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RADIO LABELING AND RADIO NUMBER FOR GENERALIZED CATERPILLAR GRAPHS†

SAIMA NAZEER, M. SAQIB KHAN, IMRANA KOUSAR, WAQAS NAZEER*

Abstract. A Radio labeling of the graph G is a function g from the vertex set $V(G)$ of G to $\mathbb{Z}^+$ such that $|g(u)-g(v)| \geq \text{diam}(G)+1-d_G(u,v)$, where $\text{diam}(G)$ and $d(u,v)$ are diameter and distance between u and v in graph G respectively. The radio number $\text{rn}(G)$ of G is the smallest number k such that G has radio labeling with $\max\{g(v) : v \in V(G)\} = k$. We investigate radio number for some families of generalized caterpillar graphs.

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1. Introduction

Radio labeling is an extension of distance two labeling, which is used to assign channels to the transmitters of radio network such that the network satisfies all the interference constraints. This assignment of channels to the transmitters is popularly known as channel assignment problem which was introduced by Hale [6] in 1980. For the solution of channel assignment problem, the interference graph is developed and assignment of channels converted into graph labeling (a graph labeling is an assignment of label to each vertex according to certain rule). In interference graph, the vertices are used to represent transmitters, and there is a major interference between two transmitters if the corresponding pair of vertices are adjacent. While there is minor interference between transmitters if corresponding vertices are at distance two, and there is no interference between transmitters if they are at distance three or beyond it. In other words, very close transmitters are represented by adjacent vertices, and close transmitters are represented by the vertices which are at distance two apart.
In 1991, Roberts [21], suggested a solution for channel assignment problem and

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proposed that a pair of transmitters having minor interference must receive different channels and a pair of transmitters having major interference must receive channels that are at least two apart. Motivated through this Griggs and Yeh [4] introduced distance two labeling, which is also known as $L(2,1)-$labeling and is defined as follows:

Definition 1.1. A distance two labeling (or $L(2,1)-$labeling) of a graph $G = (V(G), E(G))$ is a function g from vertex set $V(G)$ to the set of nonnegative integers such that the following conditions are satisfied:
(1) $|g(u) - g(v)| \geq 2$ if $d(u,v) = 1$
(2) $|g(u) - g(v)| \geq 1$ if $d(u,v) = 2$.

The difference between the largest and the smallest label assigned by g is called the span of g and the minimum span over all $L(2,1)-$labeling of G is called the $\lambda-$number of G, denoted by $\lambda(G)$. The $L(2,1)-$labeling has explored in the past two decades by many researchers like Yeh [30, 31], Georges and Mauro [3], Sabaki [22], Chang and Kuo [2], Wang [28], Vaidya and Bantva [24], and Vaidya *et al.* [25]. For more literture, we suggest the readers [9, 10, 11, 17, 7, 18, 19] and the reference therein.
But as time passed, practically it has been observed that the interference among transmitters might go beyond two levels. Radio labeling extends the number of interference level considered in $L(2,1)-$labeling from two to largest possible interference among transmitter, i.e. the diameter of G which is defined as follows:

Definition 1.2. The diameter of a graph is denoted by $\text{diam}(G)$ and defined as the maximum distance between any two vertices.
i.e $\text{diam}(G) = \max\{d(u,v); u, v \in G\}$.

Where $d(u,v)$ is distance between u and v which is defined as follows:

Definition 1.3. Let G be a connected graph, the distance $d(u,v)$ between any pair of vertices u, v is the length of the shortest path between them.

Definition 1.4. A radio labeling which is also known as multilevel distance labeling of G is a function $g : V(G) \to \mathbb{Z}^+$ such that the inequality $|g(u) - g(v)| \geq \text{diam}(G) + 1 - d(u,v)$ holds for any pair of distinct vertices u, v. The span of g is the difference of the largest and the smallest channels used, $\max_{u,v \in V(G)}\{g(v) - g(u)\}$. The radio number of G is denoted by $\text{rn}(G)$ and is defined as the minimum span of radio labeling of G.

Note that when $\text{diam}(G) = 2$ than radio labeling and distance two labeling are identical. The radio labeling is studied in the past decade by many authors like Liu [15], Liu and Xie [12, 13], Liu and Zhu [14] and Vaidya and Vihol [27]. Moreover, the radio number for path and cycles was determined in [14], for the square of paths was studied by Liu and Xie [13], for the square of a cycle [12]. Radio Number for generalized prism graph was studied in [16] and a generalized gear graph was discussed in [20], where lower bound of radio number is determined. Radio labeling for some cycle related graphs are studied by S.K.

Vadiya and P.L. Vihol [26]. Radio number for caterpillar graphs and caterpillar related graphs has been determined in [23, 24, 25].

Definition 1.5. The caterpillar graph CP_n is a graph which is obtained from the path P_n by attaching a new terminal vertex to each non terminal vertex.

In [5] Ruxandra Marinescu-Ghemeci find radio number of caterpillar graphs CP_n, i.e.

$$rn(CP_n) = \begin{cases} 4k^2 - 6k + 4, & \text{if } n = 2k; \\ 4k^2 - 2k + 4, & \text{if } n = 2k + 1. \end{cases}$$

First we define Generalized Caterpillar graphs.

Definition 1.6. $C_{(m,0)}P_n$ is Generalized Caterpillar obtained from P_n by attaching m vertices of degree one to each vertex of degree two of P_n.
$C_{(m,1)}P_n$ is Generalized Caterpillar obtained from P_n by attaching m vertices of degree two to each vertex of degree two of P_n

In this paper, we completely determine the radio number for some families of Generalized Caterpillar graphs defined above.

2. Main Results

Let T be a rooted tree. For any $u \in V(T)$, the status of u in T is defined by

$$S_T(u) = \sum_{w \in V(T)} d(u, w).$$

The status of T is minimum status among all vertices of T:

$$S(T) = \min\{S_T(u) | u \in V(T)\}.$$

Let $w* \in V(T)$. If $S_T(w*) = S(T)$ then $w*$ is called a weight center of T.

