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DIFFERENTIAL EQUATIONS ASSOCIATED WITH TANGENT NUMBERS

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Abstract. In this paper, we study differential equations arising from the generating functions of tangent numbers. We give explicit identities for the tangent numbers.

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1. Introduction

Recently, many mathematicians have studied in the area of the Bernoulli numbers, Euler numbers, Genocchi numbers, and tangent numbers (see [2, 3, 4, 6, 7, 8]). Tangent numbers T_n and polynomials, $T_n(x)(n \geq 0)$, were introduced by Ryoo (see [5]). The tangent numbers T_n are defined by the generating function:

$$F = F(t) = \frac{2}{e^{2t}+1} = \sum_{n=0}^{\infty} T_n \frac{t^n}{n!}. \tag{1.1}$$

We introduce the tangent polynomials $T_n(x)$ as follows:

$$\left(\frac{2}{e^{2t}+1}\right) e^{xt} = \sum_{n=0}^{\infty} T_{n,q}(x) \frac{t^n}{n!}. \tag{1.2}$$

In [6], Tangent numbers of higher order, $T_n^{(k)}$ are defined by means of the following generating function

$$\sum_{n=0}^{\infty} T_n^{(k)} \frac{t^n}{n!} = \left(\frac{2}{e^{2t}+1}\right)^k. \tag{1.3}$$

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The first few of them are

$$\begin{aligned}
T_0^{(k)} &= 1,\\
T_1^{(k)} &= -k,\\
T_2^{(k)} &= -k + k^2,\\
T_3^{(k)} &= 3k^2 - k^3,\\
T_4^{(k)} &= 2k + 3k^2 - 6k^3 + k^4,\\
T_5^{(k)} &= -10k^2 - 15k^3 + 10k^4 - k^5,\\
T_6^{(k)} &= -16k - 30k^2 + 15k^3 + 45k^4 - 15k^5 + k^6.\\
T_7^{(k)} &= 112k^2 + 210k^3 + 35k^4 - 105k^5 + 21k^6 - k^7,\\
T_8^{(k)} &= 272k + 588k^2 - 28k^3 - 735k^4 - 280k^5 + 210k^6 - 28k^7 + k^8,\\
T_9^{(k)} &= -2448k^2 - 5292k^3 - 2436k^4 + 1575k^5 + 1008k^6 - 378k^7 + 36k^8 - k^9.
\end{aligned}$$

Nonlinear differential equations arising from the generating functions of special polynomials are studied by T. Kim and D. Kim in order to give explicit identities for special polynomials(see [1, 4]). In this paper, we study differential equations arising from the generating functions of tangent numbers. We give explicit identities for the tangent numbers.

2. Differential equations associated with tangent numbers

In this section, we study linear differential equations arising from the generating functions of tangent numbers. Let

$$F = F(t) = \frac{2}{e^{2t}+1}. \tag{2.1}$$

Then, by (2.1), we have

$$\begin{aligned}
F^{(1)} &= \frac{d}{dt}F(t) = \frac{d}{dt}\left(\frac{2}{e^{2t}+1}\right) = -2\left(\frac{2}{e^{2t}+1}\right) + \left(\frac{2}{e^{2t}+1}\right)^2\\
&= -2F + F^2,\\
F^{(2)} &= \frac{d}{dt}F^{(1)} = -2F^{(1)} + 2FF^{(1)}\\
&= (-1)^2 4F + (-1)6F^2 + 2F^3,
\end{aligned} \tag{2.2}$$

and

$$\begin{aligned}
F^{(3)} = \frac{d}{dt}F^{(2)} &= (-1)^2 4F^{(1)} + (-1)12FF^{(1)} + 6F^2F^{(1)}\\
&= (-1)^3 8F + (-1)^2 28F^2 + (-1)24F^3 + 6F^4.
\end{aligned} \tag{2.3}$$

Continuing this process, we can guess that

$$\begin{aligned} F^{(N)} &= \left(\frac{d}{dt}\right)^N F(t) \\ &= (-1)^N \sum_{i=1}^{N+1} a_i(N) F^i, \quad (N = 0, 1, 2, \ldots). \end{aligned} \tag{2.4}$$

Taking the derivative with respect to t in (2.4), we have

$$\begin{aligned} F^{(N+1)} &= \frac{dF^{(N)}}{dt} \\ &= (-1)^N \sum_{i=1}^{N+1} i a_i(N) F^{i-1} F^{(1)} \\ &= (-1)^N \sum_{i=1}^{N+1} i a_i(N) F^{i-1} \left(-2F + F^2\right) \\ &= (-1)^N \sum_{i=1}^{N+1} (-2) i a_i(N) F^i + (-1)^N \sum_{i=1}^{N+1} i a_i(N) F^{i+1} \\ &= (-1)^{N+1} \left(\sum_{i=1}^{N+1} 2 i a_i(N) F^i - \sum_{i=2}^{N+2} (i-1) a_{i-1}(N) F^i \right). \end{aligned} \tag{2.5}$$

