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EXTRA-GRADIENT METHODS FOR QUASI-NONEXPANSIVE OPERATORS

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ABSTRACT. *In this paper, we propose an Ishikawa-type extra-gradient iterative algorithm for finding a solution of split feasibility, fixed point problems and equilibrium problems of quasi-nonexpansive mappings. It is proven that under suitable conditions, the sequences generated by the proposed iterative algorithms converge weakly to a solution of the split feasibility, fixed point problems and equilibrium problems. An example is given to illustrate the main result of this paper.*

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1. Introduction

Let H_1 and H_2 be two real Hilbert spaces and $C \subset H_1$ and $Q \subset H_2$ be two empty closed convex sets. The symbols $\mathbb{N}$ and $\mathbb{R}$ are used to denote the set of positive integers and real numbers, respectively. Let $A : H_1 \to H_2$ be a bounded linear operator with its adjoint A^*. Let $T : C \to C$ be a nonlinear mapping. The set of fixed points of T is denoted by $\mathrm{Fix}(T)$. Let F be a bi-function from $Q \times Q$ into $\mathbb{R}$. The classical equilibrium problem is to find $u \in Q$ such that

$$F(u, v) \geq 0, \quad \forall v \in Q. \tag{1.1}$$

The symbol $\mathrm{EP}(F)$ is used to denote the set of all solutions of the problem (1.1), that is,

$$\mathrm{EP}(F) = \{u \in Q : F(u, v) \geq 0, \quad \forall v \in Q\}.$$

The purpose of this paper is to study the following split feasibility, fixed point problems and equilibrium problems:

$$\text{Find} \quad x^* \in C \cap \mathrm{Fix}(T) \quad \text{such that} \quad Ax^* \in Q \cap \mathrm{EP}(F). \tag{1.2}$$

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We use Γ to denote the set of solutions of (1.2), that is,

$$\Gamma = \{x^* : x^* \in C \cap \mathrm{Fix}(T), \quad Ax^* \in Q \cap \mathrm{EP}(F)\}.$$

In the sequel, we assume $\Gamma \neq \phi$. A special case of the split feasibility, fixed point problems and equilibrium problems is the split feasibility problem:

$$\text{Fixed} \quad x^* \in C \quad \text{such that} \quad Ax^* \in Q.$$

Recently, it has been found that the split feasibility problem can be applied to study intensity-modulated radiation therapy (see [1-5,9]).

In 1976, to study the saddle point problem, Korpelevich [11] introduced the so-called extra-gradient method:

$$\begin{cases} y_n = P_C(x_n - \lambda A x_n), \\ x_{n+1} = P_C(x_n - \lambda A y_n), \quad n \geq 0, \end{cases}$$

where $\lambda > 0$, A is a strongly monotone and Lipschitz continuous mapping and P_C is a projection operator from H_1 into C.

In 2012, He et al. [8] studied the split feasibility, fixed point problems and equilibrium problems. They proposed an iterative algorithm in the following manner:

$$\begin{cases} y_n = P_C(x_n + \varepsilon A^*(T_{r_n}^F - I)Ax_n), \\ x_{n+1} = (1-\alpha_n)y_n + \alpha_n T y_n, \quad \forall n \in \mathbb{N}, \end{cases} \tag{1.3}$$

where $T : C \to C$ is a quasi-nonexpansive mapping, A^* is the adjoint of A, $T_{r_n}^F$ is the mapping defined in Lemma 2.1, $\{r_n\} \subset (0, +\infty)$ with $\liminf_{n\to\infty} r_n > 0$, $\varepsilon \in (0, \frac{1}{\|A\|^2})$, $\eta > 0$ is a constant and $\alpha_n \in [\eta, 1-\eta]$ for $n \in \mathbb{N}$. The authors proved that the sequences generated by (1.3) converge weakly to an element $x \in \Gamma = \{x^* : x^* \in C \cap \mathrm{Fix}(T), \quad Ax^* \in Q \cap \mathrm{EP}(F)\}$.

In this paper, motivated by the work of He et al. [8], we propose an Ishikawa-type extra-gradient iterative algorithm for finding a solution of the split feasibility, fixed point problems and equilibrium problems involved quasi-nonexpansive mappings. We establish weak convergence theorems for the sequences generated by the proposed iterative algorithms. Our results extend and develop the corresponding results in [8].

2. Preliminaries

Let H be a real Hilbert space whose inner product and norm are denoted by $\langle \cdot, \cdot \rangle$ and $\|\cdot\|$, respectively. Let C be a nonempty closed convex subset of H. We write $x_n \rightharpoonup x$ to indicate that the sequence $\{x_n\}$ converges weakly to x. Moreover, we use $\omega_w(x_n)$ to denote the weak ω-limit set of the sequence $\{x_n\}$, that is,

$$\omega_w(x_n) = \{x : x_{n_i} \rightharpoonup x \ \text{ for some subsequence } \ \{x_{n_i}\} \text{ of } \{x_n\}\}.$$

Recall that the projection from H onto C, denoted by P_C, is defined in such a way that for each $x \in H$, $P_C x$ is the unique point in C with the property:

$$\|x - P_C x\| = \min\{\|x - y\| : y \in C\}.$$

Some important properties of projections are gathered in the following proposition.