Theorem 2.1 ([15])**.** *Let T be a tree with n vertices and diameter d. Then*

$$rn(T) \geq (n-1)(d+1) + 2 - 2S(T).$$

Moreover, the equality holds if and only if for every weight center $w*$ there exists a radio labeling g with $g(w_1) = 1 < g(w_2) < \cdots < g(w_n)$ for which all following properties hold, for every j with $1 \leq j \leq n-1$:

(1) *w_j and w_{j+1} belong to different branches, unless one of them is $w*$;*
(2) *$\{w_1, w_n\} = \{w*, v\}$, where $v \in V(T)$ such that $d(w*, v) = 1$;*
(3) *$g(w_{j+1}) = g(w_j) + d + 1 - d(v*, w_j) - d(v*, w_{j+1})$.*

Now, we discussed the radio labeling of Generalized Caterpillar graphs $C_{(m,0)}P_n$ which is obtained from P_n by attaching m vertices of degree one to each vertex of degree two of P_n. We have $M = |V(C_{(m,0)}P_n)| = n + (n-2)m$ and diameter $d = n - 1$.

Theorem 2.2. *Let $C_{(m,0)}P_n$ be the graph for $m > 2$ and with $n = 2k+1$, $k \geq 2$. Then the radio number for $C_{(m,0)}P_n$ is $2k^2m - 2km + 2k^2 + m + 2$.*

Proof. By using Theorem 1, we achieved the lower bound for the radio number of generalized caterpillar $C_{(m,0)}P_n$ graphs, which is obtained by attaching the terminal vertices $v_{i,\rho}$ for $i \leq \rho \leq m$ to the vertex v_i, for each $2 \leq i \leq 2k$. For the sake of convenience we consider $v_{i,\rho}$ for $i \leq \rho \leq m$ in anticlockwise direction (See Figure 1).

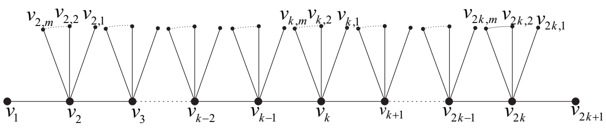

FIGURE 1. $C_{(m,0)}P_{2k+1}$

We have $V(C_{(m,0)}P_n) = n+(n-2)m = (2k+1)+(2k-1)m = M$ and $d = 2k$. Now, we compute the status function of $C_{(m,0)}P_n$. It is clear that $C_{(m,0)}P_n$ has weight center $v_k + 1$ in the case $n = 2k + 1$. We have

$$\begin{aligned} S(C_{(m,0)}P_n) &= S_{C_{(m,0)}P_n}(v_{k+1}) \\ &= \sum_{v \in V(C_{(m,0)}P_n)} d(v_{k+1}, v) \\ &= m.1 + m.2 + m.3 + ...m.k + 2.1 + 2.2 \\ &\quad + ...2.k + 2.m + 3.m + 4.m + ...k.m \\ &= (m+1)(k^2+k) - m \end{aligned}$$

It follows from Theorem 1. that

$$\begin{aligned} rn(C_{(m,0)}P_n) &\geq (V(C_{(m,0)}P_n) - 1)(d+1) + 2 - 2S(C_{(m,0)}P_n) \\ &= [(2k+1) + (2k-1)m - 1](2k+1) + 2 - 2[(m+1)(k^2+k) - m] \\ &= 2k^2m - 2km + 2k^2 + m + 2. \end{aligned}$$

Moreover, in order to prove equality, it suffices to find a radio labeling g for $C(m,0)P_n$ that fulfil the properties(1)-(3) in Theorem 1 for weight center with span$(g) = 2k^2m - 2km + 2k^2 + m + 2$.
For that, we order the vertices of $C_{(m,0)}P_n$ as follows:

$$\begin{aligned} &v_{k+1} \to v_1 \to v_{k+1,1} \to v_{2k,1} \to v_{k,1} \to v_{2k-1,1} \to v_{k-1,1} \to v_{2k-2,1} \to v_{k-2,1} \\ &\to ... \to v_{k+2,1} \to v_{2,1} \to v_{k+1,2} \to v_{2k,2} \to v_{k,2} \to v_{2k-1,2} \to v_{k-1,2} \to v_{2k-2,2} \\ &\to v_{k-2,2} \to ... \to v_{k+2,2} \to v_{2,2} \to v_{k+1,3} \to v_{2k,3} \to v_{k,3} \to v_{2k-1,3} \to v_{k-1,3} \\ &\to v_{2k-2,3} \to v_{k-2,3} \to ... \to v_{k+2,3} \to v_{2,3} \to v_{k+1,4} \to ... \to v_{k+1,m} \to v_{2k,m} \\ &\to v_{k,m} \to v_{2k-1,m} \to v_{k-1,m} \to ... \to v_{k+2,m} \to v_{2,m} \to v_{2k+1} \to v_k \to v_{2k} \\ &\to v_{k-1} \to v_{2k-1} \to ... \to v_{k-(k-2)} = v_2 \to v_{2k-(k-2)} = v_{k+2}. \end{aligned}$$

We rename the vertices of $C_{(m,0)}P_n$ in the above ordering by $u_1, u_2, u_3, ...u_M$. A labeling g for $C_{(m,0)}P_n$ by using the rules given in (2) and (3) from Theorem

1, is defined as follows: $g(u_1) = 1$, $g(u_{i+1}) = g(u_i) + d + 1 - d(u_{i+1}, u_i)$ for $1 \le i \le M-1$. The order in which the vertices are labeled and their labels are shown in Fig.2 for $k = 3, m = 3$. Since we have the following distances:

