J. Appl. Math. & Informatics Vol. **34***(2016), No. 5 - 6, pp. 495 - 508*
http://dx.doi.org/10.14317/jami.2016.495

A SCHWARZ METHOD FOR FOURTH-ORDER SINGULARLY PERTURBED REACTION-DIFFUSION PROBLEM WITH DISCONTINUOUS SOURCE TERM

M. CHANDRU* AND V. SHANTHI

ABSTRACT. A singularly perturbed reaction-diffusion fourth-order ordinary differential equation(ODE) with discontinuous source term is considered. Due to the discontinuity, interior layers also exist. The considered problem is converted into a system of weakly coupled system of two second-order ODEs, one without parameter and another with parameter ε multiplying highest derivatives and suitable boundary conditions. In this paper a computational method for solving this system is presented. A zero-order asymptotic approximation expansion is applied in the second equation. Then, the resulting equation is solved by the numerical method which is constructed. This involves non-overlapping Schwarz method using Shishkin mesh. The computation shows quick convergence and results presented numerically support the theoretical results.

AMS Mathematics Subject Classification : 65L11, CR G1.7.
Key words and phrases : Singularly Perturbed Problems, Ordinary Differential Equations, Reaction-Diffusion, Interior layer, Higher Order.

1. Introduction

Singular Perturbation Problems (SPPs) arises in several branches of engineering and applied mathematics, including fluid flow at high Reynolds numbers, heat and mass transfer at high Péclet numbers, chemical reaction, control theory, semi conductor devices, nuclear physics, etc. It is well-known fact that the solution of these problems have multi-scale character. That is, there are thin layers where the solution varies rapidly, while away from the layer(s) the solutions behaves regularly and varies slowly. So, the numerical treatment of SPPs gives major computational difficulties and in recent years a large number of special purpose methods have been proposed to provide accurate numerical

Received March 5, 2016. Revised June 13, 2016. Accepted June 21, 2016. *Corresponding author.

solutions. For more detailed discussion on the analytical and numerical treatment of SPPs we may refer the reader to the books [1] - [5] and a very recent literature survey [6]. Miller et al. [8] considered a parameter-uniform Schwarz method for a singularly perturbed reaction-diffusion problem with an interior layer. Kopteva et al. [9] discussed an overlapping Schwarz method for a singularly perturbed semi-linear reaction-diffusion problem with multiple solutions. They have used Bakhvalov and Shishkin type meshes to obtain second-order convergence. Chandru and Shanthi applied a boundary value technique for singularly perturbed boundary value problem of reaction-diffusion type with discontinuous source term in [10]. The same type of problem evaluated using hybrid difference scheme to obtain second-order convergence [11]. A fitted mesh method for singularly perturbed Robin type boundary value problem with non-smooth data discussed in [12]. Recently, Chandru et al. evaluated a two parameter singularly perturbed problem with discontinuous source term using hybrid difference scheme [13]. But only in few papers, numerical methods for higher-order differential equations with smooth and non-smooth cases are developed. Some methods are available in the literature in order to obtain numerical solution to singularly perturbed fourth-order differential equations when f is smooth on $\Omega = (0, 1)$ [7], [14] - [17] and f is non-smooth on Ω [18]. Motivated by the works of [8, 18], a fourth-order singularly perturbed reaction-diffusion boundary value problem with discontinuous source term is considered:

$$-\varepsilon y^{iv}(x) + b(x)y''(x) - c(x)y(x) = -f(x), \; x \in \Omega^- \cup \Omega^+, \tag{1}$$

$$y(0) = p, y(1) = q, y''(0) = -r, y''(1) = -s, \tag{2}$$

where $0 < \varepsilon << 1$ is a singular perturbation parameter. Define $\Omega^- = (0, d)$, $\Omega^+ = (d, 1)$, $d \in \Omega$, to indicate the jump at d in any function $[w](d) = w(d+) - w(d-)$, $b(x)$ on $(\Omega^- \cup \Omega^+)$ and $c(x)$ on $\overline{\Omega} = [0, 1]$ such that

$$b(x) \geq \beta > 0, \quad \textit{for some positive constant } \beta, \tag{3}$$

$$0 \geq c(x) \geq -\gamma, \;\; \gamma > 0, \tag{4}$$

$$\beta - \theta\gamma \geq \eta > 0, \;\; \textit{for some } \theta \textit{ arbitrarily close to 2.} \tag{5}$$

Further it is assumed that f is sufficiently smooth on $\overline{\Omega} \setminus \{d\}$; a single discontinuity in the source term $f(x)$ occurs at a point $d \in \Omega$; *f(x)* and its derivatives have jump discontinuity at the same point. In general this discontinuity gives rise to interior layers in the second derivative of the exact solution of the problem. As f is discontinuous at d the solution y of (1)-(2) does not necessarily have a continuous fourth derivative at the point d. Thus $y \notin C^4(\Omega)$. However, the third derivative of the solution exists and is continuous.

This paper is organized as follows. Section 2 presents analytic behavior of the solution of the system of SPP (1)-(5). Some analytical and numerical results for second-order singularly perturbed boundary value problem with discontinuous source terms are described in Section 3.1, numerical scheme in Section 3.2

and truncation error analysis estimated in Section 3.3. The computational technique for the considered problem is discussed in Section 4. Section 5 explains the error estimates for the numerical solution. Numerical example is solved in Section 6. The paper ends with a conclusion. Throughout this paper, C denotes a generic positive constant that is independent of nodal point (i), number of mesh point (N) and the singular perturbation parameter ε. We use the norm $\|w\| = \sup_{x\in\Omega} |w(x)|$. Further $|\overline{y}(x)|$ means $(|y_1(x)|, |y_2(x)|)^T$.

