J. Appl. Math. & Informatics Vol. **34***(2016), No. 5 - 6, pp. 443 - 450*
http://dx.doi.org/10.14317/jami.2016.443

IDENTITIES INVOLVING TANGENT NUMBERS AND POLYNOMIALS

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Abstract. In this paper we give some properties, explicit formulas, several identities, a connection with tangent numbers and polynomials, and some integral formulas.

AMS Mathematics Subject Classification : 11B68, 11S40, 11S80.
Key words and phrases : Tangent numbers and polynomials, Gamma function, Beta function, Eulerian integral.

1. Introduction

Recently, many mathematicians have studied in the area of the Bernoulli numbers, Euler numbers, Genocchi numbers, and tangent numbers (see [1, 2, 3, 4, 5, 8, 9 10, 11]). Throughout this paper, we always make use of the following notations: $\mathbb{N}$ denotes the set of natural numbers and $\mathbb{Z}_+ = \mathbb{N} \cup \{0\}$, and $\mathbb{C}$ denotes the set of complex numbers. The tangent numbers T_n are defined by the generating function:

$$F(t) = \frac{2}{e^{2t}+1} = \sum_{n=0}^{\infty} T_n \frac{t^n}{n!}, (|t| < \frac{\pi}{2}),$$

where we use the technique method notation by replacing T^n by $T_n (n \geq 0)$ symbolically [6, 7]. We consider the tangent polynomials $T_n(x)$ as follows:

$$F(x,t) = \left(\frac{2}{e^{2t}+1}\right) e^{xt} = \sum_{n=0}^{\infty} T_n(x) \frac{t^n}{n!}. \tag{1.1}$$

Note that $T_n(x) = \sum_{k=0}^{n} \binom{n}{k} T_k x^{n-k}$. Numerous properties of tangent number are known. More studies and results in this subject we may see references [6, 7,

Received March 20, 2016. Revised May 16, 2016. Accepted May 23, 2016. *Corresponding author.

8]. About extensions for the tangent numbers can be found in [6, 7, 8]. Because

$$\frac{\partial F}{\partial x}(x,t) = tF(x,t) = \sum_{n=0}^{\infty} \frac{dT_n}{dx}(x)\frac{t^n}{n!},$$

it follows the important relation

$$\frac{dT_n}{dx}(x) = nT_{n-1}(x).$$

Since

$$\begin{aligned}\int_0^x T_n(t)dt &= \sum_{l=0}^{n} \binom{n}{l} T_l \int_0^x t^{n-l} dt \\ &= \sum_{l=0}^{n} \binom{n}{l} T_l \left. \frac{t^{n-l+1}}{n-l+1} \right|_0^x \\ &= \frac{1}{n+1} \sum_{l=0}^{n+1} \binom{n+1}{l} T_l \left. t^{n-l+1} \right|_0^x,\end{aligned}$$

we see that

$$\int_0^x T_n(t)dt = \frac{T_{n+1}(x) - T_{n+1}(0)}{n+1}. \tag{1.2}$$

Since $T_n(0) = T_n$, by (1.2), we have the following theorem.

Theorem 1.1. *For $n \in \mathbb{N}$, we have*

$$T_n(x) = T_n + n\int_0^x T_{n-1}(t)dt.$$

From (1.1), we can derive the following equation:

$$\begin{aligned}\sum_{n=0}^{\infty} T_n(2-x)\frac{(-t)^n}{n!} &= \frac{2}{e^{-2t}+1} e^{(2-x)(-t)} \\ &= \frac{2}{e^{2t}+1} e^{xt} = \sum_{n=0}^{\infty} T_n(x)\frac{t^n}{n!}.\end{aligned} \tag{1.3}$$

By comparing the coefficients on both sides of (1.3), we have the following theorem.

Theorem 1.2. *For any positive integer n, we have*

$$T_n(x) = (-1)^n T_n(2-x). \tag{1.4}$$

Now we observed that

$$\begin{aligned}\sum_{n=0}^{\infty} T_n(2-x)\frac{t^n}{n!} &= \frac{2}{e^{2t}+1}e^{(2-x)t} \\ &= \frac{2}{e^{2t}+1}e^{2t}e^{(-x)t} \\ &= \left(\sum_{n=0}^{\infty} T_n(2)\frac{t^n}{n!}\right)\left(\sum_{m=0}^{\infty}(-x)^m\frac{t^m}{m!}\right) \\ &= \sum_{n=0}^{\infty}\left(\sum_{m=0}^{n}\binom{n}{m}T_{n-m}(2)(-1)^m x^m\right)\frac{t^n}{n!}.\end{aligned} \tag{1.5}$$

By (1.5), we have the following theorem.