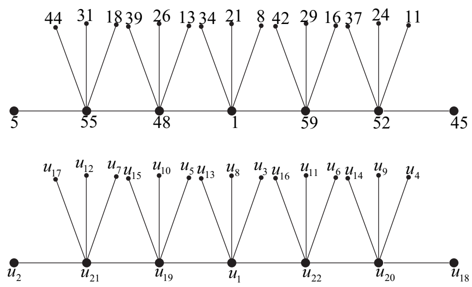

FIGURE 2. $C_{(3,0)}P_7$

$$\begin{aligned} d(v_{k+1-j,\rho}, v_{2k-j,\rho}) &= k+1 \ \ for \ \ 0 \le j \le k-1 \ \ and \ \ 1 \le \rho \le m; \\ d(v_j, v_{k+j}) &= k \ \ for \ \ 2 \le j \le k+1; \\ d(v_1, v_{k+1}) &= k; \\ d(v_1, v_{k+1,1}) &= k+1; \\ d(v_{2,m}, v_{2k+1}) &= k; \\ d(v_{2,m}, v_{2k+1}) &= 2k. \end{aligned}$$

We get:

$$\begin{aligned} sp(g) = g(u_M) = g(v_{k+2}) &= g(u_1) + (M-1)(d+1) - \sum_{i=1}^{M-1} d(u_{i+1}, u_i) \\ &= 2k^2m - 2km + 2k^2 + m + 2. \end{aligned}$$

The following relations also hold:

$$\begin{aligned} g(v_j) &= g(v_{j+1}) + 2k + 1 \ \ for \ 1 \le j \le k-2; \\ g(v_j) &= g(v_{j+1}) + 2k + 1 \ \ for \ k+1 \le j \le 2k-2; \\ g(v_1) &= g_{v_{k+1}} + k + 1; \\ g(v_{k+1,1}) &= g(v_1) + k. \end{aligned}$$

Similarly,

$$\begin{aligned} g(v_{j,\rho}) &= g(v_{j+1,\rho}) + 2k - 1 \ \ for \ 2 \le j \le k, \ and \ 1 \le \rho \le m; \\ g(v_{j,\rho}) &= g(v_{j+1,\rho}) + 2k - 1 \ \ for \ k+2 \le j \le 2k-1, \ and \ 1 \le \rho \le m. \end{aligned}$$

$|g(v_i) - g(v_j)| \ge 2k+1$, $|g(v_{i,\rho}) - g(v_{j,\rho})| \ge 2k-1$ for $1 \le \rho \le m$, if v_i, v_j and $v_{i.\rho}, v_{j,\rho}$ for $1 \le \rho \le m$ are not consecutive in the order previously established.

Consecutive vertices in the ordering satisfy radio condition by construction. For every pair of distinct vertices u and v (both vertices are from P_n, both are terminal or they are of different type), it is easy to verify that radio condition are satisfied. So, g is a radio labeling for $C_(m,0)P_n$. Moreover, g was defined in such a way, that it satisfies the properties (1) to (3) in Theorem 1 for the weight center v_{k+1}. Since the vertices u_i and u_{i+1} belongs to different branches for $2 \leq i \leq m-1$, $u_1 = v_{k+1}$ and $u_M = v_{k+2}$, with $L_{v_{k+1}}(v_{k+2}) = d(v_{k+1}, v_{k+2}) = 1$. $\square$

Theorem 2.3. *Let $C_{(m,0)}P_n$ be the graph for $m \geq 2$ and with $n = 2k$, $k \geq 2$. Then the radio number for $(C_{(m,0)}P_n)$ is $2k^2m - 4km + 2k^2 - 2k + 2m + 2$.*

Proof. In order to prove one inequality, we use Theorem 1 . The caterpillar $C_{(m,0)}P_n$ is obtained by attaching the terminal vertices $v_{i,\rho}$ for $i \leq \rho \leq m$ to the vertex v_i, for each $2 \leq i \leq 2k-1$. For the sake of convenience we consider $v_{i,\rho}$ for $i \leq \rho \leq m$ in anticlockwise direction (See Figure 3).

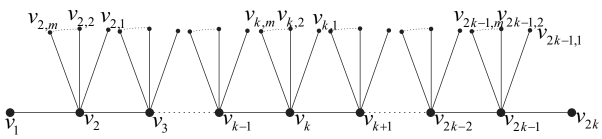

FIGURE 3. $C_{(m,0)}P_{2k}$

We have $V(C_{(m,0)}P_n) = n + (n-2)m = 2k + (2k-2)m = M$ and $d = 2k-1$. Now, we compute the status function of $C_{(m,0)}P_n$. It is clear that there are two weight centers for $C_{(m,0)}P_n$ in the case $n = 2k$. We have

$$\begin{aligned} S(C_{(m,0)}P_n) = S_{C_{(m,0)}P_n}(v_k) &= \sum_{u \in V(C_{(m,0)}P_n)} d(v_k, u) \\ &= m.1 + m.2 + m.3 + ...mk + 1.1 + 1.2 + ...1.k + \\ &\quad 2.m + 3.m + 4.m + ... + (k-1).m + 1.1 + 1.2 + ...1.(k-1) \\ &= (m+1)k^2 - m \end{aligned}$$

It follows from Theorem 1. that

$$\begin{aligned} rn(C_{(m,0)}P_n) &\geq (V(C_{(m,0)}P_n) - 1)(d+1) + 2 - 2S(C_{(m,0)}P_n) \\ &= [2k + (2k-2)m - 1](2k) + 2 - 2[(m+1)k^2 - m] \\ &= 2k^2m - 4km + 2k^2 - 2k + 2m + 2. \end{aligned}$$