2. Some analytical results

In this section we derive a maximum principle for the following problem. Then using this principle, a stability result for the same problem is derived. Further, an asymptotic expansion approximation is constructed for the solution. Using the transformation $y = y_1$ and $y_1'' = -y_2$, the SPBVP (1)-(2) can be transformed into an equivalent problem of the form

$$\begin{cases} P_1\overline{y}(x) = -y_1''(x) - y_2(x) = 0, \ x \in \Omega \\ P_2\overline{y}(x) = -\varepsilon y_2''(x) + b(x)y_2(x) + c(x)y_1(x) = f(x), \ x \in (\Omega^- \cup \Omega^+), \end{cases} \tag{6}$$

$$\begin{cases} y_1(0) = p, \ y_1(1) = q, \\ y_2(0) = r, \ y_2(1) = s, \end{cases} \tag{7}$$

where $\overline{y} = (y_1, y_2)^T, \ y_1 \in C^2(\overline{\Omega}) \cap C^3(\Omega) \cap C^4(\Omega^- \cup \Omega^+), \ y_2 \in C^0(\overline{\Omega}) \cap C^1(\Omega) \cap C^2(\Omega^- \cup \Omega^+)$ [18]. The proof of the Theorem 1-3 are obtained by following the steps defined in [18].

Theorem 1. *The BVP (1)-(2) has a solution* $y \in C^2(\overline{\Omega}) \cap C^3(\Omega) \cap C^4(\Omega^- \cup \Omega^+)$.

Theorem 2 (Maximum principle)**.** *Suppose that* $\overline{y} = (y_1, y_2)^T, y_1 \in C^2(\overline{\Omega}) \cap C^3(\Omega) \cap C^4(\Omega^- \cup \Omega^+), \ y_2 \in C^0(\overline{\Omega}) \cap C^1(\Omega) \cap C^2(\Omega^- \cup \Omega^+)$, *satisfies* $\overline{y}(0) \geq \overline{0}, \ \overline{y}(1) \geq \overline{0}$ *and* $P_1\overline{y}(x) \geq 0, \ \forall \ x \in \Omega, P_2\overline{y}(x) \geq 0, \ \forall \ x \in \Omega^- \cup \Omega^+$ *and* $[y_2]'(d) \leq 0$. *Then* $\overline{y}(x) \geq \overline{0}, \ \forall \ x \in \overline{\Omega}$.

Theorem 3 (Stability result)**.** *Consider the BVPs (6)-(7) subject to conditions (3)-(5). If* $y_1 \in C^2(\overline{\Omega}) \cap C^3(\Omega) \cap C^4(\Omega^- \cup \Omega^+), \ y_2 \in C^0(\overline{\Omega}) \cap C^1(\Omega) \cap C^2(\Omega^- \cup \Omega^+)$, *then*

$$|y_i(x)|_{\overline{\Omega}} \leq Cmax\{|y_1(0)|, |y_1(1)|, |y_2(0)|, |y_2(1)|, \|P_1\overline{y}\|_{\Omega}, \|P_2\overline{y}\|_{\Omega^- \cup \Omega^+}\}, i = 1, 2.$$

2.1. Some asymptotic expansion approximation.

Consider the BVP (6)-(7). Using the perturbation method defined in [2, 15], we can construct an asymptotic expansion for the solution of the BVP (6)-(7) as follows. Let $\overline{u}_{l0} = (u_{l01}, u_{l02})$, and $\overline{u}_{r0} = (u_{r01}, u_{r02})$ be the solutions of the reduced problem (6)-(7).

$$\begin{cases} -u_{l01}''(x) - u_{l02}(x) = 0, \\ b(x)u_{l02}(x) + c(x)u_{l01}(x) = f(x), \quad x \in \Omega^-, \end{cases} \tag{8}$$

$$\begin{cases} -u''_{r01}(x) - u_{r02}(x) = 0, \\ b(x)u_{r02}(x) + c(x)u_{r01}(x) = f(x), \quad x \in \Omega^+, \end{cases} \tag{9}$$

subject to the conditions

$$u_{l01}(0) = p, \; u_{l01}(d-) = u_{r01}(d+), \tag{10}$$

$$u'_{l01}(d-) = u'_{r01}(d+), \; u_{r01}(1) = q \tag{11}$$

and $\overline{v}_{l0} = (v_{l01}, v_{l02}),\; \overline{v}_{r0} = (v_{r01}, v_{r02})$ are layer correction terms given by

$$v''_{l01}(x) = -v_{l02}, \; v_{l02} = k_1 e^{-x\sqrt{b(0)/\varepsilon}},$$

$$v''_{r01}(x) = -v_{r02}, \; v_{r02} = k_2 e^{-(x-d)\sqrt{b(d)/\varepsilon}},$$

Here,

$$\begin{aligned}
k_1 &= [y_2(0) - u_{l02}(0)] - k_2 e^{-d\sqrt{b(d)/\varepsilon}}, \\
k_2 &= \frac{k_{21}\left\{\sqrt{b(d)} + \sqrt{b(1)}e^{-(1-d)\sqrt{(b(d)+b(1))/\varepsilon}}\right\}}{k_{24}+k_{25}} \\
&\quad + \frac{(k_{22}+k_{23})\left\{1 - e^{-(1-d)\sqrt{(b(d)+b(0))/\varepsilon}}\right\}}{k_{24}+k_{25}}, \\
k_{21} &= \left\{[y_2(1) - u_{r02}(1)]e^{-(1-d)\sqrt{b(1)/\varepsilon}} - [y_2(0) - u_{l02}(0)]e^{-d\sqrt{b(0)/\varepsilon}}\right\} \\
&\quad + [u_{r02}(d+) - u_{l02}(d-)], \\
k_{22} &= \sqrt{b(1)}[y_2(1) - u_{r02}(1)]e^{-(1-d)\sqrt{b(1)/\varepsilon}} + \sqrt{b(0)}[y_2(0) - u_{l02}(0)] \\
&\quad e^{-d\sqrt{b(0)/\varepsilon}}, \\
k_{23} &= \sqrt{\varepsilon}[u'_{r02}(d+) - u'_{l02}(d-)], \\
k_{24} &= \left(1 - e^{-d\sqrt{(b(0)+b(d))/\varepsilon}}\right)\left(\sqrt{b(d)} + \sqrt{b(1)}e^{-(1-d)\sqrt{(b(d)+b(1))/\varepsilon}}\right), \\
k_{25} &= \left(1 - e^{-(1-d)\sqrt{(b(d)+b(1))/\varepsilon}}\right)\left(\sqrt{b(d)} + \sqrt{b(0)}e^{-d\sqrt{(b(0)+b(d))/\varepsilon}}\right).
\end{aligned}$$