On the other hand, by replacing N by $N+1$ in (2.4), we get

$$F^{(N+1)} = (-1)^{N+1} \sum_{i=1}^{N+2} a_i(N+1) F^i. \tag{2.6}$$

By (2.5) and (2.6), we have

$$\sum_{i=1}^{N+1} 2 i a_i(N) F^i - \sum_{i=2}^{N+2} (i-1) a_{i-1}(N) F^i = \sum_{i=1}^{N+2} a_i(N+1) F^i. \tag{2.7}$$

Comparing the coefficients on both sides of (2.7), we obtain

$$2a_1(N) = a_1(N+1), \quad a_{N+2}(N+1) = -(N+1) a_{N+1}(N), \tag{2.8}$$

and

$$a_i(N+1) = 2 i a_i(N) - (i-1) a_{i-1}(N), (2 \leq i \leq N+1). \tag{2.9}$$

In addition, by (2.4), we get

$$F = F^{(0)} = a_1(0) F. \tag{2.10}$$

Thus, by (2.10), we obtain

$$a_1(0) = 1. \tag{2.11}$$

It is not difficult to show that

$$\begin{aligned} -2F+F^2 &= F^{(1)} \\ &= (-1)\left(\sum_{i=1}^{2} a_i(1)F^i\right) \\ &= -a_1(1)F - a_2(1)F^2. \end{aligned} \tag{2.12}$$

Thus, by (2.12), we also get

$$a_1(1)=2, \quad a_2(1)=-1. \tag{2.13}$$

From (2.8), we note that

$$\begin{aligned} a_1(N+1) &= 2a_1(N) \\ &= 2^2 a_1(N-1) \\ &= \cdots \\ &= 2^N a_1(1) \\ &= 2^{N+1}, \end{aligned} \tag{2.14}$$

and

$$\begin{aligned} a_{N+2}(N+1) &= -(N+1)a_{N+1}(N) \\ &= (-1)^2(N+1)Na_N(N-1) \\ &= \cdots \\ &= (-1)^{N+1}(N+1)N(N-1)\cdots 3\cdot 2. \end{aligned}$$

For $i=2,3,4$ in (2.9), we have

$$a_2(N+1) = (-1)\sum_{k=0}^{N}(2-1)(2\cdot 2)^k a_1(N-k),$$

$$a_3(N+1) = (-1)\sum_{k=0}^{N-1}(3-1)(2\cdot 3)^k a_2(N-k),$$

and

$$a_4(N+1) = (-1)\sum_{k=0}^{N-2}(4-1)(2\cdot 4)^k a_3(N-k).$$

Continuing this process, we can deduce that, for $2 \le i \le N+1$,

$$a_i(N+1) = (-1)\sum_{k=0}^{N-i+2}(i-1)(2\cdot i)^k a_{i-1}(N-k). \tag{2.15}$$

Here, we note that the matrix $a_i(j)_{1\le i\le N+2, 0\le j\le N+1}$ is given by

$$\begin{pmatrix} 1 & 2 & 2^2 & 2^3 & \cdots & 2^{N+1} \\ 0 & (-1)1! & \cdot & \cdot & \cdots & \cdot \\ 0 & 0 & (-1)^2 2! & \cdot & \cdots & \cdot \\ 0 & 0 & 0 & (-1)^3 3! & \cdots & \cdot \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & (-1)^{N+1}(N+1)! \end{pmatrix}$$

Now, we give explicit expressions for $a_i(N+1)$. By (2.14) and (2.15), we get

$$\begin{aligned} a_2(N+1) &= (-1)\sum_{k_1=0}^{N}(2-1)(2\cdot 2)^k a_1(N-k_1) \\ &= (-1)\sum_{k_1=0}^{N}(2-1)(2\cdot 2)^{k_1}2^{N-k_1}, \end{aligned}$$

$$\begin{aligned} a_3(N+1) &= (-1)\sum_{k_2=0}^{N-1}(3-1)(2\cdot 3)^{k_2}a_2(N-k_2) \\ &= (-1)^2\sum_{k_2=0}^{N-1}\sum_{k_1=0}^{N-k_2-1}(2-1)(3-1)(2\cdot 2)^{k_1}(2\cdot 3)^{k_2}2^{N-k_2-k_1-1}, \end{aligned}$$

and

$$\begin{aligned} &a_4(N+1) \\ &= (-1)\sum_{k_3=0}^{N-2}(4-1)(2\cdot 4)^{k_3}a_3(N-k_3) \\ &= (-1)^3\sum_{k_3=0}^{N-2}\sum_{k_2=0}^{N-k_3-2}\sum_{k_1=0}^{N-k_3-k_2-2}1\cdot 2\cdot 3(2\cdot 4)^{k_3}(2\cdot 3)^{k_2}(2\cdot 2)^{k_1}2^{N-k_3-k_2-k_1-2}. \end{aligned}$$

Continuing this process, we have

$$\begin{aligned} a_i(N+1) = (-1)^{i-1}&\sum_{k_{i-1}=0}^{N-i+2}\sum_{k_{i-2}=0}^{N-i+2-k_{i-1}}\cdots\sum_{k_1=0}^{N-i+2-k_{i-1}-\cdots-k_2}(i-1)! \\ &\times\left(\prod_{l=2}^{i}(2\cdot l)^{k_{l-1}}\right)2^{N-i+2-k_{i-1}-k_{i-2}-\cdots-k_2-k_1}. \end{aligned} \tag{2.16}$$

Therefore, by (2.16), we obtain the following theorem.