Moreover, in order to prove equality, it is sufficient to find a radio labeling g for $C_(m,0)P_n$ that fulfil the properties(1)-(3) in Theorem 1 for every weight center with span$(g) = 2k^2m - 2km + 2k^2 + m + 2$. Because $C_{(m,0)}P_n$ is symmetrical, it is suffices to find a radio labelling $C_{(m,0)}P_n$ with these properties only for weight

center v_k.
For that, we order the vertices of $C_{(m,0)}P_n$ as follows:

$$v_k \to v_{2k-1,1} \to v_{2,1} \to v_{k+1,1} \to v_{3,1} \to v_{k+2,1} \to \dots \to v_{k-1,1} \to v_{2k-2,1}$$
$$\to v_{k,1} \to v_{2k-2,1} \to v_{k,1} \to v_{2k-1,2} \to v_{2,2} \to v_{k+1,2} \to v_{3,2} \to v_{k+2,2}$$
$$\to \dots \to v_{k-1,2} \to v_{2k-2,2} \to v_{k,2} \to v_{2k-1,3} \to v_{2,3} \to v_{k+1,3} \to v_{3,3} \to v_{k+2,3}$$
$$\to \dots \to v_{k-1,3} \to v_{2k-2,3} \to \dots \to v_{k,m-1} \to v_{2k-1,m} \to v_{2,m} \to v_{k+1,m} \to v_{3,m}$$
$$\to v_{k+2,m} \to \dots \to v_{k-1,m} \to v_{2k-2,m} \to v_{k,m} \to v_{2,m} \to v_{2k+1} \to v_k \to v_{2k}$$
$$\to v_{k-1} \to v_{2k} \to v_{k-1} \to v_{2k-1} \to v_{k-2} \to v_{k-(k-1)} = v_1 \to v_{2k-(k-1)} = v_{k+1}.$$

We rename the the vertices of $C_{(m,0)}P_n$ in the above ordering by $u_1, u_2, u_3, ...u_M$. A labeling g for $C_{(m,0)}P_n$ by using the rules given in (2) and (3) from Theorem 1, is defined as follows: $g(u_1) = 1$, $g(u_{i+1}) = g(u_i) + d + 1 - d(u_{i+1}, u_i)$ for $1 \le i \le M-1$. The order in which the vertices are labeled and their labels are shown in Fig.4 for $k = 4, m = 3$.

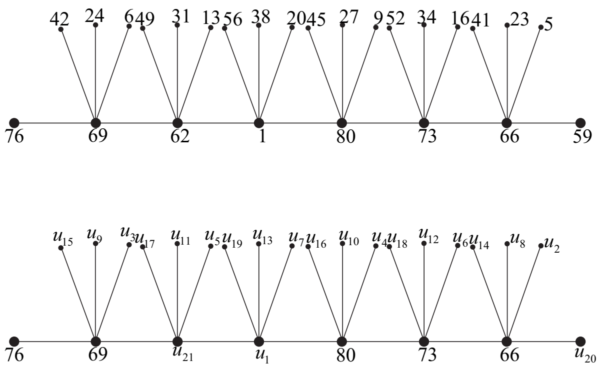

FIGURE 4. $C_{(3,0)}P_8$

Since we have the following distances:

$$d(v_{j,\rho}, v_{k+j-1,\rho}) = k+1; \;\; for \;\; 2 \le j \le k-1 \;\; and \;\; 1 \le \rho \le m;$$
$$d(v_{2,\rho}, v_{2k-1,\rho}) = 2k-1; \;\; for \;\; 1 \le \rho \le m;$$
$$d(v_{k,\rho}, v_{2k-2,\rho}) = k; \;\; for \;\; 1 \le \rho \le m;$$
$$d(v_{k-j-1}, v_{2k-j}) = k, \;\; for \;\; 1 \le j \le k-1;$$
$$d(v_k, v_{2k-1,1}) = k;$$
$$d(v_1, v_{2k}) = 2k-1;$$
$$d(v_{2,1}, v_{2k-1,1}) = 2k-1;$$
$$d(v_{k-1}, v_{2k}) = k+1;$$
$$d(v_{k,m}, v_{2k}) = k+1;$$

$$d(v_1, v_{k+1}) = k;$$
$$d(v_k, v_{2k-1}) = k.$$

We get:

$$sp(g) = g(u_m) = g(v_{k+1}) = g(u_1) + (M-1)(d+1) - \sum_{i=1}^{M-1} d(u_{i+1}, u_i)$$
$$= 2k^2m - 4km + 2k^2 - 2k + 2m + 2.$$

The following relations also hold:

$$g(v_j) = g(v_{j+1}) + 2k - 1 \;\; for \; 1 \leq j \leq k-2;$$
$$g(v_j) = g(v_{j+1}) + 2k - 1 \;\; for \; k+1 \leq j \leq 2k-2;$$
$$g(v_{2k}) = g(v_{k,m}) + k - 1;$$
$$g(v_{2k-1,1}) = g(v_k) + k.$$

Similarly,

$$g(v_{j+1,\rho}) = g(v_{j,\rho}) + 2k - 1 \;\; for \; 2 \leq j \leq k-1, \; and \; 1 \leq \rho \leq m;$$
$$g(v_{j+1,\rho}) = g(v_{j,\rho}) + 2k - 1 \;\; for \; k+1 \leq j \leq 2k-3, \; and \; 1 \leq \rho \leq m;$$
$$g(v_{2,\rho}) = g(2k, \rho) - 1: \;\; for \;\; 1 \leq \rho \leq m.$$

$|g(v_i) - g(v_j)| \geq 2k - 1$ and $|g(v_{i,\rho}) - g(v_{j,\rho})| \geq 2k - 1$ if v_i, v_j and $v_{i,\rho}, v_{j,\rho}$ are not consecutive in the order previously established. Consecutive vertices in the ordering satisfy radio condition by construction. For every pair of distinct vertices u and v (both vertices are from P_n, both are terminal or they are of different type), it is easy to verify that radio condition are satisfied. So, g is a radio labeling for $C(m,0)P_n$. Moreover, g was defined in such a way, that it satisfies the properties (1) to (3) in Theorem 1 for the weight center v_k. Since the vertices u_i and u_{i+1} belongs to different branches for $2 \leq i \leq m-1$, $u_1 = v_k$ and $u_M = v_{k+1}$, with $L_{v_k}(v_{k+1}) = d(v_k, v_{k+1}) = 1$. □

Now, we discussed the radio labeling of Generalized Caterpillar graphs $C_{(m,1)}P_n$ which is obtained from P_n by attaching m vertices of degree two to each vertex of degree two of P_n. We have $M = |V(C_{(m,1)}P_n)| = n + (n-2)2m$ and diameter $d = n + 1$.