Proposition 2.1. *Given $x \in H$ and $z \in C$,*

(1) $z = P_C x \quad \Leftrightarrow \quad \langle x - z, y - z\rangle \leq 0$ for all $y \in C$;

(2) $z = P_C x \Leftrightarrow \|x - z\|^2 \leq \|x - y\|^2 - \|y - z\|^2$ for all $y \in C$;

(3) $\langle x - y, P_C x - P_C y\rangle \geq \|P_C x - P_C y\|^2$ for all $y \in H$;

(4) $\|P_C x - P_C y\|^2 \leq \|x - y\|^2 - \|(I - P_C)x - (I - P_C)y\|^2$ for all $x, y \in H$.

For all $x, y \in H$, the following conclusions hold:

$$\|tx + (1-t)y\|^2 = t\|x\|^2 + (1-t)\|y\|^2 - t(1-t)\|x - y\|^2, \quad t \in [0,1], \tag{2.1}$$

and

$$\|x + y\|^2 = \|x\|^2 + 2\langle x, y\rangle + \|y\|^2. \tag{2.2}$$

Recall that a mapping $T : C \to C$ is said to be quasi-nonexpansive if $\mathrm{Fix}(T) \neq \phi$ and $\|Tx - Tp\| \leq \|x - p\|$ for all $x \in C$ and $p \in \mathrm{Fix}(T)$.

A Banach space $(E, \|\cdot\|)$ is said to satisfy Opial's condition if for each sequence $\{x_n\}$ in E which converges weakly to a point $x \in E$, we have

$$\liminf_{n\to\infty} \|x_n - x\| < \liminf_{n\to\infty} \|x_n - y\|, \quad \forall y \in E, y \neq x.$$

It is well known that any Hilbert space satisfies Opial's condition.

Definition 2.2. Let C be a nonempty closed convex subset of a real Hilbert space H and T be a mapping from C into C. The mapping T is called demiclosed if for any sequence $\{x_n\}$ which weakly converges to $\overline{x}$ and if the sequence $\{T(x_n)\}$ strongly converges to z, then $T(\overline{x}) = z$.

Let F be a bifunction of $C \times C$ into $\mathbb{R}$ satisfying the following conditions:

(A1) $F(x, x) = 0$ for all $x \in C$;

(A2) F is monotone, *i.e.*, $F(x, y) + F(y, x) \leq 0$ for all $x, y \in C$;

(A3) for each $x, y, z \in C$, $\lim_{t\to 0} F(tz + (1-t)x, y) \leq F(x, y)$;

(A4) for each $x \in C$, $y \mapsto F(x, y)$ is convex and lower semi-continuous.

Lemma 2.3 ([7])**.** *Let C be a nonempty closed convex subset of H and let F be a bi-function of $C \times C$ into $\mathbb{R}$ satisfying (A1)-(A4). For $r > 0$, define a mapping $T_r^F : H \to C$ as follows:*

$$T_r^F(x) = \{z \in C : F(z, y) + \frac{1}{r}\langle y - z, z - x\rangle \geq 0. \quad \forall y \in C\}$$

for all $x \in H$. Then the following hold:

(i) T_r^F is single-valued and $Fix(T_r^F) = EP(F)$ for any $r > 0$ and $EP(F)$ is closed and convex;

(ii) T_r^F *is firmly nonexpansive, i.e., for any* $x, y \in H$,

$$\|T_r^F x - T_r^F y\|^2 \leq \langle T_r^F x - T_r^F y, x - y\rangle.$$

Lemma 2.4 ([10]). *Let* H *be a real Hilbert space and let* $\{x_n\}$ *be a bounded sequence in* H *such that there exists a nonempty closed convex set* C *of* H *satisfying:*

(1) for every $w \in C$, $\lim_{n\to\infty} \|x_n - w\|$ *exists;*

(2) each weak-cluster point of the sequence $\{x_n\}$ *is in* C.

Then $\{x_n\}$ *converges weakly to a point in* C.

Lemma 2.5 ([6]). *Assume that* $F : C \times C \to \mathbb{R}$ *satisfies Assumption 2.1 and let* T_r^F *be defined as in Lemma 2.3. Let* $x, y \in H$ *and* $r_1, r_2 > 0$. *Then*

$$\|T_{r_2}^F(y) - T_{r_1}^F(x)\| \leq \|y - x\| + |\frac{r_2 - r_1}{r_2}|\|T_{r_2}^F(y) - y\|.$$