Now let $\overline{w}_{l0} = (w_{l01}, w_{l02}), \overline{w}_{r0} = (w_{r01}, w_{r02})$ be the right-layer corrections given by

$$w''_{l01} = -w_{l02}, \qquad w_{l02} = k_3 e^{-(d-x)\sqrt{b(d)/\varepsilon}},$$

$$w''_{r01} = -w_{r02}, \qquad w_{r02} = k_4 e^{-(1-x)\sqrt{b(1)/\varepsilon}},$$

where

$$\begin{aligned}
k_3 &= \frac{k_{31} - k_{32}}{k_{33}}, \\
k_{31} &= [y_2(0) - u_{l02}(0)]e^{-d\sqrt{b(0)/\varepsilon}} + k_2\left(1 - e^{-d\sqrt{(b(0)+b(d))/\varepsilon}}\right),
\end{aligned}$$

$$k_{32} = [y_2(1) - u_{r02}(1)]e^{-(1-d)\sqrt{b(1)/\varepsilon}} + [u_{r02}(d+) - u_{l02}(d-)],$$
$$k_{33} = \left(1 - e^{-(1-d)\sqrt{(b(d)+b(1))/\varepsilon}}\right),$$
$$k_4 = [y_2(1) - u_{r02}(1)] - k_3 e^{-(1-d)\sqrt{b(d)/\varepsilon}}.$$

The values of k_1, k_2, k_3, k_4 are determined by imposing the following boundary and continuity conditions:

$$y_{2,as}(0) = y_2(0),\ y_{2,as}(1) = y_2(1),$$
$$y_{2,as}(d-) = y_{2,as}(d+),\ y'_{2,as}(d-) = y'_{2,as}(d+).$$

Let

$$\overline{y}_{as} = (y_{1,as}, y_{2,as}),\ \text{where} \quad \begin{cases} y_{1,as} = u_{01} + v_{01} + w_{01}, \\ y_{2,as} = u_{02} + v_{02} + w_{02}, \end{cases} \quad \forall\ x \in \overline{\Omega},$$

where $u_{01} \in C^2(\overline{\Omega})$,

$$u_{01} = \begin{cases} u_{l01}(x), & x \in \Omega^-, \\ u_{l01}(d-) = u_{r01}(d+), & x = d \\ u_{r01}(x), & x \in \Omega^+, \end{cases} \qquad u_{02} = \begin{cases} u_{l02}(x), & x \in \Omega^-, \\ u_{l02}(d-) = u_{r02}(d+), & x = d \\ u_{r02}(x), & x \in \Omega^+, \end{cases}$$

$$v_{01} = \begin{cases} v_{l01}(x), & x \in \Omega^-, \\ v_{l01}(d-) = v_{r01}(d+), & x = d \\ v_{r01}(x), & x \in \Omega^+, \end{cases} \qquad w_{01} = \begin{cases} w_{l01}(x), & x \in \Omega^-, \\ w_{l01}(d-) = w_{r01}(d+), & x = d \\ w_{r01}(x), & x \in \Omega^+. \end{cases}$$

Remark 1. If (u_{l01}, u_{r02}) are the solution of (8)-(11), then u_{01} is the solution of the BVP

$$-u''_{01} + [c(x)/b(x)]u_{01}(x) = f(x)/b(x), \quad \forall\ x \in \Omega^- \cup \Omega^+ \tag{12}$$
$$u_{01}(0) = p, u_{01}(1) = q, u_{01}(d-) = u_{01}(d+), u'_{01}(d-) = u'_{01}(d+). \tag{13}$$

In the following it is assumed that BVP (12)-(13) can be solved exactly and closed form solution is available. This problem has a unique solution $u_{01} \in C^0(\overline{\Omega}) \cap C^1(\Omega) \cap C^2(\Omega^- \cup \Omega^+)$ [18].

Theorem 4. *The zero-order asymptotic expansion approximation $\overline{y}_{as}$ of the solution $\overline{y}(x)$ of (6)-(7) satisfies the inequality*

$$|y_i(x) - y_{i,as}(x)| \le C\sqrt{\varepsilon},\ x \in \overline{\Omega},\ i = 1, 2,$$

where $y_{1,as} \in C^2(\overline{\Omega}) \cap C^3(\Omega) \cap C^4(\Omega^- \cup \Omega^+)$, $y_{2,as} \in C^0(\overline{\Omega}) \cap C^1(\Omega) \cap C^2(\Omega^- \cup \Omega^+)$.

Proof. It is easy to verify that $y_{1,as} \in C^2(\overline{\Omega}) \cap C^3(\Omega) \cap C^4(\Omega^- \cup \Omega^+)$ and $y_{2,as} \in C^0(\overline{\Omega}) \cap C^1(\Omega) \cap C^2(\Omega^- \cup \Omega^+)$. Defining barrier functions $\overline{\phi}^{\pm} = \left(\phi_1^{\pm}, \phi_2^{\pm}\right)$ as

$$\phi_1^{\pm} = C(1+\delta)\left(1 - \frac{x^2}{2}\right)\sqrt{\varepsilon} \pm (y_1 - u_{01})(x)$$
$$\phi_2^{\pm} = C\sqrt{\varepsilon} \pm (y_2 - y_{2,as})(x),$$

it is easy to verify that

$$P_1\overline{\phi}^{\pm} \geq 0,\ P_2\overline{\phi}^{\pm} \geq 0, \quad \forall\ x \in (\Omega^- \cup \Omega^+),$$

