiences of mathematics. Teaching experiments and clinical interviews provide a means for the teacher-researcher to infer students' available operations and structures on the basis of their physical activity (including verbalizations) and to begin harmonizing with their mathematical experience.

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WHAT IS MATHEMATICS

Mathematics has been characterized as the science of number, patterns, and relationships. It has also been characterized as a language. But certain aspects of mathematics distinguish it among all other sciences and all other languages. For example, no other science approaches the stability of mathematics. Whereas the replacement of one theory for another is common in science, in mathematics theories are only expanded and generalized,

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[1] A draft version of the article was presented at the 2015 KSME International Conference on Mathematics Education held at Seoul National University, Seoul 08826, Korea; November 6–8, 2015 (*cf.* Norton, 2015).

or further specified and justified. For its apparent timelessness and certainty, mathematics is unique among all human activity. So, what endows mathematics with these qualities, and how can we characterize human experience in this domain?

"I like the way mathematics builds on itself." "In mathematics, there is always a right answer." Some mathematics educators might cringe at such clichés from their students, but these notions have a basis in the history of mathematics and the nature of mathematical experience—not just some formalist perspective foisted upon students in our educational system. Consider, for example, the historical development of the cubic formula.

In 16th century Italy, Western mathematics underwent a Renaissance of its own, but was not born out of nothing (Burton, 2007). Rather, mathematicians showed renewed interest in the problems and methods presented by Diaphantus centuries earlier. Even by Diaphantus' time, in 3rd century Greece, scholars knew how to solve quadratic equations, using geometric methods similar to completing the square. Aided by further development of algebraic notation, del Ferro[2] and Tartaglia[3] began making progress in developing formulas for finding solutions to generic cubic equations. By making the clever substitution, $x=a-b$, they leveraged the algebraic identity, $(a-b)^3+3ab(a-b)=a^3-b^3$, to solve all cubic equations of the form $x^3-px=q$ (with p and q positive rational numbers) in terms of a and b. As such, they were taking full advantage of numerous mathematical developments up to that time, including algebraic methods developed over the centuries, from the ancient Greeks, to Diaphantus, to al-Khowârizmî, and Vièta.

Cardan built upon this work further by showing that one more simple substitution could reduce all cubic equations (again, with positive rational coefficients) to the special form, thus providing a general cubic formula. A few years later, Ferrari built on that work still further, to develop a quartic formula. His method relied on completing the square to reduce quartic equations to cubic equations. And in the midst of this Italian mathematical flourish, Bombelli used these methods and solutions to build a foundation for imaginary numbers, by showing that imaginary numbers could sometimes generate real solutions.

Imaginary numbers did not replace real numbers but, rather, extended them, just as Galois Theory extended the work of del Ferro, Tartaglia, Ferrari, and Cardan, in demon-

[2] (Added by Editors) Scipione del Ferro (1465–1526) was an Italian mathematician who first discovered a method to solve the depressed cubic equation). Extracted from: https://en.wikipedia.org/wiki/Scipione_del_Ferro

[3] (Added by Editors) Niccolò Fontana Tartaglia (1499/1500–1557) was an Italian mathematician, an engineer, a surveyor and a bookkeeper from the then-Republic of Venice (now part of Italy). He published many books, including the first Italian translations of an Archimedes and Euclid, and an acclaimed compilation of mathematics. Tartaglia was the first to apply mathematics to the investigation of the paths of cannonballs, known as ballistics, in his Nova Scientia, "A New Science;" his work was later partially validated and partially superseded by Galileo's studies on falling bodies). Extracted from: https://en.wikipedia.org/wiki/Niccol%C3%B2_Fontana_Tartaglia

strating the futility of seeking a quintic formula. Although we rarely use the cubic and quartic formulas, these methods are just as valid today as they were five centuries ago—generating perfectly accurate solutions (“right answers”) every time. Contrast this with Newton’s mechanics, which is still useful in limited circumstances, but which was invalidated by relativity and the Standard Model of physics. Relativity did not build on Newton’s mechanics like the quartic formula built on the quadratic formula. Here we have an illustration of what distinguishes mathematics from the sciences.

We might say that the uniqueness and apparent certainty of mathematics owes to its non-empirical nature. As Bertrand Russell quipped, “mathematics may be defined as the subject in which we never know what we are talking about, nor whether what we are saying is true.” This is because mathematics does not make ontological claims, its only criterion being internal consistency. Although Gödel demonstrated that we could never prove the internal consistency of the system as a whole, we consistently find consistency in its development, and this consistency owes to the psychology of mathematics.

Tall, Thomas, Davis, Gray & Simpson (1999) noted a common theme in many of the theories explaining how students construct mathematics. Whether in terms of Piaget’s reflective abstraction, Sfard’s (1991) theory of reification, or Dubinsky’s (1991) APOS, students’ progress from performing actions on objects, to treating those actions as objects upon which to act. As a simple example, consider the construction of the mathematical object, 5.