3. Main results

Theorem 3.1. *Let* H_1, H_2 *be two real Hilbert spaces and let* $C \subset H_1$, $Q \subset H_2$ *be two nonempty closed convex sets. Let* $A : H_1 \to H_2$ *be a bounded linear operator with its adjoint* A^*. *Let* $T : C \to C$ *be a quasi-nonexpansive mapping and let* $F : Q \times Q \to \mathbb{R}$ *be a bi-function with* $\Gamma = \{x^* : x^* \in C \cap Fix(T), \quad Ax^* \in Q \cap EP(F)\} \neq \phi$. *Suppose* $T - I$ *is demiclosed at* 0. *For* $x_0 \in H_1$ *arbitrarily, let* $\{x_n\}$ *be a sequence defined by the following Ishikawa-type extra-gradient iterative algorithm:*

$$\begin{cases} y_n = P_C(x_n - \varepsilon A^*(I - T_{r_n}^F)Ax_n), \\ z_n = P_C(x_n - \varepsilon A^*(I - T_{r_n}^F)Ay_n), \\ w_n = (1 - \alpha_n)z_n + \alpha_n T z_n, \\ x_{n+1} = (1 - \beta_n)z_n + \beta_n T w_n, \quad n \geq 1, \end{cases} \tag{3.1}$$

where $\{r_n\} \subset (0, +\infty)$, $\varepsilon \in (0, \frac{1}{2\|A\|^2})$ *and* $\{\alpha_n\}$, $\{\beta_n\} \subset (0, 1)$ *are two real numbers satisfying* $0 < b < \alpha_n, \beta_n < a < 1$.

Then the sequence $\{x_n\}$ *generated by algorithm (3.1) converges weakly to an element of* Γ.

Proof. Let $x^* \in \Gamma$. Then we have $x^* \in C \cap \mathrm{Fix}(T)$ and $Ax^* \in Q \cap \mathrm{EP}(F)$. For simplicity, we write $u_n = x_n - \varepsilon A^*(I - T_{r_n}^F)Ax_n$ for all $n \geq 1$. For each $n \geq 1$, by Lemma 2.1, we have

$$\begin{aligned}\|T_{r_n}^F Ax_n - Ax^*\| &\leq \langle T_{r_n}^F Ax_n - T_{r_n}^F Ax^*, Ax_n - Ax^*\rangle \\ &= \frac{1}{2}(\|T_{r_n}^F Ax_n - Ax^*\|^2 + \|Ax_n - Ax^*\|^2 - \|T_{r_n}^F Ax_n - Ax_n\|^2).\end{aligned}$$

So, we obtain

$$\|T_{r_n}^F Ax_n - Ax^*\|^2 \leq \|Ax_n - Ax^*\|^2 - \|T_{r_n}^F Ax_n - Ax_n\|^2 \tag{3.2}$$

for any $n \geq 1$. By (2.2), we have

$$\|u_n - x^*\|^2 = \|x_n - x^*\|^2 + 2\varepsilon\langle x_n - x^*, A^*(T_{r_n}^F - I)Ax_n\rangle + \varepsilon^2\|A^*(I - T_{r_n}^F)Ax_n\|^2. \tag{3.3}$$

Since A is a linear operator with its adjoint A^*, we have

$$\begin{aligned}
&\langle x_n - x^*, A^*(T_{r_n}^F - I)Ax_n\rangle \\
&= \langle Ax_n - Ax^*, T_{r_n}^F Ax_n - Ax_n\rangle \\
&= \langle Ax_n - Ax^* + (T_{r_n}^F Ax_n - Ax_n) - (T_{r_n}^F Ax_n - Ax_n), T_{r_n}^F Ax_n - Ax_n\rangle \\
&= \langle T_{r_n}^F Ax_n - Ax^*, T_{r_n}^F Ax_n - Ax_n\rangle - \|T_{r_n}^F Ax_n - Ax_n\|^2.
\end{aligned} \tag{3.4}$$

Again using (2.2), we obtain

$$\begin{aligned}
&\langle T_{r_n}^F Ax_n - Ax^*, T_{r_n}^F Ax_n - Ax_n\rangle \\
&= \frac{1}{2}(\|T_{r_n}^F Ax_n - Ax^*\|^2 + \|T_{r_n}^F Ax_n - Ax_n\|^2 - \|Ax_n - Ax^*\|^2).
\end{aligned} \tag{3.5}$$

From (3.2), (3.4) and (3.5), we get

$$\begin{aligned}
&\langle x_n - x^*, A^*(T_{r_n}^F - I)Ax_n\rangle \\
&= \frac{1}{2}(\|T_{r_n}^F Ax_n - Ax^*\|^2 + \|T_{r_n}^F Ax_n - Ax_n\|^2 - \|Ax_n - Ax^*\|^2) \\
&\quad - \|T_{r_n}^F Ax_n - Ax_n\|^2 \\
&\leq \frac{1}{2}(\|Ax_n - Ax^*\|^2 - \|T_{r_n}^F Ax_n - Ax_n\|^2 + \|T_{r_n}^F Ax_n - Ax_n\|^2 \\
&\quad - \|Ax_n - Ax^*\|^2) - \|T_{r_n}^F Ax_n - Ax_n\|^2 \\
&= -\|T_{r_n}^F Ax_n - Ax_n\|^2.
\end{aligned} \tag{3.6}$$