and

$$\phi_1^{\pm}(0) \geq 0, \phi_1^{\pm}(1) \geq 0, \phi_2^{\pm}(0) \geq 0, \phi_2^{\pm}(1) \geq 0, [\phi_2^{\pm}]'(d) = 0$$

for a suitable selection of C. Then, by Theorem 2 we have the required result. □

3. Some analytical and numerical results for SPBVP for second-order ODEs with discontinuous source terms

We state some results for the following SPBVP which are needed in the rest of the paper. Consider the SPBVP

$$Ly_2^*(x) = -\varepsilon {y_2^*}''(x) + b(x)y_2^*(x) = f(x) - c(x)u_{01}(x),\ x \in (\Omega^- \cup \Omega^+), \tag{14}$$

$$y_2^*(0) = r, y_2^*(1) = s, \tag{15}$$

Remark 2. The BVP (14)-(15) has a unique solution $y_2^* \in C^0(\overline{\Omega}) \cap C^1(\Omega) \cap C^2(\Omega^- \cup \Omega^+)$ [19].

3.1. Analytical results.

Theorem 5. *If (y_1, y_2) and y_2^* are solutions of the BVPs (6)-(7) and (14)-(15), respectively, then*

$$|(y_2 - y_2^*)(x)| \leq C\sqrt{\varepsilon},\ x \in \overline{\Omega}.$$

Proof. Since (y_1, y_2) are the solution of (6)-(7), then y_2 satisfies the BVP

$$-\varepsilon y_2''(x) + b(x)y_2(x) = f(x) - c(x)y_1(x),\ x \in (\Omega^- \cup \Omega^+),$$
$$y_2(0) = r,\ y_2(1) = s.$$

Further, the function $w = y_2 - y_2^*$ satisfies the BVP

$$-\varepsilon w''(x) + b(x)w(x) = f(x) - c(x)[y_1(x) - u_{01}(x)],\ x \in (\Omega^- \cup \Omega^+),$$
$$w(0) = 0,\ w(1) = 0,\ [w]'(d) = 0.$$

From Theorem 4 and the definition of v_{01} and w_{01}, we have

$$\begin{aligned} |(y_1 - u_{01})(x)| &\leq |y_1(x) - (u_{01} + v_{01} + w_{01})(x)| + |(v_{01} + w_{01})(x)| \\ &\leq C\sqrt{\varepsilon} + C\varepsilon, \end{aligned}$$

that is,

$$|(y_1 - u_{01})(x)| \leq C\sqrt{\varepsilon}.$$

From this inequality and the stability result given in [19] we have

$$|w(x)| \leq C\sqrt{\varepsilon}$$

that is,

$$|(y_2 - y_2^*)(x)| \leq C\sqrt{\varepsilon}.$$

□

The proof of the Theorem 6-7 are obtained by following the steps defined in [5].

Theorem 6. *Let $y_2^*(x)$ be the solution of (L) and $y_2^{*[k]}(x)$ be the corresponding sequences of Schwarz iterates. Then, for all $k \geq 1$*

$$\|y_2^{*[k]} - y_2^*\|_{\overline{\Omega}} \leq Cq^k$$

where C *is independent of* k *and* ε *and*

$$q = e^{-\alpha\left(\xi^+ - \xi^-\right)/\varepsilon} < 1$$

The solution $y_2^*(x)$ of (L) is decomposed as

$$y_2^*(x) = u_{02}(x) + v_{02}(x) + w_{02}(x)$$

where $u_{02}(x)$ is the smooth component and $v_{02}(x)$ and $w_{02}(x)$ are the singular components. Each of the Schwarz iterates is also decomposed in an analogous manner. Thus, for $k \geq 1$,

$$y_2^{*[k]}(x) = u_{02}^{[k]} + v_{02}^{[k]} + w_{02}^{[k]}$$

Theorem 7. *Let u_{02} be the smooth component and v_{02}, w_{02} are singular components of $y_2^*(x)$, and let $u_{02}^{[k]}$ and $v_{02}^{[k]}$, $w_{02}^{[k]}$ be the corresponding sequences of Schwarz iterates. Then, for all $k \geq 1$ and for all $x \in \overline{\Omega}$,*

$$\|u_{02}^{[k]} - u_{02}\|_{\overline{\Omega}} \leq Cq^k,$$

$$\|v_{02}^{[k]} - v_{02}\|_{\overline{\Omega}} \leq Cq^k \text{ and } \|w_{02}^{[k]} - w_{02}\|_{\overline{\Omega}} \leq Cq^k$$

where C *is independent of* k *and* ε *and*

$$q = e^{-\alpha\left(\xi^+ - \xi^-\right)/\varepsilon} < 1.$$

3.2. Numerical scheme. A non-overlapping Schwarz iterative process for the BVP (14)-(15) is now described. On $(\Omega^- \cup \Omega^+)$ a piecewise uniform mesh of N mesh interval is constructed as follows. The interval Ω^-; is subdivided into the three subintervals $[0, \tau_1), [\tau_1, d - \tau_1)$ and $[d - \tau_1, d)$ for some τ_1 that satisfies $0 < \tau_1 \leq d/4$. On $[0, \tau_1)$ and $[d - \tau_1, d)$ a uniform mesh with N/8 mesh intervals is placed, while $[\tau_1, d - \tau_1)$ has a uniform mesh with N/4 mesh intervals. The subinterval $[d, d+\tau_2), [d+\tau_2, 1-\tau_2)$ and $[1-\tau_2, 1]$ of Ω^+ are treated analogously for some τ_2 satisfying $0 < \tau_2 \leq (1 - d)/4$. The interior points of the mesh are denoted by

$$\Omega_\varepsilon^N = \left\{ x_i : 1 \leq i \leq \frac{N}{2} - 1 \right\} \cup \left\{ x_i : \frac{N}{2} + 1 \leq i \leq N - 1 \right\}.$$