Theorem 2.1. *For $N = 0, 1, 2, \dots,$ the functional equations*

$$F^{(N)} = (-1)^N\left(\sum_{i=1}^{N+1}a_i(N)F^i\right)$$

have a solution

$$F = F(t) = \frac{2}{e^{2t}+1},$$

where

$$\begin{aligned}
&a_1(N) = 2^N,\\
&a_{N+1}(N) = (-1)^N N(N-1)\cdots 3\cdot 2,\\
&a_i(N) = (-1)^{i-1} \sum_{k_{i-1}=0}^{N-i+1} \sum_{k_{i-2}=0}^{N-i+1-k_{i-1}} \cdots \sum_{k_1=0}^{N-i+1-k_{i-1}-\cdots-k_2} (i-1)!\\
&\qquad\qquad \times \left(\prod_{l=2}^{i} (2\cdot l)^{k_{l-1}}\right) 2^{N-i+1-k_{i-1}-k_{i-2}-\cdots-k_2-k_1}.
\end{aligned}$$

Here is a plot of the surface for this solution. In Figure 1, we plot of the shape

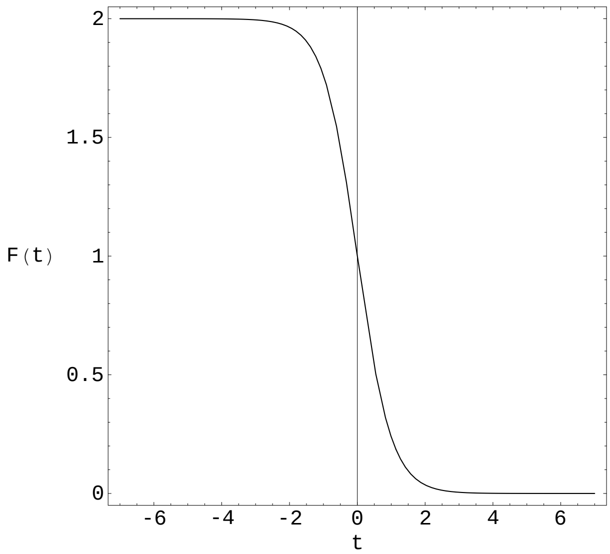

FIGURE 1. The shape for the solution $F(t)$

for this solution.

From (1.1), we note that

$$F^{(N)} = \left(\frac{d}{dt}\right)^N F(t) = \sum_{k=0}^{\infty} T_{k+N} \frac{t^k}{k!}. \tag{2.17}$$

From Theorem 1, (1.3), and (2.17), we can derive the following equation:

$$\begin{aligned}\sum_{k=0}^{\infty} T_{k+N}\frac{t^k}{k!} &= F^{(N)} \\ &= (-1)^N \sum_{i=1}^{N+1} a_i(N)\left(\frac{2}{e^{2t}+1}\right)^i \\ &= (-1)^N \sum_{i=1}^{N+1} a_i(N)\left(\sum_{k=0}^{\infty} T_k^{(i)}\frac{t^k}{k!}\right) \\ &= \sum_{k=0}^{\infty}\left((-1)^N \sum_{i=1}^{N+1} a_i(N)T_k^{(i)}\right)\frac{t^k}{k!}.\end{aligned} \tag{2.18}$$

By comparing the coefficients on both sides of (2.18), we obtain the following theorem.

Theorem 2.2. *For $k, N = 0, 1, 2, \ldots,$ we have*

$$T_{k+N} = (-1)^N \sum_{i=1}^{N+1} a_i(N)T_k^{(i)}, \tag{1.19}$$

$$\begin{aligned} a_1(N) &= 2^N, \\ a_{N+1}(N) &= (-1)^N N(N-1)\cdots 3\cdot 2, \\ a_i(N) &= (-1)^{i-1}\sum_{k_{i-1}=0}^{N-i+1}\ \sum_{k_{i-2}=0}^{N-i+1-k_{i-1}}\cdots\sum_{k_1=0}^{N-i+1-k_{i-1}-\cdots-k_2}(i-1)! \\ &\quad\times\left(\prod_{l=2}^{i}(2\cdot l)^{k_{l-1}}\right)2^{N-i+1-k_{i-1}-k_{i-2}-\cdots-k_2-k_1}. \end{aligned}$$

Let us take $k = 0$ in (2.19). Then, we have the following corollary.

Corollary 2.3. *For $N = 0, 1, 2, \ldots,$ we have*

$$T_N = (-1)^N \sum_{i=1}^{N+1} a_i(N).$$

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