Substituting (3.6) into (3.3), we deduce

$$\begin{aligned}
\|u_n - x^*\|^2 &\leq \|x_n - x^*\|^2 - 2\varepsilon\|T_{r_n}^F Ax_n - Ax_n\|^2 \\
&\quad + \varepsilon^2\|A^*(I - T_{r_n}^F)Ax_n\|^2 \\
&= \|x_n - x^*\|^2 - \varepsilon(2 - \varepsilon\|A\|^2)\|T_{r_n}^F Ax_n - Ax_n\|^2 \\
&\leq \|x_n - x^*\|^2.
\end{aligned} \tag{3.7}$$

Since P_C is nonexpansive, we get

$$\|y_n - x^*\|^2 \leq \|u_n - x^*\|^2 \leq \|x_n - x^*\|^2. \tag{3.8}$$

Since $T_{r_n}^F$ is nonexpansive, we know that $I - T_{r_n}^F$ is $\frac{1}{2}$-inverse strongly monotone. Therefore, it is easy to see that $A^*(I - T_{r_n}^F)A$ is $\frac{1}{2\|A\|^2}$-inverse strongly monotone, that is,

$$\begin{aligned}
&\langle x - y, A^*(I - T_{r_n}^F)Ax - A^*(I - T_{r_n}^F)Ay\rangle \\
&\geq \frac{1}{2\|A\|^2}\|A^*(I - T_{r_n}^F)Ax - A^*(I - T_{r_n}^F)Ay\|^2
\end{aligned} \tag{3.9}$$

and

$$A^*(T_{r_n}^F - I)Ax^* = 0. \tag{3.10}$$

It follows from Proposition 2.1, (2.2), (3.9) and (3.10) that

$$\begin{aligned}
&\|z_n - x^*\|^2 \\
&\le \|x_n - \varepsilon A^*(I - T_{r_n}^F)Ay_n - x^*\|^2 - \|x_n - \varepsilon A^*(I - T_{r_n}^F)Ay_n - z_n\|^2 \\
&= \|x_n - x^*\|^2 - \|x_n - z_n\|^2 + 2\varepsilon\langle A^*(I - T_{r_n}^F)Ay_n, x^* - z_n\rangle \\
&= \|x_n - x^*\|^2 - \|x_n - z_n\|^2 + 2\varepsilon(\langle A^*(I - T_{r_n}^F)Ay_n - A^*(I - T_{r_n}^F)Ax^*, x^* - y_n\rangle \\
&\quad + \langle A^*(I - T_{r_n}^F)Ay_n, y_n - z_n\rangle) \\
&\le \|x_n - x^*\|^2 - \|x_n - z_n\|^2 + 2\varepsilon\langle A^*(I - T_{r_n}^F)Ay_n, y_n - z_n\rangle \\
&= \|x_n - x^*\|^2 - \|x_n - y_n\|^2 - \|y_n - z_n\|^2 \\
&\quad + 2\langle x_n - \varepsilon A^*(I - T_{r_n}^F)Ay_n - y_n, z_n - y_n\rangle.
\end{aligned}$$

From Proposition 2.1 and (3.9), we have

$$\begin{aligned}
&\langle x_n - \varepsilon A^*(I - T_{r_n}^F)Ay_n - y_n, z_n - y_n\rangle \\
&= \langle x_n - \varepsilon A^*(I - T_{r_n}^F)Ax_n - y_n, z_n - y_n\rangle \\
&\quad + \varepsilon\langle A^*(I - T_{r_n}^F)Ax_n - A^*(I - T_{r_n}^F)Ay_n, z_n - y_n\rangle \\
&\le \varepsilon\langle A^*(I - T_{r_n}^F)Ax_n - A^*(I - T_{r_n}^F)Ay_n, z_n - y_n\rangle \\
&\le \varepsilon\|A^*(I - T_{r_n}^F)Ax_n - A^*(I - T_{r_n}^F)Ay_n\|\|z_n - y_n\| \\
&\le 2\varepsilon\|A\|^2\|x_n - y_n\|\|z_n - y_n\|.
\end{aligned}$$

From the assumption of ε, we obtain

$$\begin{aligned}
\|z_n - x^*\|^2 &\le \|x_n - x^*\|^2 - \|x_n - y_n\|^2 - \|y_n - z_n\|^2 \\
&\quad + 4\varepsilon\|A\|^2\|x_n - y_n\|\|z_n - y_n\| \\
&\le \|x_n - x^*\|^2 - \|x_n - y_n\|^2 - \|y_n - z_n\|^2 \\
&\quad + 4\varepsilon^2\|A\|^4\|x_n - y_n\|^2 + \|z_n - y_n\|^2 \\
&= \|x_n - x^*\|^2 - (1 - 4\varepsilon^2\|A\|^4)\|x_n - y_n\|^2 \\
&\le \|x_n - x^*\|.
\end{aligned} \tag{3.11}$$

Similarly, we have

$$\|z_n - x^*\|^2 \le \|x_n - x^*\|^2 - (1 - 4\varepsilon^2\|A\|^4)\|z_n - y_n\|^2. \tag{3.12}$$