Theorem 2.4. *Let $C_{(m,0)}P_n$ be a graph for $m \geq 2$ and with $n = 2k+1$, $k \geq 2$. Then the radio number for $C_{(m,0)}P_n$ is $4k^2m + 2k^2 + 4k + 2$.*

Proof. We achieved the lower bound for the radio number of generalized caterpillar $C_{(m,1)}P_n$ graphs by using Theorem 1. The generalized caterpillar $C_{(m,1)}P_n$ is obtained by attaching the vertices of degree two $v_{i,\rho}$ for $i \leq \rho \leq m$ to the vertex v_i for each $2 \leq i \leq 2k$ and $v'_{i,\rho}$ for $i \leq \rho \leq m$ are the terminal vertices. For the sake of convenience we consider $v_{i,\rho}$ and $v_{i,\rho}$ for $i \leq \rho \leq m$ in anticlockwise direction (See Figure 5).

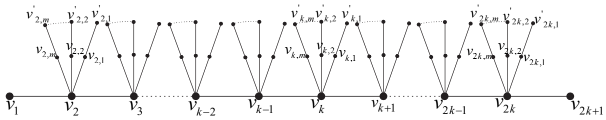

FIGURE 5. $C_{(m,1)}P_{2k+1}$

We have $V(C_{(m,1)}P_n) = n + (n-2)2m = (2k+1) + (2k-1)2m = M$ and $d = 2k+2$. Now, we compute the status function of $C_{(m,1)}P_n$. It is clear that $C_{(m,1)}P_n$ has weight center $v_k + 1$ in the case $n = 2k+1$. We have

$$\begin{aligned}S(C_{(m,1)}P_n) = S_{C_{(m,1)}P_n}(v_{k+1}) &= \sum_{v\in V(C_{(m,1)}P_n)} d(v_{k+1}, v)\\ &= m.1 + m.2 + m.3 + ...m.k + m.2 + m.3 + ... + m.k + 2.1 + 2.2\\ &\quad + ...2.k + m.2 + m.3 + m.4 + ...m.k + m.3 + m.4 + ... + m.(k+1)\\ &= m(k^2+k) + m[k^2+3k+2] + (k^2+k) - 5m\\ &= (2m+1)k^2 + (4m+1)k - 3m.\end{aligned}$$

It follows from Theorem 1, that

$$\begin{aligned}rn(C_{(m,1)}P_n) &\geq (V(C_{(m,1)}P_n) - 1)(d+1) + 2 - 2S(C_{(m,1)}P_n)\\ &= [(2k+1) + (2k-1)2m - 1](2k+3) + 2\\ &\quad - 2[(2m+1)k^2 + (4m+1)k) - 3m]\\ &= 4mk^2 + 2k^2 + 4k + 2.\end{aligned}$$

Moreover, in order to prove equality, it suffices to find a radio labeling g for $C_(m,1)P_n$ that satisfies the properties(1)-(3) in Theorem 1 for weight center with span$(g) = 4mk^2 + 2k^2 + 4k + 2$.
For that, we order the vertices of $C_{(m,1)}P_n$ as follows:

$v_{k+1} \to v'_{2k,1} \to v'_{k,1} \to v'_{2k-1,1} \to v'_{k-1,1} \to ... \to v'_{2k-(k-2),1} = v'_{k+2,1} \to v'_{k-(k-2),1}$
$= v'_{2,1} \to v'_{k+1,1} \to v'_{2k,2} \to v'_{k,2} \to v'_{2k-1,2} \to v'_{k-1,2} \to ... \to v'_{2k-(k-2),2} = v'_{k+2,2}$
$\to v'_{k-(k-2),2} = v'_{2,2} \to .... \to v'_{k+1,m-1} \to v'_{2k,m} \to v'_{k,m} \to v'_{2k-1,m} \to v'_{k-1,m} \to ...$
$\to v'_{2k-(k-2),m} = v'_{k+2,m} \to v'_{k-(k-2),m} = v'_{2,m} \to v'_{k+1,m} \to v_{2k+1} \to v_k \to v_{2k}$
$\to v_{k-1} \to v_{2k-1} \to ... \to v_{2k+1-(k-1)} = v_{k+2} \to v_{k-(k-1)} = v_1 \to v_{2k,1} \to v_{k,1}$
$\to v_{2k-1,1} \to v_{k-1,1} \to ... \to v_{2k-(k-2),1} = v_{k+2,1} \to v_{k-(k-2),1} = v_{2,1} \to v_{k+1,1}$
$\to v_{2k,2} \to v_{k,2} \to v_{2k-1,2} \to v_{k-1,2} \to ... \to v_{2k-(k-2),2} = v_{k+2,2} \to v_{k-(k-2),2} =$
$v_{2,2} \to ... \to v_{k+1,m-1} \to v_{2k,m} \to v_{k,m} \to v_{2k-1,m} \to v_{k-1,m} \to ... \to v_{2k-(k-2),m} =$
$v_{2,m} \to v_{k+1,m}$.