Theorem 1.3. *For any positive integer n, we have*

$$T_n(2-x) = \sum_{k=0}^{n}\binom{n}{k}(-1)^k T_{n-k}(2)x^k. \tag{1.6}$$

The beta integral is defined for $\mathrm{Re}(x) > 0$, $\mathrm{Re}(y) > 0$ by

$$B(x,y) = \int_0^1 t^{x-1}(1-t)^{y-1}dt. \tag{1.7}$$

For $\mathrm{Re}(x) > 0$, the gamma function $\Gamma(x)$ is defined by

$$\Gamma(x) = \int_0^{\infty} t^{x-1}e^{-t}dt. \tag{1.8}$$

The above integral for $\Gamma(x)$ is sometimes called the Eulerian integral of the second kind. Thus, by (1.7) and (1.8), we have

$$\Gamma(x+1) = x\Gamma(x), \quad B(x,y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}. \tag{1.9}$$

Our aim in this paper is to give some properties, explicit formulas, several identities, a connection with tangent numbers and polynomials, and some integral formulas.

2. Identities involving tangent numbers and polynomials

In this section, we obtain several new and interesting identities involving tangent numbers and polynomials.

By (1.6), we get

$$\begin{aligned}\int_0^1 T_n(2-x)x^n dx &= \sum_{k=0}^{n}\binom{n}{k}(-1)^k T_{n-k}(2)\int_0^1 x^{k+n}dx. \\ &= \sum_{k=0}^{n}\binom{n}{k}(-1)^k\frac{T_{n-k}(2)}{n+k+1}.\end{aligned} \tag{2.1}$$

Since

$$\sum_{n=0}^{\infty} T_n(x+y)\frac{t^n}{n!} = \frac{2}{e^{2t}+1}e^{xt}e^{yt} = \left(\sum_{n=0}^{\infty} T_n(x)\frac{t^n}{n!}\right)\left(\sum_{m=0}^{\infty} y^m \frac{t^m}{m!}\right)$$
$$= \sum_{n=0}^{\infty}\left(\sum_{m=0}^{n}\binom{n}{m}T_m(x)y^{n-m}\right)\frac{t^n}{n!},$$

we have the following theorem.

Theorem 2.1. *For $n \in \mathbb{Z}_+$, we have*

$$T_n(x+y) = \sum_{k=0}^{n}\binom{n}{k}T_k(x)y^{n-k}. \tag{2.2}$$

By (2.2), we note that

$$\begin{aligned}\int_0^1 y^n T_n(x+y)dy &= \int_0^1 y^n \sum_{l=0}^{n}\binom{n}{l}T_{n-l}(x)y^l dy \\ &= \sum_{l=0}^{n}\binom{n}{l}T_{n-l}(x)\int_0^1 y^{n+l}dy \\ &= \sum_{l=0}^{n}\binom{n}{l}T_{n-l}(x)\frac{1}{n+l+1}.\end{aligned} \tag{2.3}$$

From (1.4) and (2.3), we note that

$$\begin{aligned}&\int_0^1 y^n T_n(x+y)dy = y^n \frac{T_{n+1}(x+y)}{n+1}\Big|_0^1 - \int_0^1 ny^{n-1}\frac{T_{n+1}(x+y)}{n+1}dy \\ &= \frac{T_{n+1}(x+1)}{n+1} - \frac{n}{n+1}\int_0^1 y^{n-1}T_{n+1}(x+y)dy \\ &= \frac{T_{n+1}(x+1)}{n+1} - \frac{n}{n+1}\int_0^1 (-1)^{n+1}y^{n-1}T_{n+1}(2-(x+y))dy \\ &= \frac{T_{n+1}(x+1)}{n+1} \\ &\qquad - \frac{n}{n+1}\int_0^1 (-1)^{n+1}y^{n-1}\sum_{l=0}^{n+1}\binom{n+1}{l}T_{n+1-l}(1-x)(1-y)^l dy \\ &= \frac{T_{n+1}(x+1)}{n+1} \\ &\qquad - \frac{n}{n+1}(-1)^{n+1}\sum_{l=0}^{n+1}\binom{n+1}{l}T_{n+1-l}(1-x)\int_0^1 y^{n-1}(1-y)^l dy \\ &= \frac{T_{n+1}(x+1)}{n+1} + \frac{n}{n+1}\sum_{l=0}^{n+1}\binom{n+1}{l}(-1)^{1-l}T_{n+1-l}(x+1)B(n,l+1)\end{aligned} \tag{2.4}$$

Therefore, by (2.3) and (2.4), we obtain the following theorem.