From (2.1), we have

$$\begin{aligned}
\|Tw_n - x^*\|^2 &\le \|(1 - \alpha_n)z_n + \alpha_n Tz_n - x^*\|^2 \\
&= (1 - \alpha_n)\|z_n - x^*\|^2 + \alpha_n\|Tz_n - x^*\|^2 \\
&\quad - \alpha_n(1 - \alpha_n)\|z_n - Tz_n\|^2
\end{aligned}$$

$$\begin{aligned}&\leq \|z_n - x^*\|^2 - \alpha_n(1-\alpha_n)\|z_n - Tz_n\|^2\\&\leq \|z_n - x^*\|^2.\end{aligned} \tag{3.13}$$

From (2.1), (3.1), (3.11) and (3.13), we have

$$\begin{aligned}\|x_{n+1} - x^*\|^2 &= (1-\beta_n)\|z_n - x^*\|^2 + \beta_n\|Tw_n - x^*\|^2\\&\quad - \beta_n(1-\beta_n)\|z_n - Tw_n\|^2\\&\leq (1-\beta_n)\|z_n - x^*\|^2 + \beta_n\|z_n - x^*\|^2\\&\quad - \beta_n(1-\beta_n)\|z_n - Tw_n\|^2\\&= \|z_n - x^*\|^2 - \beta(1-\beta_n)\|z_n - Tw_n\|^2\\&\leq \|z_n - x^*\|^2\\&\leq \|x_n - x^*\|^2.\end{aligned} \tag{3.14}$$

The inequality (3.14) implies that $\lim_{n\to\infty}\|x_n - x^*\|^2$ exists. This implies that $\{x_n\}$ is bounded. Additionally, we get the boundedness of $\{y_n\}$ and $\{z_n\}$ from (3.8) and (3.11) immediately. Returning to (3.11), (3.12) and (3.14), we have

$$\begin{aligned}\|x_{n+1} - x^*\|^2 &\leq \|z_n - x^*\|^2\\&\leq \|x_n - x^*\|^2 - (1-4\varepsilon^2\|A\|^4)\|x_n - y_n\|^2\end{aligned}$$

and

$$\begin{aligned}\|x_{n+1} - x^*\|^2 &\leq \|z_n - x^*\|^2\\&\leq \|x_n - x^*\| - (1-4\varepsilon^2\|A\|^4)\|z_n - y_n\|^2.\end{aligned}$$

Hence

$$(1-4\varepsilon^2\|A\|^4)\|x_n - y_n\|^2 \leq \|x_n - x^*\|^2 - \|x_{n+1} - x^*\|^2$$

and

$$(1-4\varepsilon^2\|A\|^4)\|z_n - y_n\|^2 \leq \|x_n - x^*\|^2 - \|x_{n+1} - x^*\|^2,$$

which imply that

$$\lim_{n\to\infty}\|x_n - y_n\| = 0 \tag{3.15}$$

and

$$\lim_{n\to\infty}\|z_n - y_n\| = 0. \tag{3.16}$$

From (3.7) and (3.8), we have

$$\begin{aligned}\varepsilon(2-\varepsilon\|A\|^2)\|T^F_{r_n}Ax_n - Ax_n\|^2 &\leq \|x_n - x^*\|^2 - \|y_n - x^*\|^2\\&= (\|x_n - x^*\| + \|y_n - x^*\|)\|x_n - y_n\|,\end{aligned}$$

which implies that

$$\lim_{n\to\infty}\|T^F_{r_n}Ax_n - Ax_n\| = 0. \tag{3.17}$$

By (3.11) and (3.14), we have

$$\begin{aligned}\|x_{n+1} - x^*\|^2 &\leq \|z_n - x^*\|^2 - \beta_n(1-\beta_n)\|z_n - Tw_n\|^2 \\ &\leq \|x_n - x^*\|^2 - \beta_n(1-\beta_n)\|z_n - Tw_n\|^2.\end{aligned}$$

It follows that

$$\beta_n(1-\beta_n)\|z_n - Tw_n\|^2 \leq \|x_n - x^*\|^2 - \|x_{n+1} - x^*\|^2.$$

This implies that

$$\lim_{n\to\infty} \|z_n - Tw_n\| = 0. \tag{3.18}$$

From (3.13), we obtain

$$\begin{aligned}\alpha_n(1-\alpha_n)\|z_n - Tz_n\|^2 &\leq \|z_n - x^*\|^2 - \|Tw_n - x^*\|^2 \\ &= (\|z_n - x^*\| + \|Tw_n - x^*\|)\|z_n - Tw_n\|\end{aligned}$$

for any $n \in \mathbb{N}$. It is from (3.18) that we get

$$\lim_{n\to\infty} \|z_n - Tz_n\| = 0. \tag{3.19}$$

Since the sequence $\{x_n\}$ is bounded, we can choose a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that $x_{n_i} \rightharpoonup \hat{x}$. Consequently, we derive from (3.15)-(3.17) that

$$\begin{cases} y_{n_i} \rightharpoonup \hat{x}, \\ z_{n_i} \rightharpoonup \hat{x}, \\ Ax_{n_i} \rightharpoonup A\hat{x}, \\ T^F_{r_n} Ax_{n_i} \rightharpoonup A\hat{x}. \end{cases} \tag{3.20}$$

Since $I - T$ is demiclosed at 0 and (3.19), we deduce $\hat{x} \in \mathrm{Fix}(T)$.