Clearly $x_{N/2} = d$ and $\overline{\Omega}_\varepsilon^N = \{x_i\}_0^N$. Note that this mesh is a uniform mesh when $\tau_1 = d/4$ and $\tau_2 = (1 - d)/4$. It is fitted to the BVP (14)-(15) by choosing τ_1 and τ_2 to be the following functions of N and ε.

$$\tau_1 = \left\{ \frac{d}{4}, 2\sqrt{\frac{\varepsilon}{\beta}} \ln N \right\} \text{ and } \tau_2 = \left\{ \frac{1-d}{4}, 2\sqrt{\frac{\varepsilon}{\beta}} \ln N \right\}$$

The discretization of (14)-(15) and the procedure for the non-overlapping Schwarz iterative technique is described below

$$y_{2,i}^{*[0]}(x) = 0,\ \ x \in (0,d) \cup (d,1), \tag{16}$$

$$y_{2,i}^{*[0]}(0) = r,\ \ y_{2,i}^{*[0]}(d) = y_2^*(d),\ \ y_{2,i}^{*[0]}(1) = s. \tag{17}$$

Here the proper choice of an initial guess $y_{2,i}^{*[0]}(d)$ for the unknown $y_2^*(d)$. Then, we solve the following two finite difference subproblems for the mesh functions $y_{2,i_l}^{*[k]}, y_{2,i_r}^{*[k]},\ k \geq 1$:

$$L^N y_{2,i_l}^{*[k]} = -\varepsilon\delta^2 y_{2,i_l}^{*[k]} + b(x_i)y_{2,i_l}^{*[k]} + c(x_i)u_{01,i} = f(x_i), 1 < i < N/2, \tag{18}$$

$$y_{2,i_l}^{*[k]}(0) = y_{2,i_l}^{*[k-1]}(0),\ \ y_{2,i_l}^{*[k]}(d) = y_{2,i_l}^{*[k-1]}(d), \tag{19}$$

$$L^N y_{2,i_r}^{*[k]} = -\varepsilon\delta^2 y_{2,i_r}^{*[k]} + b(x_i)y_{2,i_r}^{*[k]} + c(x_i)u_{01,i} = f(x_i), 1 < i < N/2, \tag{20}$$

$$y_{2,i_r}^{*[k]}(0) = y_{2,i_r}^{*[k-1]}(d),\ \ y_{2,i_r}^{*[k]}(1) = y_{2,i_r}^{*[k-1]}(1). \tag{21}$$

where,

$$\delta^2 Z_i = \frac{D^+Z_i - D^-Z_i}{(x_{i+1} - x_{i-1})/2},\ \ D^+Z_i = \frac{Z_{i+1} - Z_i}{x_{i+1} - x_i}\ \ and\ D^-Z_i = \frac{Z_i - Z_{i-1}}{x_i - x_{i-1}}.$$

After these two sub-problems are solved, the approximation to $y_2^*(d)$ is updated using the average of the computed values at the two neighboring nodes of d. That is,

$$y_{2,i}^{*[k]}(d) = \frac{y_{2,i_l}^{*[k-1]}(x_{(N/2)-1}) + y_{2,i_r}^{*[k-1]}(x_{(N/2)+1})}{2}. \tag{22}$$

We define the k^{th} approximation to y_2^* as

$$y_{2,i}^{*[k]} = \begin{cases} \overline{y}_{2,i_l}^{*[k]}, & x < d \\ \frac{y_{2,i_l}^{*[k-1]}(x_{(N/2)-1}) + y_{2,i_r}^{*[k-1]}(x_{(N/2)+1})}{2}, & x = d, \\ \overline{y}_{2,i_r}^{*[k]}, & x > d, \end{cases} \tag{23}$$

where $\overline{y}_{2,i_l}^{*[k]}$ and $\overline{y}_{2,i_r}^{*[k]}$ are the continuous linear interpolant of $y_{2,i}^{*[k]}$ on Ω^- and Ω^+ respectively. The corresponding mesh functions are defined in the earlier stage of this section.

Each of the iterates $y_{2,i}^{*[k]}$ is decomposed into a smooth component $u_{2,i}^{*[k]}$ and singular components $v_{2,i}^{*[k]},\ w_{2,i}^{*[k]}$. Thus

$$y_{2,i}^{*[k]} = u_{2,i}^{*[k]} + v_{2,i}^{*[k]} + w_{2,i}^{*[k]}$$

where $u_{2,i}^{*[k]},\ v_{2,i}^{*[k]}$ and $w_{2,i}^{*[k]}$ are defined, for all $k \geq 1$,

$$u_{2,i}^{*[k]} = \begin{cases} u_{2,i_l}^{*[k]}, & \text{for } x \in \Omega^- \\ u_{2,i_r}^{*[k]}, & \text{for } x \in \Omega^+ \end{cases} \quad \text{and } v_{2,i}^{*[k]} = \begin{cases} v_{2,i_l}^{*[k]}, & \text{for } x \in \Omega^- \\ v_{2,i_r}^{*[k]}, & \text{for } x \in \Omega^+ \end{cases}$$

$$w_{2,i}^{*[k]} = \begin{cases} w_{2,i_l}^{*[k]}, & \text{for } x \in \Omega^- \\ w_{2,i_r}^{*[k]}, & \text{for } x \in \Omega^+ \end{cases}$$

3.3. Error analysis. The error at each mesh point $x_i \in \overline{\Omega}_\varepsilon^N$ is denoted by $|(y_2^{*[k]} - y_{2,i}^{*[k]})(x)|$.