We rename the vertices of $C_{(m,1)}P_n$ in the above ordering by $u_1, u_2, u_3, ...u_M$. A labeling g for $C_{(m,1)}P_n$ by using the rules given in (2) and (3) from Theorem

1, is defined as follows: $g(u_1) = 1$, $g(u_{i+1}) = g(u_i) + d + 1 - d(u_{i+1}, u_i)$ for $1 \leq i \leq M - 1$. The order in which the vertices are labeled and their labels are shown in Fig.6 for $k = 3, m = 3$. Since we have the following distances:

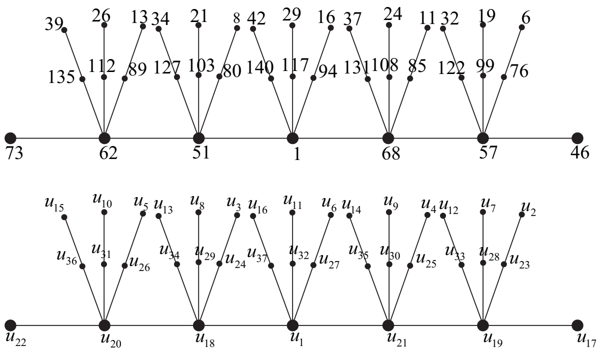

FIGURE 6. $C_{(3,1)}P_7$

$$\begin{aligned}
d(v'_{k-j,\rho}, v'_{2k-j,\rho}) &= k + 4 \ for 0 \leq j \leq k - 2 \, and \, 1 \leq \rho \leq m; \\
d(v_{k-j,m}, v_{2k-j,m}) &= k + 2 \ for \ 0 \leq j \leq k - 2; \\
d(v_{k-j}, v_{2k+1-j}) &= k + 1 \ for \, 0 \leq j \leq k - 1; \\
d(v_{k+1}, v'_{2k,1}) &= k + 1; \\
d(v'_{2,m}, v'_{k+1,m}) &= k + 3; \\
d(v'_{k+1,m}, v_{2k+1}) &= k + 2; \\
d(v_1, v_{2k,1}) &= 2k; \\
d(v_{2,m}, v_{k+1,m}) &= k + 1;
\end{aligned}$$

We get:

$$\begin{aligned}
sp(g) = g(u_M) = g(v_{k+1,m}) &= g(u_1) + (M-1)(d+1) - \sum_{i=1}^{M-1} d(u_{i+1}, u_i) \\
&= 4mk^2 + 2k^2 + 4k + 2.
\end{aligned}$$

The following relations also hold:

$$\begin{aligned}
g(v_j) &= g(v_{j+1}) + 2k + 5 \ \ for \ 1 \leq j \leq k - 1; \\
g(v_j) &= g(v_{j+1}) + 2k + 5 \ \ for \ k + 2 \leq j \leq 2k - 1;
\end{aligned}$$

similarly,

$$\begin{aligned}
g(v_{j,\rho}) &= g(v_{j+1,\rho}) + 2k + 3 \ \ for \ 2 \leq j \leq k - 1, \ and \ 1 \leq \rho \leq m; \\
g(v_{j,\rho}) &= g(v_{j+1,\rho}) + 2k + 3 \ \ for \ k + 1 \leq j \leq 2k - 1, \ and \ 1 \leq \rho \leq m.
\end{aligned}$$

$$g(v'_{j,\rho}) = g(v'_{j+1,\rho}) + 2k - 1 \ \ for\ 2 \le j \le k-1,\ and\ 1 \le \rho \le m;$$
$$g(v'_{j,\rho}) = g(v'_{j+1,\rho}) + 2k - 1 \ \ for\ k+1 \le j \le 2k-1,\ and\ 1 \le \rho \le m.$$
$$g(v'_{2k,1}) = g(v_k) + k + 2;$$
$$g(v_{2k+1}) = g(v'_{k+1,m}) + k + 1;$$
$$g(v_{2k,1}) = g(v_1) + 3;$$

$|g(v_i) - g(v_j)| \ge 2k+5$, $|g(v_{i,\rho}) - g(v_{j,\rho})| \ge 2k+3$ and $|g(v'_{i,\rho}) - g(v'_{j,\rho})| \ge 2k-1$ for $1 \le \rho \le m$ if v_i, v_j , $v_{i.\rho}, v_{j,\rho}$ and $v'_{i.\rho}, v'_{j,\rho}$ for $1 \le \rho \le m$ are not consecutive in the order previously established. Consecutive vertices in the ordering satisfy radio condition by construction. For every pair of distinct vertices u and v (both vertices are from P_n, both are terminal and both vertices of degree two or they are of different type), it is easy to verify that radio condition are satisfied. So, g is a radio labeling for $C_(m,1)P_n$. Moreover, g was defined in such a way, that it satisfies the properties (1) to (3) in Theorem 1 for the weight center v_{k+1}. Since the vertices u_i and u_{i+1} belongs to different branches for $2 \le i \le m-1$, $u_1 = v_{k+1}$ and $u_M = v_{k+1,m}$, with $L_{v_{k+1}}(v_{k+1,m}) = d(v_{k+1}, v_{k+1,m}) = 1$. □

Theorem 2.5. *Let $C_{(m,1)}P_n$ be a graph for $m \ge 2$ and with $n = 2k$, $k \ge 2$. Then the radio number for $C_{(m,1)}P_n$ is $4k^2m - 4km + 2k^2 + 2k$.*

Proof. In order to prove one inequality, we use Theorem 1 . The caterpillar $C_{(m,1)}P_n$ is obtained by attaching the vertices of degree two $v_{i,\rho}$ for $i \le \rho \le m$ to the vertex v_i for each $2 \le i \le 2k-1$ and $v'_{i,\rho}$ for $i \le \rho \le m$ are the terminal vertices. For the sake of convenience we consider $v_{i,\rho}$ and $v_{i,\rho}$ for $i \le \rho \le m$ in anticlockwise direction (See Figure 8).