Next we claim $T^F_r A\hat{x} = A\hat{x}$ for any $r > 0$. Suppose that $T^F_r A\hat{x} \neq A\hat{x}$ for any $r > 0$. From (3.17) and Lemma 2.3, we have

$$\begin{aligned}\liminf_{j\to\infty} \|Ax_{n_j} - A\hat{x}\| &< \liminf_{j\to\infty} \|Ax_{n_j} - T^F_r A\hat{x}\| \\ &\leq \liminf_{j\to\infty} (\|Ax_{n_j} - T^F_{r_{n_j}} Ax_{n_j}\| + \|T^F_{r_{n_j}} Ax_{n_j} - T^F_r A\hat{x}\|) \\ &= \liminf_{j\to\infty} \|T^F_{r_{n_j}} Ax_{n_j} - T^F_r A\hat{x}\| \\ &\leq \liminf_{j\to\infty} (\|Ax_{n_j} - A\hat{x}\| + \frac{|r_{n_j} - r|}{r_{n_j}} \|T^F_{r_{n_j}} Ax_{n_j} - Ax_{n_j}\|) \\ &= \liminf_{j\to\infty} \|Ax_{n_j} - A\hat{x}\|,\end{aligned}$$

which lead to a contradiction. So, $A\hat{x} \in \mathrm{Fix}(T^F_r) = \mathrm{EP}(F)$. Note that $y_{n_i} = P_C u_{n_i} \in C$ and $T^F_{r_n} Ax_n \in Q$. From (3.20), we deduce $\hat{x} \in C$ and $A\hat{x} \in Q$. To this end, we deduce $\hat{x} \in C \cap \mathrm{Fix}(T)$ and $A\hat{x} \in Q \cap \mathrm{EP}(F)$. That is to say $\hat{x} \in \Gamma$. This shows that $\omega_w(x_n) \subset \Gamma$. Since the $\lim_{n\to\infty} \|x_n - x^*\|$ exists for every $x^* \in \Gamma$, the weak convergence of the whole sequence $\{x_n\}$ follows by applying Lemma 2.2. This completes the proof. □

Furthermore, we can immediately obtain the following weak convergence result.

Corollary 3.2. *Let H_1 and H_2 be two real Hilbert spaces and let $C \subset H_1$ and $Q \subset H_2$ be two nonempty closed convex sets. Let $A : H_1 \to H_2$ be a bounded linear operator with its adjoint A^*. Let $T : C \to C$ be a quasi-nonexpansive mapping and let $F : Q \times Q \to \mathbb{R}$ be a bi-function with $\Gamma = \{x^* : x^* \in C \cap Fix(T), Ax^* \in Q \cap EP(F)\} \neq \phi$. Suppose $T - I$ is demiclosed at 0. For $x_0 \in H_1$ arbitrarily, let $\{x_n\}$ be a sequence defined by the following Ishikawa-type iterative algorithm:*

$$\begin{cases} y_n = P_C(x_n - \varepsilon A^*(I - T^F_{r_n})Ax_n, \\ z_n = (1-\alpha_n)y_n + \alpha_n T y_n, \\ x_{n+1} = (1-\beta_n)y_n + \beta_n T z_n, \quad n \geq 1, \end{cases} \tag{3.21}$$

where $\{r_n\} \subset (0, +\infty)$, $\{\alpha_n\}, \{\beta_n\} \subset (0,1)$ such that $0 < b < \alpha_n, \beta_n < a < 1$ and $\varepsilon \in (0, \frac{1}{2\|A\|^2})$ is a constant. Then the sequence $\{x_n\}$ generated by algorithm (3.21) converges weakly to an element of Γ.

Proof. Let $x^* \in \Gamma$. Then we have $x^* \in C \cap \mathrm{Fix}(T)$ and $Ax^* \in Q \cap \mathrm{EP}(F)$. For simplicity, we write $u_n = x_n - \varepsilon A^*(I - T^F_{r_n})Ax_n$ for all $n \geq 1$. By (3.7) and (3.8), we have

$$\|u_n - x^*\|^2 \leq \|x_n - x^*\|^2 - \varepsilon(2 - \varepsilon\|A\|^2)\|T^F_{r_n}Ax_n - Ax_n\|^2 \tag{3.22}$$