Theorem 8. *Let $u_{02}^{[k]}$ and $u_{2,i}^{*[k]}$ denote the smooth components of $y_2^{*[k]}$ and $y_{2,i}^{*[k]}$ respectively. Then, for all $k \geq 1$ and $x_i \in \overline{\Omega}$*

$$\|(u_{2,i}^{*[k]} - u_{02}^{[k]})(x_i)\| \leq CN^{-1}$$

Proof. Using the result in [5](Lemma 1 pg.21),

$$| - \varepsilon \left(\frac{d^2}{dx^2} - \delta \right) u_{02}^{[k]}(x_i)| \leq \frac{1}{3}(x_{i+1} - x_{i-1})|u_{02}^{[k]}|_3 \quad \text{for } x_i \in \Omega^-$$

and

$$|L^N(u_{2,i_l}^{*[k]} - u_{02}^{[k]})(x_i)| \leq C\sqrt{\varepsilon} \leq CN^{-1} \quad \text{for } x_i \in \Omega^-.$$

Similarly for $x_i \in \Omega^+$

$$|L^N(u_{2,i_r}^{*[k]} - u_{02}^{[k]})(x_i)| \leq CN^{-1} \quad \text{for } x_i \in \Omega^+.$$

Then it leads to the required estimate

$$|(u_{2,i}^{*[k]} - u_{02}^{[k]})(x_i)| \leq CN^{-1} \quad \text{for } x_i \in \overline{\Omega}.$$

□

Theorem 9. *Let $v_{02}^{[k]}$, $w_{02}^{[k]}$ and $v_{2,i}^{*[k]}$, $w_{2,i}^{*[k]}$ denote the singular components of $y_2^{[k]}$ and $y_{2,i}^{*[k]}$ respectively. Then, for all $k \geq 1$*

$$\|(v_{2,i}^{*[k]} - v_{02}^{[k]})(x_i)\| \leq CN^{-1}\ln N, \quad \textit{for } x_i \in \Omega^-$$
$$\|(w_{2,i}^{*[k]} - w_{02}^{[k]})(x_i)\| \leq CN^{-1}\ln N, \quad \textit{for } x_i \in \Omega^+$$

Proof. The argument lying on $(0, \tau_1)$ and $(d - \tau_1, d)$, the local truncation error of the singular part of the solution is estimated as follows

$$\left| -\varepsilon \left(\frac{d^2}{dx^2} - \delta^2 \right) v_2^{[k]}(x_i) \right| \quad \leq \quad \varepsilon(x_{i+1} - x_{i-1})|v_2^{[k]}(x_i)|_3 \tag{24}$$

and for $(\tau_1, d - \tau_1)$

$$\left| -\varepsilon \left(\frac{d^2}{dx^2} - \delta^2 \right) v_2^{[k]}(x_i) \right| \quad \leq \quad 2\varepsilon \max_{x_i \in [x_{i+1}, x_{i-1}]} |v_2^{[k]}(x_i)| \tag{25}$$

Using (24) outside the layers and at $x_i = \tau_1,\ x_i = d - \tau_1$ on Ω^- gives

$$\left| -\varepsilon \left(\frac{d^2}{dx^2} - \delta^2 \right) v_2^{[k]}(x_i) \right| \leq 2\varepsilon C\varepsilon^{-1} \max_{x_i \in [x_{i+1}, x_{i-1}]} e^{-\sqrt{\beta}x_i/\sqrt{\varepsilon}} \leq CN^{-1}$$

Using (25) outside the layers and at $x_i = \tau_1,\ x_i = d - \tau_1$ on Ω^- gives

$$\left|-\varepsilon\left(\frac{d^2}{dx^2} - \delta^2\right) v_2^{[k]}(x_i)\right| \le C\varepsilon 2\frac{8\tau_1}{N}\varepsilon^{-\frac{3}{2}} e_1(x_i) \le C\tau_1\varepsilon^{-\frac{1}{2}} N^{-1} \le CN^{-1}\ln N.$$

Hence

$$\left|-\varepsilon\left(\frac{d^2}{dx^2} - \delta^2\right) v_2^{[k]}(x_i)\right| \le CN^{-1}\ln N, \quad \text{for } x_i \in \Omega^-$$

Then it leads to the required estimate

$$|v_{2,i}^{*[k]} - v_{02}^{[k]}(x_i)| \le CN^{-1}\ln N, \quad \text{for } x_i \in \Omega^-.$$

Similarly, the local truncation error of the singular component lying on $(d + \tau_2)$ and $(1 - \tau_2, 1)$ is

$$|w_{2,i}^{*[k]} - w_{02}^{[k]}(x_i)| \le CN^{-1}\ln N, \quad \text{for } x_i \in \Omega^+.$$

□

At the point $x_i = d$.

$$\begin{aligned}
&|L^N(y_{2,i}^{*[k]} - y_2^{*[k]})(d)| \\
&= y_{2,i}^{*[k]} + \frac{\varepsilon}{h^2}\int_{t=d}^{d+h}\int_{s=d}^{t} y_2^{*[k]''}(s)dsdt \\
&\quad - \frac{\varepsilon}{h^2}\int_{t=d-h}^{d}\int_{s=d}^{t} y_2^{*[k]''}(s)dsdt - a(d)y_2^{*[k]''}(d) \\
&= \frac{1}{h^2}\int_{t=d}^{d+h}\int_{s=d}^{t}\int_{r=s}^{d+h}(f - ay_2^{*[k]})'(r)drdsdt \\
&\quad + \frac{1}{h^2}\int_{t=d-h}^{d}\int_{s=d}^{t}\int_{r=d-h}^{s}(f - ay_2^{*[k]})'(r)drdsdt \\
&\quad - a(d)y_2^{*[k]''}(d) + \frac{1}{2}a(d-h)y_2^{*[k]}(d-h) + \frac{1}{2}a(d+h)y_2^{*[k]}(d+h) \\
&= \frac{1}{2}\int_{t=d}^{d+h}\int_{s=d}^{t}\int_{r=s}^{d+h}(f - ay_2^{*[k]})'(r)drdsdt \\
&\quad + \frac{1}{h^2}\int_{t=d-h}^{d}\int_{s=d}^{t}\int_{r=d-h}^{s}(f - ay_2^{*[k]})'(r)drdsdt \\
&\quad + \frac{1}{2}\int_{t=d}^{d-h}\left[a(t)y_2^{*[k]}(t)\right]' dt + \frac{1}{2}\int_{t=d}^{d+h}\left[a(t)y_2^{*[k]}(t)\right]' dt \\
&|L^N(y_{2,i}^{*[k]} - y_2^{*[k]})(d)| \le CN^{-1}\ln N. \qquad (26)
\end{aligned}$$