Young children initially understand 5 as a member of a verbal counting sequence, not so different from the letter E in the alphabet. The number 5 starts to take on a cardinal meaning when the child begins to coordinate pointing acts with her verbal utterances, in one-to-one correspondence. Still, the child has to reproduce each instantiation of 5 through that coordinated activity (Steffe, 1992). For children at this stage, answering the question, “how many 1’s are there in 5,” is a genuine problem, which they would likely solve by counting out five fingers. Only after this activity is interiorized, does 5 become an object for them. At this point, questions that involve acting on 5 begin to make sense (e.g., how much is four 5’s?).

The basic nature of this development is illustrated in Figure 1. The number 5, which had existed only through the students’ coordinated activity, becomes an object (top arrow) that, as an object, the student can now act upon it (bottom arrow).

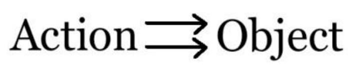

Figure 1. Mathematics as objectified action.

In fact, we can use this simple illustration to trace a rough progression from elementary to advanced mathematics, as outlined in Table 1.

Table 1. An action-object progression in algebra.

Actions	Objects	Sources
Coordinated counting	Whole numbers	Steffe, 1992
Composing whole numbers (as in addition or multiplication)	Composite numbers: additive and multiplicative structures	Steffe & Olive, 2010; Ulrich, 2012
Mapping additive to multiplicative structures	Exponential relationships	Confrey & Smith, 1995
Performing additive, multiplicative, and exponential operations on unknowns	Algebraic expressions and equations	Hackenberg, 2005
Mapping algebraic expressions to unknowns	Algebraic functions	Sfard & Linchevski, 1994
Composing and structuring (reversible) operations	Abstract algebraic structures (e.g., groups, rings, and fields)	Dubinsky, Dautermann, Leron, & Zazkis, 1994
Mapping algebraic structures to algebraic structures	Homomorphisms	Leron, Hazzan, & Zazkis, 1995

When high school algebra teachers ask their students to graph the function, $y=2x^2-7$, they want the students to understand the graph and its equation as representing the same *object*—an invariant relationship between to varying quantities. However, for many of their students, even the expression $2x^2-7$, remains a command for *action*, and not a transformation that might map x to a new variable, y. Thus, the task of graphing the function becomes an activity of point plotting, with no consideration of the invariant relationship the resulting graph might represent.

If we cannot sympathize with the struggles such students experience, try considering homomorphisms as objects. We can remind ourselves of the definition of a homomorphism, just as we can define function for our students: A group homomorphism, say, is a mapping from one group to another that maps the product of two elements to product of their mappings. If this definition is not helpful, we could consider an example: $f(x)=e^x$ is a homomorphism (and, furthermore, an isomorphism) from the group of integers under addition to the group of positive real numbers under multiplication because $e(a+b)=e^a \bullet e^b$. Of course, making sense of the definition and example demands that we have already constructed groups as objects on which to act. But even if groups are objects for us, and even if $f(x)=e^x$ is an object for us, as a function, are homomorphsisms objects for us? Can we yet consider groups and mappings of homomorophsisms? "For an experienced mathematician, it is possible to think of a homomorphism as a single unified concept, indeed, as a single object" (Leron, Hazzan, & Zazkis, 1995, p. 153). Even if I can objectify homomorphisms, the progression continues, building further empathy for the experience of algebra students: For every professional algebraist in the world, a cohomology is also an object to act upon, but for me it is only a diagram of transformations to act within.

The objectification of action is the unifying theme in mathematical experience that makes the experience mathematical. If we want to understand the lived mathematical experiences of students, we need to consider their mental actions, the objects they act upon, and the origins of those objects, as objectified actions. As researchers, we have no access to students' mental actions, any more than we have access students' mathematical experiences themselves. However, through our interactions with students, we can infer those actions, build structured models of their organization, and then rely on these models to understand how students experience mathematics.

BUILDING MODELS OF STUDENTS' MATHEMATICS

According to Piaget's *Structuralism* (1970), actions form the basis for all logico-mathematical knowledge. Beyond becoming internalized as mental actions (operations) that students can carry out in imagination, what renders an action mathematical is its organization within a structure of related operations. This is not to say that the student is aware of the structure, but rather that, from an observer's perspective, the student composes the operations with one another in a way that follows logical rules, similar to those of a mathematical group. In particular, mathematical operations are closed, in the sense that composing two operations within the same structure results in another operation within that same structure; and mathematical operations are reversible, in the sense that, for every operation, there is an inverse operation that undoes it.

For example, consider the group of displacements, which children construct in their first year of life, along with their construction object permanence and space (Piaget, 1967/1948). Children learn that any potential action involving the movement of an object from one location to another can be combined with other movements to produce new movements, including an inverse movement that returns the object to its original location. Children know nothing of mathematical groups, but from observations of their physical actions, we can infer such an underlying structure that explains and predicts how they might reason with space.