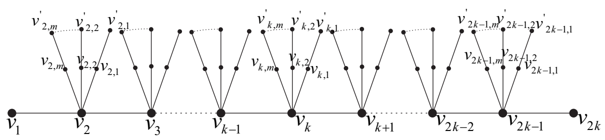

FIGURE 7. $C_{(m,1)}P_{2k}$

We have $V(C_{(m,1)}P_n) = n + (n-2)m = 2k + (2k-2)2m = M$ and $d = 2k+1$. Now, we compute the status function of $C_{(m,1)}P_n$. It is clear that there are two weight centers for $C_{(m,1)}P_n$ in this case. We have

$$\begin{aligned} S(C_{(m,1)}P_n) = S_{C_{(m,1)}P_n}(v_k) &= \sum_{u \in V(C_{(m,1)}P_n)} d(v_k, u) \\ &= m.1 + m.2 + m.3 + ... + m.k + m.2 + m.3 + ... + m.(k+1)1.1 \\ &\quad + 1.2 + ...1.(k-1) + m.2 + m.3 + m.4 + ... + m.(k-1) \\ &\quad + m.3 + m.4 + m.5 + ... + m.k + 1.1 + 1.2 + ...1.k \end{aligned}$$

$$= m(2k^2 + 2k - 4) + k^2.$$

It follows from Theorem 1, that

$$\begin{aligned} rn(C_{(m,1)}P_n) &\geq (V(C_{(m,1)}P_n) - 1)(d+1) + 2 - 2S(C_{(m,1)}P_n) \\ &= [2k + (2k-2)2m - 1](2k+2) + 2 - 2[2mk^2 + 2mk - 4m + k^2] \\ &= 4k^2m - 4km + 2k^2 + 2k. \end{aligned}$$

Moreover, in order to prove equality, it is sufficient to find a radio labeling g for $C_(m,1)P_n$ that fulfil the properties(1)-(3) in Theorem 1 for every weight center with span$(g) = 4k^2m - 4km + 2k^2 + 2k$. Because $C_{(m,1)}P_n$ is symmetrical, so find a radio labelling $C_{(m,1)}P_n$ with these properties only for weight center v_k. For that, we order the vertices of $C_{(m,1)}P_n$ as follows:

$$\begin{aligned} & v_k \to v'_{2k-1,1} \to v'_{2,1} \to v'_{k+1,1} \to v'_{3,1} \to v'_{k+2,1} \to \ldots \to v'_{k-1),1} \to v'_{k+(k-2),1} \\ & = v'_{2k-2,1} \to v'_{k,1} \to v'_{2k-1,2} \to v'_{2,2} \to v'_{k+1,2} \to v'_{3,2} \to v'_{k+2,2} \to \ldots \to v'_{k-1,2} \\ & \to v'_{2k-2,2} \to v'_{k,2} \to v'_{2k-1,3} \to v'_{2,3} \to v'_{k+1,3} \to v'_{3,3} \to v'_{k+2,3} \to \ldots \to v'_{k-1,3} \\ & \to v'_{2k-2,3} \to \ldots \to v'_{k,m-1} \to v'_{2k-1,m} \to v'_{2,m} \to v'_{k+1,m} \to v'_{3,m} \to v'_{k+2,m} \to \ldots \\ & \to v'_{k-1,m} \to v'_{k+(k-2),m} = v'_{2k-2,m} \to v'_{k,m} \to v_{2k-1,1} \to v_{2,1} \to v_{k+1,1} \to v_{3,1} \\ & \to v_{k+2,1} \to \ldots \to v_{2+(k-3),1} = v_{k-1,1} \to v_{(k+1)+(k-3),1} = v_{2k-2,1} \to v_{k,1} \to \\ & v_{2k-1,2} \to v_{2,2} \to v_{k+1,2} \to v_{3,2} \to v_{k+2,2} \to \ldots \to v_{2+(k-3),2} = v_{k-1,2} \to \\ & v_{(k+1)+(k-3),2} = v_{2k-2,2} \to v_{k,2} \to \ldots \to v_{k,m-1} \to v_{2k-1,m} \to v_{2,m} \to v_{k+1,m} \\ & \to v_{3,m} \to v_{3,m} \to v_{k+2,m} \to \ldots \to v_{2+(k-3),m} = v_{k-1,m} \to v_{(k+1)+(k-3),m} \\ & = v_{2k-2,m} \to v_{k,m} \to v_{2k} \to v_{k-1} \to v_{2k-1} \to v_{k-2} \to \ldots \to v_{2k-(k-2)} = v_{k+2} \to \\ & v_{k-(k-1)} = v_1 \to v_{2k-(k-1)} = v_{k+1}. \end{aligned}$$

We rename the the vertices of $C_{(m,1)}P_n$ in the above ordering by $u_1, u_2, u_3, \ldots .u_M$. A labeling g for $C_{(m,1)}P_n$ by using the rules given in (2) and (3) from Theorem 1, is defined as follows: $g(u_1) = 1$, $g(u_{i+1}) = g(u_i) + d + 1 - d(u_{i+1}, u_i)$ for $1 \leq i \leq M - 1$. The order in which the vertices are labeled and their labels are shown in Fig.7 for $k = 4, m = 3$.