Theorem 10. *The error in using the scheme (18)-(21) to solve the BVP problem (14)-(15) at the inner grid points $\{x_i, i = 1, 2, ..., N-1\}$ satisfies*

$$|(y_2^{*[k]} - y_{2,i}^{*[k]})(x)| \le CN^{-1}\ln N \quad \textit{for } x \in \overline{\Omega}_\varepsilon^N.$$

Proof. From Theorem 8, 9 and (26) the above result can be obtained. □

4. Computational method

Consider the BVP (6)-(7). Let $u_{01}(x)$ be the reduced problem solution of the BVP (12)-(13) . From the Theorem 5 we get $|y_1(x) - u_{01}(x)| \leq C\sqrt{\varepsilon}$. The first step in the computational method is to replace y_1 by u_{01} in the second equation of the system (6)(as we have said earlier it is assumed that the closed from solution is available). Hence the system (6) gets decoupled. In the second step, we find the numerical approximation solution for y_2 by applying the scheme (18)-(21). Then find y_1 of (6) by using current y_2 in the similar manner. This iterative process is repeated until successive iterates are sufficiently close at each point of $\overline{\Omega}_\varepsilon^N$, in the sense that they satisfy the converging criterion

$$\max_{0<i<N} |(y_{2,i}^{*[k]} - y_{2,i}^{*[k-1]})(x_i)| < 10^{-10}.$$

5. Error estimate

Theorem 11. *Let (y_1, y_2) be the solution of (6)-(7). Further, let $y_{2,i}^{*[k]}$ be its numerical solution (14)-(15) obtained by numerical scheme. Then*

$$|y_2^{[k]}(x_i) - y_{2,i}^{*[k]}(x_i)| \leq C[N^{-1}\ln N + \sqrt{\varepsilon}].$$

Proof. Using Theorem 5 and 10 and the triangle inequality, we conclude that,

$$\begin{aligned} |y_2^{[k]}(x_i) - y_{2,i}^{*[k]}(x_i)| &\leq |y_2^{[k]}(x_i) - y_2^{*[k]}(x_i)| + |y_2^{*[k]}(x_i) - y_{2,i}^{*[k]}| \\ &\leq CN^{-1}\ln N + C\sqrt{\varepsilon} \\ |y_2^{[k]}(x_i) - y_{2,i}^{*[k]}(x_i)| &\leq C[N^{-1}\ln N + \sqrt{\varepsilon}]. \end{aligned}$$

□

Remark 3. There are two boundary layers ($x = 0$ and $x = 1$) and an interior layer at $x = d$. If the boundary conditions happen to have values such that no boundary layer occurs at a boundary point and do the necessary modifications in the distribution of the mesh points[8].

Remark 4. So far, it has been assumed that the exact solution u_{01} of the BVP (12)-(13) is available. If not, one has to obtain a numerical solution for u_{01} by a suitable finite difference method with a piecewise uniform mesh of N mesh interval described in Section 3.2. As done earlier, in the second equation the values of y_1 at the above grid points will be taken as $u_{01,i}$, then the resulting equations are solved for $y_{2,i}$.

6. Numerical results

In this section an example is solved for the particular problem of the type (1)-(2).

Example 1.

$$b(x) = \begin{cases} 2x+1, & x \le 0.5, \\ 2(1-x)+1, & x > 0.5, \end{cases}, \quad c(x) = 1.0, \quad f(x) = \begin{cases} -0.5, & x \le 0.5, \\ 0.5, & x > 0.5, \end{cases}$$

and $y(0) = 1.0, y(1) = 1.0, y''(0) = y''(1) = f(0)$.

which validate the theoretical results established in the previous result. The maximum pointwise errors and number of iterations are evaluated using the double mesh principle.

$$D_\varepsilon^N = \max_{x_i \in \overline{\Omega}_\varepsilon^N} \mid (Y^N - Y^{2N})(x_i) \mid \text{ and } D^N = \max_\varepsilon D_\varepsilon^N$$

where $Y^N(x_i)$ and $Y^{2N}(x_i)$ denote the numerical solutions obtained using N and *2N* mesh intervals. In addition, the order of convergence is calculated from

$$\rho^N = \log_2 \left(\frac{D^N}{D^{2N}} \right)$$

The solution is presented for various values of N and ε in Table 1.

7. Conclusion

A fourth-order singularly perturbed two point boundary value problem for ODEs with discontinuous source term is considered. The suitable boundary conditions are used to reduce the fourth-order differential equation into a system of two second-order equations and also established maximum principle, stability result and other necessary estimates. An iterative numerical method is used to solve the given example and numerical result is in agreement with the theoretical results.

Acknowledgements

The authors wish to thank Department of Science and Technology, New Delhi for financial support of project SR/FTP/MS-039/2012 and also thankful to anonymous referees for their valuable suggestions.

References

1. E.P. Doolan, J.J.H. Miller and W.H.A. Schilders, *Uniform numerical methods for problems with initial and boundary layers*, Boole press. Dublin, Ireland, 1980.
2. A.H. Nayfeh, *Introduction to perturbation methods*, John Wiley and Sons, New York, 1981.
3. R.E. O'Malley, *Singularly perturbation method for ordinary differential equation*, Springer Verlag, New York, 1991.
4. H.G. Roos, M. Stynes and L. Tobiska, *Numerical methods for singularly perturbed differential equations Springer*, New York, 1996.
5. J.J. H. Miller, E. O'Riordan and G.I. Shishkin, *Fitted numerical methods for singular perturbation problem*, World Scientific, Singapore, 1996.
6. M.K. Kadalbajoo and V. Gupta, *A brief survey of numerical methods for solving singularly perturbed problems*, Applied Mathematics and Computation **217** (2010), 3641–3716.