and

$$\|y_n - x^*\|^2 \leq \|u_n - x^*\|^2 \leq \|x_n - x^*\|^2. \tag{3.23}$$

From (2.1), (3.21) and (3.23), we have

$$\begin{aligned} \|Tz_n - x^*\|^2 &\leq \|z_n - x^*\|^2 \\ &= \|(1-\alpha_n)(y_n - x^*) + \alpha_n(Ty_n - x^*)\|^2 \\ &= (1-\alpha_n)\|y_n - x^*\|^2 + \alpha_n\|Ty_n - x^*\|^2 \\ &\quad - \alpha_n(1-\alpha_n)\|y_n - Ty_n\|^2 \\ &\leq \|y_n - x^*\|^2 - \alpha_n(1-\alpha_n)\|y_n - Ty_n\|^2 \end{aligned} \tag{3.24}$$

and

$$\begin{aligned} \|x_{n+1} - x^*\|^2 &= \|(1-\beta_n)y_n + \beta_n T z_n - x^*\|^2 \\ &= (1-\beta_n)\|y_n - x^*\|^2 + \beta_n\|Tz_n - x^*\|^2 \\ &\quad - \beta_n(1-\beta_n)\|y_n - Tz_n\|^2 \\ &\leq (1-\beta_n)\|y_n - x^*\|^2 + (\|y_n - x^*\|^2 - \alpha_n(1-\alpha_n)\|y_n - Ty_n\|^2) \\ &= \|y_n - x^*\|^2 - \alpha_n\beta_n(1-\alpha_n)\|y_n - Ty_n\|^2 \\ &\leq \|y_n - x^*\|^2 \\ &\leq \|x_n - x^*\|^2. \end{aligned} \tag{3.25}$$

So, we obtain $\lim_{n\to\infty} \|x_n - x^*\|^2$ exists immediately. This implies that $\{x_n\}$ is bounded. Additionally, we get the boundedness of $\{y_n\}$ and $\{z_n\}$ from (3.23) and (3.24) immediately. Returning to (3.22), (3.23) and (3.25), we have

$$\begin{aligned}\|x_{n+1} - x^*\|^2 &\le \|y_n - x^*\|^2 - \alpha_n\beta_n(1-\alpha_n)\|y_n - Ty_n\|^2 \\ &\le \|x_n - x^*\|^2 - \varepsilon(2-\varepsilon\|A\|^2)\|T^F_{r_n}Ax_n - Ax_n\|^2 \\ &\quad - \alpha_n\beta_n(1-\alpha_n)\|y_n - Ty_n\|^2.\end{aligned}$$

Hence,

$$\begin{aligned}&\varepsilon(2-\varepsilon\|A\|^2)\|T^F_{r_n}Ax_n - Ax_n\|^2 + \alpha_n\beta_n(1-\alpha_n)\|y_n - Ty_n\|^2 \\ &\le \|x_n - x^*\|^2 - \|x_{n+1} - x^*\|^2,\end{aligned}$$

which implies that

$$\lim_{n\to\infty} \|T^F_{r_n}Ax_n - Ax_n\| = 0 \quad \text{and} \quad \lim_{n\to\infty} \|y_n - Ty_n\|^2 = 0.$$

Using the firm nonexpansiveness of P_C, Proposition 2.1 and (3.22), we have

$$\begin{aligned}\|y_n - x^*\|^2 &= \|P_C u_n - P_C x^*\|^2 \\ &\le \|u_n - x^*\|^2 - \|(I-P_C)u_n\|^2 \\ &\le \|x_n - x^*\|^2.\end{aligned}$$

Therefore, all conditions in Theorem 3.1 are satisfied. The conclusion of Corollary 3.1 can be obtained from Theorem 3.1 immediately. □

Remark 3.1. Theorem 3.1 extends and develops the corresponding one of He and Du [8] in the following aspects:

The corresponding iterative algorithms in [8], Theorem 3.1 and Corollary 3.1 are extended for developing our Ishikawa-type extra-gradient iterative algorithms for the split common solution problem in Theorem 3.1 and Corollary 3.1.

Example 3.3. Let $H_1 = H_2 = \mathbb{R}$ with the inner product defined by $\langle x, y\rangle = xy$ for all $x, y \in \mathbb{R}$ and the absolute valued norm $|\cdot|$. Let $C = [0, +\infty)$, $Q = (-\infty, 0]$ and $Tx = \frac{x^2+2}{1+x}$ for all $x \in C$. Obviously, $\mathrm{Fix}(T) = \{2\}$. It is easy to see that

$$\begin{aligned}|Tx - 2| &= \frac{x}{x+1}|x-2| \\ &\le |x-2|\end{aligned}$$

for all $x \in C$. Thus, T is a continuous quasi-nonexpansive mapping.

Let $\{x_n\}$ be a sequence in C such that $x_n \rightharpoonup z \in C$ and $\|x_n - Tx_n\| \to 0$ as $n \to \infty$. Then $z \in \mathrm{Fix}(T) = \{2\}$. Therefore, T is zero-demiclosed.