Once constructed, students can use their operations in purposeful activity, in an attempt to control the world they experience. In fact, as humans, we all do this every day, often outside of our own awareness. For example, when making spaghetti for my family, I estimate the amount of noodles to boil based on how much each person would get if it were split evenly. I do not make this estimate this by imagining the desired share for one person and then measuring out, or iterating, four of these shares. Instead, I operate in reverse, by continually partitioning the amount thrown in the pot until the four parts reach the desired amount, knowing that this will yield the same result. I know this action will yield the same result because partitioning and iterating are inverse operations for me—

part of a group-like structure described in Wilkins & Norton (2011). What makes the experience mathematical for me, is not that I considered it mathematical at the time, but rather that I relied upon a structure for composing mental actions that I did not need to perform; and I was, thus, acting on my mental actions (see Figure 1) within that structure.

By working intensively with individual or small groups (usually pairs) of students, teaching experiments and clinical interviews provide a means for researchers to infer when a student's actions indicate a similar reliance on mathematical structures. Teaching experiments (Steffe & Thompson, 2000) in particular provide opportunities for longitudinal interactions between the teacher-researcher and the students. The teacher-researcher's goal during these interactions is to provoke student activity and to build *second-order models* of the students' thinking that explain that activity. Over the course of a teaching experiment (usually several weeks), the researcher makes use of retrospective analysis (usually using video recordings) to modify and refine the explanatory models. The teacher-researcher also designs tasks for future sessions with which to test these models by predicting how the students will respond to the tasks. This iterative process of refining and testing the model continues until the researcher's models reach stability, consistently explaining observations of the students' actions and verbalizations across sessions.

The models mathematics education researchers construct often rely upon a second kind of Piagetian structure—operational schemes—in which mathematical operations form the key elements (Steffe & Olive, 2010). Operational schemes describe students' ways of operating, including a description of the situations that students assimilate into that way of operating, the combination of operations activated in those situations, and the expected result of operating. Whereas group-like structures can be used to describe how operations can be organized and composed with each other (and, thus, acted upon), schemes describe how those operations can be used to resolve problematic situations.

Because the models developed from teaching experiments explain and predict observations, they qualify as scientific models. Because we cannot access to students' mathematical experiences directly, we can do no better. On the other hand, we might rely on these scientific models to interpret aspects of students' lived mathematical experiences, beyond descriptions of structures that explain and predict their mathematical activity.

LIVED MATHEMATICAL EXPERIENCE

The first section of this paper characterized mathematics as the objectification of action: Mathematical actions are actions that can be interiorized as objects to act upon. The second section shared Piaget's structuralism as a theoretical framework for building models of students' mathematics. Thus, we have identified a means for determining whether an experience is mathematical and for explaining how students might operate within those experiences. In this section, we consider approaches to understanding the mathematical experiences in which students operate.

The difficulty in understanding the mathematical experience of another person stems from the fact that this experience precedes any form of language by which it might be communicated. Even when we attempt to understand our own mathematical experience, through introspection, we have to dig beneath our explanation of it. As Norretranders (1998) described in "*The User Illusion*," lived experience precedes the narratives that define our conscious lives, and we often lie to ourselves in constructing those narratives. Attempting to access and share raw experience is the domain of art and poetry. However, the models we construct of students' mathematics can provide starting points for empathy.

In building models of students' mathematics, we validate and stipulate their ways of operating. As such, we put ourselves in a position to participate in their mathematical activity. On the other hand, our affective responses to this activity and the particular imagery it invokes—aspects of experience inextricably tied to cognition—will likely differ considerably from those of the students. Understanding how students experience mathematics requires that we attempt to not only participate in, but to co-participate with the students' mathematical activity. Hackenberg (2010) has described such attempts in terms of *mathematical caring relations*: "a quality of interaction between a student and a teacher that conjoins affective and cognitive realms in the process of aiming for mathematical learning" (p. 237). Mathematical caring relations rely on models of students' mathematics to inform the teacher-researcher's balance between challenging students' ways of operating and harmonizing with those ways of operating.

We should also recognize that students are building models of us, as teacher-researchers, even as we build models of them. In fact, communication in general can be understood as a process of reciprocal model building (von Glasersfeld, 1995), where each participant in a conversation chooses her words and gestures to evoke mental actions, images, and affective responses from the other participants, in an attempt to convey a thought or feeling—attempts to be understood. Furthermore, these participants make use of queues read through eye contact and body language, for example, in order to test and refine *third-order models* of the other participants' models of the speaker herself. In the context of a teaching experiment, we can foster mathematical communication and harmony by describing our models of students' mathematics to the students themselves, so they can react to them, thus supporting refinements to the students' third-order models of our second-order models of them, as well as our third-order models of the students' second-order models of us.

Students want to be understood as much as they want to understand. Building second-order models of students' mathematics provides a starting point for understanding their mathematical experience. These second-order models prescribe mathematical activity that we can participate in, and they help us identify that activity as mathematical. Although raw experience may remain elusive, even to ourselves, we can begin to understand how students experience mathematical activity if we co-participate in that activity. And we can further harmonize with the experience of that activity by communicating our second-order models to the students themselves.

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