TABLE 1. Maximum point-wise errors E_ε^N and iteration counts for various N and ε for the Problem 1.

ε/N	2^5	2^6	2^7	2^8	2^9
2^0	5.6298e-03	2.7985e-03	1.3868e-03	6.8202e-04	3.2990e-04
	134	216	506	974	1873
2^{-2}	1.0819e-02	5.2972e-03	2.6051e-03	1.2763e-03	6.1622e-04
	101	195	375	721	1383
2^{-4}	7.8880e-03	3.7361e-03	1.8072e-03	8.7810e-04	4.2220e-04
	56	107	204	391	749
2^{-6}	1.5534e-03	6.8062e-04	3.1652e-04	1.5077e-04	7.1777e-05
	26	48	91	173	330
2^{-8}	3.2783e-03	8.3046e-04	2.0828e-04	5.2075e-05	1.2985e-05
	11	19	34	62	115
2^{-10}	1.3183e-02	3.4879e-03	8.8511e-04	2.2197e-04	5.5383e-05
	3	3	4	6	16
2^{-12}	3.0706e-02	1.3610e-02	3.6105e-03	9.1647e-04	2.2938e-04
	2	1	1	1	1
2^{-14}	3.1224e-02	1.4857e-02	5.2468e-03	1.7713e-03	5.5468e-04
	2	1	1	1	1
2^{-16}	3.1494e-02	1.4992e-02	5.2999e-03	1.7940e-03	5.5619e-04
	2	1	1	1	1
2^{-18}	3.1635e-02	1.5057e-02	5.3248e-03	1.8026e-03	5.5896e-04
	2	1	1	1	1
2^{-20}	3.1707e-02	1.5091e-02	5.3375e-03	1.8070e-03	5.6036e-04
	2	1	1	1	1
2^{-22}	3.1743e-02	1.5107e-02	5.3438e-03	1.8092e-03	5.6106e-04
	1	1	1	1	1
2^{-24}	3.1761e-02	1.5116e-02	5.3470e-03	1.8103e-03	5.6142e-04
	1	1	1	1	1
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
2^{-50}	3.1779e-02	1.5124e-02	5.3499e-03	1.8110e-03	5.6142e-04
	1	1	1	1	1
ρ^N	1.071233	1.499255	1.562725	1.689634	1.779789

7. M. Paramasivam, S. Valarmathi and J.J.H. Miller, *Second-order parameter uniform convergence for finite difference method for singularly perturbed linear differential equation*, Mathematical Communications **15** (2010), 587–612.
8. J.J.H. Miller, E. O'Riordan, G.I. Shishkin and S. Wang, *A parameter-uniform Schwarz method for a singularly perturbed reaction-diffusion problem with an interior layer*, Applied Numerical Mathematics **35** (2000), 323–337.
9. N. Kopteva, M. Pickett and H. Purtill, *A robust overlapping Schwarz method for a singularly perturbed semilinear reaction-diffusion problem with multiple solutions*, Int. J. Numer. Anal. Model. **6** (2009), 680–695.

10. M. Chandru and V. Shanthi, *A boundary value technique for singularly perturbed boundary value problem of reaction-diffusion with non-smooth data*, Journal of Engineering Science and Technology, Special Issue on ICMTEA2013 Conference (2014), 32–45.
11. M. Chandru, T. Prabha and V. Shanthi, *A Hybrid Difference Scheme For A second-order singularly perturbed reaction-diffusion problem with non-smooth data*, Int. J. Appl. Comput. Math. **1** (2015), 87–100.
12. M. Chandru and V. Shanthi, *Fitted mesh method for singularly perturbed Robin type boundary value problem with discontinuous source term*, Int. J. Appl. Comput. Math. **1** (2015), 491–501.
13. M. Chandru, T. Prabha and V. Shanthi, *A parameter robust higher order numerical method for singularly perturbed two parameter problems with non-smooth data*, Journal of Computational and Applied Mathematics(Accepted), DOI:10.1016/j.cam.2016.06.009.
14. S. Valarmathi and N. Ramanujam, *An asymptotic numerical fitted mesh method for singularly perturbed third-order ordinary differential equations of reaction-diffusion type*, Applied Mathematics and Computation **132** (2002), 87–104.
15. V. Shanthi and N. Ramanujam, *A numerical method for boundary value problems for singularly perturbed fourth-order ordinary differential equations*, Applied Mathematics and Computation **129** (2002), 269–294.
16. V. Shanthi and N. Ramanujam, *Asymptotic numerical method for singularly perturbed fourth-order ordinary differential equations of reaction-diffusion type*, Computers & Mathematics with Applications **46** (2003), 463–478.
17. V. Shanthi and N. Ramanujam, *Computational methods for reaction-diffusion problems for fourth-order ordinary differential equations with a small parameter at the highest derivative*, Applied Mathematics and Computation **147** (2004), 97–113.
18. V. Shanthi and N. Ramanujam, *An asymptotic numerical method for fourth-order singular perturbation problems with a discontinuous source term*, International Journal of Computer Mathematics **55** (2008), 1147–1159.
19. P.A. Farrell, J.J.H. Miller, E. O'Riordan and G.I. Shishkin, *Singularly perturbed differential equations with discontinuous source terms*, In Proc. Lozenetz **2** (2000).

M. Chandru is pursuing his Ph.D. at National Institute of Technology. His area of interests are Differential Equations and Numerical Analysis.

Department of Mathematics, National Institute of Technology, Tiruchirappalli, Tamilnadu, India-620015.
`e-mail: leochandru@gmail.com`

V Shanthi is working as an Assistant Professor at National Institute of Technology, Tiruchirappalli. She received her Post Doctoral Fellow from Bharathidasan University, Tiruchirappalli. Her area of interest include Numerical Analysis and Differential Equations. She has published more than 15 research articles in reputed international mathematical journals.

Department of Mathematics, National Institute of Technology, Tiruchirappalli, Tamilnadu, India-620015.
`e-mail: vshanthi@nitt.edu`