Since we have the following distances:

$$\begin{aligned} d(v'_{j,\rho}, v'_{k+j-1,\rho}) &= k + 3 \ \ for \ \ 2 \leq j \leq k - 1 \ \ and \ \ 1 \leq \rho \leq m; \\ d(v_{j,\rho}, v_{k+j-1,\rho}) &= k + 1 \ \ for \ \ 2 \leq j \leq k - 1; \\ d(v_{k-j-1}, v_{2k-j}) &= k + 1 \ \ for \ \ 0 \leq j \leq k - 2; \\ d(v'_{k,\rho}, v'_{2k-2,\rho}) &= k + 2 \ \ for \ \ 1 \leq \rho \leq m; \\ d(v'_{2,\rho}, v'_{2k-1,\rho}) &= 2k + 1 \ \ for \ \ 1 \leq \rho \leq m; \\ d(v_{k,m}, v_{2k}) &= k + 1; \\ d(v_1, v_{k+1}) &= k; \\ d(v_k, v'_{2k-1,1}) &= k + 1; \end{aligned}$$

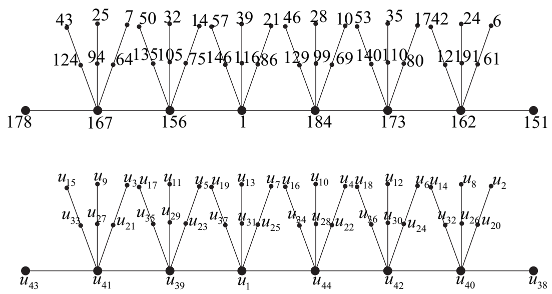

FIGURE 8. $C_{(3,1)}P_8$

$$\begin{aligned} d(v'_{2,1}, v'_{2k-1,1}) &= 2k+1; \\ d(v'_{k,m}, v_{2k-1,1}) &= k+2; \\ d(v_{2k-1,1}, v_{2,1}) &= 2k-1; \end{aligned}$$

We get:

$$\begin{aligned} sp(g) = g(u_m) = g(v_{k+1}) &= g(u_1) + (M-1)(d+1) - \sum_{i=1}^{M-1} d(u_{i+1}, u_i) \\ &= 4k^2m - 4km + 2k^2 + 2k. \end{aligned}$$

The following relations also hold:

$$\begin{aligned} g(v'_{j+1,\rho}) &= g(v'_{j,\rho}) + 2k - 1 \;\; for \;\; 2 \le j \le k-1, \;\; and \;\; 1 \le \rho \le m; \\ g(v'_{j+1,\rho}) &= g(v'_{j,\rho}) + 2k - 1 \;\; for \;\; k+1 \le j \le 2k-3, \;\; and \;\; 1 \le \rho \le m; \\ g(v_{j+1,\rho}) &= g(v_{j,\rho}) + 2k + 3 \;\; for \;\; 1 \le j \le k-1, \;\; and \;\; 1 \le \rho \le m; \\ g(v_{j+1,\rho}) &= g(v_{j,\rho}) + 2k + 3 \;\; for \;\; k+1 \le j \le 2k-3, \;\; and \;\; 1 \le \rho \le m; \\ g(v_j) &= g(v_{j+1}) + 2k + 3 \;\; for \;\; 1 \le j \le k-2; \\ g(v_j) &= g(v_{j+1}) + 2k + 3 \;\; for \;\; k+1 \le j \le 2k-1; \\ g(v_{2k-1,\rho}) &= g(v'_{2,\rho}) - 1 \;\; for \;\; 1 \le \rho \le m; \\ g(v_{2,\rho}) &= g(v_{2k-1,\rho}) + 3 \;\; for \;\; 1 \le \rho \le m; \\ g(v'_{2k-1,1}) &= g(v_1) + k + 1; \\ g(v_{2k-1,1}) &= g(v'_{k,m}) + k; \\ g(v_{2k}) &= g(v_{k,m}) + k + 1. \end{aligned}$$

Similarly, $|g(v_i) - g(v_j)| \ge 2k+3$ and $|g(v_{i,\rho}) - g(v_{j,\rho})| \ge 2k+3$, $|g(v'_{i,\rho}) - g(v'_{j,\rho})| \ge 2k+3$ for $1 \le \rho \le m$,if v_i, v_j and $v_{i,\rho}, v_{j,\rho}$, $v'_{i,\rho}, v'_{j,\rho}$ for $1 \le \rho \le m$ are not consecutive in the order previously established. Consecutive vertices in the ordering satisfy radio condition by construction. For every pair of distinct vertices u and v (both vertices are from P_n, both are terminal or they are of degree two or they are of different type), it is easy to verify that radio condition are satisfied. So, g is a radio labeling for $C_(m,1)P_n$. Moreover, g was defined in such a way, that it satisfies the properties (1) to (3) in Theorem 1 for the weight center v_k. Since the vertices u_i and u_{i+1} belongs to different branches for $2 \leq i \leq m-1$, $u_1 = v_k$ and $u_M = v_{k+1}$, with $L_{v_k}(v_{k+1}) = d(v_k, v_{k+1}) = 1$. □

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Saima Nazeer received M.Sc. from The University of Punjab and Ph.D at Lahore College for Women University. Since 2006 she has been at Lahore College for Women University. Her research interest is Graph Theory.

Lahore College for Women University, Lahore Pakistan.
e-mail: saimanazeer123@yahoo.com

M. Saqib Khan received B.Sc. from Lahore University of Managment sciences and M.S. frm Florida Atlantic University. His research interests are computational mathematics, and graph theory.

Department of Mathematics, Lahore Leads University, Lahore Pakistan.
e-mail: khan.saqib@me.com

Imrana Kausar received M.Sc. from The University of the Punjab, and Ph.D. from FAST Pakistan. She is currently associte professor at Lahore College for Women University. Her research interests are Number Theory and Graph Theory.

Department of Mathematics, Lahore College for Women University, Lahore Pakistan.
e-mail: imrana.kausar@hotmail.com

Waqas Nazeer received M.Sc. from The University of punjab, Lahored and Ph.D from Abdus Salam School of Mathematical Sciences, Goverment College University, Lahore pakistan. Since 2014 he has been at Division of Science and Technology, University of Education, Township. His research interests include Numerical Analysis, Graph Theory and Functional Analysis.

Division of Science and Technology, University of Education, Township, Lahore Pakistan.
e-mail: nazeer.waqas@ue.edu.pk