Define the mapping $A : \mathbb{R} \to \mathbb{R}$ and $F : Q \times Q \to \mathbb{R}$ as follows:

$$Ax = -2x, \quad \forall x \in \mathbb{R}$$

and

$$F(x, y) = (y - x)(x + 4), \quad \forall (x, y) \in Q \times Q.$$

Then A is a bounded linear operator with $A^* = A$ and $\|A\| = 2$, and F satisfies the conditions (A1)-(A4) with $\mathrm{EP}(F) = \{-4\}$. Obviously, $\Gamma = \{x^* : x^* \in C \cap \mathrm{Fix}(T), \quad Ax^* \in Q \cap \mathrm{EP}(F)\} = \{2\} = \mathrm{Fix}(T)$. For $r_n = 1$, $v_n = T^F_{r_n} Ax_n$ is equivalent to

$$F(v_n, v) + \langle v - v_n, v_n - Ax_n \rangle \geq 0, \quad \forall v \in Q, n \in \mathbb{N}.$$

Hence, we can easily find $v_n = -x_n - 2 \in Q$. It is not hard to compute $A^*(I - T^F_{r_n})Ax_n = 2(x_n - 2)$ for all $n \in \mathbb{N}$. Hence, for $\varepsilon = \frac{1}{16}$, we have

$$x_n - \frac{1}{16} A^*(I - T^F_{r_n})Ax_n = \frac{7}{8} x_n + \frac{1}{4} \in C$$

for all $n \in \mathbb{N}$. So, we get

$$\begin{aligned} y_n &= P_C(x_n - \frac{1}{16} A^*(I - T^F_{r_n})Ax_n) \\ &= \frac{7}{8} x_n + \frac{1}{4} \end{aligned} \tag{3.26}$$

and

$$\begin{aligned} z_n &= P_C(x_n - \frac{1}{16} A^*(I - T^F_{r_n})Ay_n) \\ &= x_n - \frac{y_n}{8} + \frac{1}{4} \end{aligned} \tag{3.27}$$

for all $n \in \mathbb{N}$. For every $n \geq 1$, from (3.26) and (3.27), we can rewrite (3.1) as follows:

$$\begin{cases} y_n = \frac{7}{8} x_n + \frac{1}{4}, \\ z_n = x_n - \frac{y_n}{8} + \frac{1}{4}, \\ w_n = (1 - \alpha_n) z_n + \alpha_n T z_n, \\ x_{n+1} = (1 - \beta_n) z_n + \beta_n T w_n, \quad n \geq 1. \end{cases}$$

Observe that for all $n \geq 1$,

$$\begin{aligned} |x_{n+1} - 2| &\leq (1 - \beta_n)|z_n - 2| + \beta_n |Tw_n - 2| \\ &\leq (1 - \beta_n |z_n - 2| + \beta_n |w_n - 2| \\ &\leq (1 - \beta_n)|z_n - 2| + \beta_n |z_n - 2| \\ &= |x_n - \frac{y_n}{8} + \frac{1}{4} - 2| \\ &= \frac{57}{64} |x_n - 2|. \end{aligned}$$

Hence we have $|x_{n+1} - 2| \leq (\frac{57}{64})^n |x_1 - 2|$ for all $n \geq 1$. This implies that $\{x_n\}$ converges to 2.

References

1. C. Byrne,*Iterative oblique projection onto convex subsets and the split feasibility problem*, Inverse Probl. **18** (2002), 441–453.
2. Y. Censor, T. Bortfield and A. Trofimov, *A unified approch for inversion problems in intensity modulated radiation therapy*, Phys. Med. Biol. **51** (2006), 2353–2365.
3. Y. Censor and T. Elfving, *A multiprojection algorithm using Bregman projections in a product space*, Numer. Algorithms **8** (1994), 221–239.
4. Y. Censor, T. Elfving, N. Kopf and T. Bortfeld, *The multiple-sets split feasibility problem and its applications for inverse problems*, Inverse Probl. **6** (2005), 2071–2084.
5. Y. Censor, A. Motova and A. Segal, *Perturbed projections and subgradient projections for the multiple-sets split feasibility problem*, J. Math. Anal. Appl. **327** (2007), 1244–1256.
6. F. Cianciaruso, G. Marino, L. Muglia and Y. Yao, *A hybrid projection algorithm for finding solutions of mixed equilibrium problem and variational inequality problem*, Fixed Point Theory Appl. **2010** (2010), Article ID 383740.
7. P.L. Combettes and A. Hirstoaga, *Equilibrium programming in Hilbert spaces*, J. Nonlinear Convex Anal. **6** (2005), 117–136.
8. Z. He and W.S. Du, *Nonlinear algorithms approach to split common solution problems*, Fixed Point Theory Appl. **2012** (2012), Article ID 130.
9. J.U. Jeong, *Strong convergence theorems for fixed point problems of asymptotically quasi-ϕ-nonexpansive mappings in the intermediate sense*, J. Appl. Math. & Informatics **32** (2014), 621–633.
10. Z.R. Kong, L.C. Ceng and C.F. Wen, *Some modified extragradient methods for solving feasibility and fixed point problems*, Abstr. Appl. Anal. **2012** (2012), Article ID 975981.
11. G. Korpelevich, *An extragradient method for finding saddle points and for other problems*, Ekon. Mat. Metody **12** (1976), 747–756.

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