Theorem 2.2. *For $n \in \mathbb{Z}_+$, we have*

$$\frac{T_{n+1}(x+1)}{n+1} = \sum_{l=0}^{n} \binom{n}{l} T_{n-l}(x) \frac{1}{n+l+1} + \frac{n}{n+1} \sum_{l=0}^{n+1} \binom{n+1}{l} (-1)^l T_{n+1-l}(x+1) B(n, l+1).$$

For $n \in \mathbb{N}$ with $n \geq 4$, we obtain

$$\begin{aligned}
\int_0^1 y^n T_n(x+y)dy &= y^n \frac{T_n(x+y)}{n+1}\bigg|_0^1 - \int_0^1 ny^{n+1} \frac{T_{n-1}(x+y)}{n+1} dy \\
&= \frac{T_n(x+1)}{n+1} - \frac{n}{n+1} \int_0^1 y^{n+1} T_{n-1}(x+y)dy \\
&= \frac{T_n(x+1)}{n+1} - \frac{nT_{n-1}(x+1)}{(n+1)(n+2)} + (-1)^2 \frac{n}{n+1} \frac{n-1}{n+2} \int_0^1 y^{n+2} T_{n-2}(x+y)dy \\
&= \frac{T_n(x+1)}{n+1} + (-1) \frac{nT_{n-1}(x+1)}{(n+1)(n+2)} + (-1)^2 \frac{n}{n+1} \frac{n-1}{n+2} \frac{T_{n-2}(x+1)}{n+3} \\
&\quad + (-1)^3 \frac{n}{n+1} \frac{n-1}{n+2} \frac{n-2}{n+3} \int_0^1 y^{n+3} T_{n-3}(x+y)dy \\
&= \frac{T_n(x+1)}{n+1} + (-1) \frac{nT_{n-1}(x+1)}{(n+1)(n+2)} + (-1)^2 \frac{n}{n+1} \frac{n-1}{n+2} \frac{T_{n-2}(x+1)}{n+3} \\
&\quad + (-1)^3 \frac{n}{n+1} \frac{n-1}{n+2} \frac{n-2}{n+3} \frac{T_{n-3}(x+1)}{n+4} \\
&\quad + (-1)^4 \frac{n}{n+1} \frac{n-1}{n+2} \frac{n-2}{n+3} \frac{n-3}{n+4} \int_0^1 y^{n+4} T_{n-4}(x+y)dy
\end{aligned}$$

Continuing this process, we obtain

$$\begin{aligned}
\int_0^1 y^n T_n(x+y)dy &= \frac{T_n(x+1)}{n+1} \\
&\quad + \sum_{l=2}^{n} \frac{n(n-1)\cdots(n-l+2)(-1)^{l-1}}{(n+1)(n+2)\cdots(n+l)} T_{n-l+1}(x+1) \\
&\quad + (-1)^n \frac{n!}{(n+1)(n+2)\cdots(2n)} \int_0^1 y^{2n} T_0(x+y)dy
\end{aligned} \tag{2.5}$$

Hence, by (2.3) and (2.5), we have the following theorem.

Theorem 2.3. *For $n \in \mathbb{N}$ with $n \geq 2$, we have*

$$\begin{aligned}\sum_{l=0}^{n} \binom{n}{l} T_{n-l}(x) \frac{1}{n+l+1} \\ = \frac{T_n(x+1)}{n+1} + \sum_{l=2}^{n} \frac{n(n-1)\cdots(n-l+2)(-1)^{l-1}}{(n+1)(n+2)\cdots(n+l)} T_{n-l+1}(x+1) \\ + (-1)^n \frac{n!}{(n+1)(n+2)\cdots(2n)} \times \frac{1}{2n+1}.\end{aligned}$$

By Theorem 2.2 and Theorem 2.3, we have the following corollary.

Corollary 2.4. *For $n \in \mathbb{N}$ with $n \geq 2$, we have*

$$\begin{aligned}T_{n+1}(x+1) - T_n(x+1) = n \sum_{l=0}^{n+1} \binom{n+1}{l} (-1)^l T_{n+1-l}(x+1) B(n, l+1) \\ + \sum_{l=2}^{n} \frac{n(n-1)\cdots(n-l+2)(-1)^{l-1}}{(n+2)\cdots(n+l)} T_{n-l+1}(x+1) \\ + (-1)^n \frac{n!}{(n+2)\cdots(2n)(2n+1)}.\end{aligned}$$

From (1.4), we have $(-1)^n T_n = T_n(2)$. Putting $x = 1$ in Theorem 2.3 gives the identity

$$\begin{aligned}\sum_{l=0}^{n} \binom{n}{l} T_{n-l}(1) \frac{1}{n+l+1} \\ = \frac{(-1)^n T_n}{n+1} + \sum_{l=2}^{n} \frac{n(n-1)\cdots(n-l+2)(-1)^{l-1}}{(n+1)(n+2)\cdots(n+l)} (-1)^{n-l+1} T_{n-l+1} \\ + (-1)^n \frac{n!}{(n+1)(n+2)\cdots(2n)} \times \frac{1}{2n+1}.\end{aligned}$$

Hence we have the following corollary.

Corollary 2.5. *For $n \in \mathbb{N}$ with $n \geq 2$, we have*

$$\begin{aligned}T_n = \sum_{l=0}^{n} (-1)^n \binom{n}{l} T_{n-l}(1) \frac{n+1}{n+l+1} - \sum_{l=2}^{n} \frac{n(n-1)\cdots(n-l+2)}{(n+2)\cdots(n+l)} T_{n-l+1} \\ - \frac{n!}{(n+2)\cdots(2n)(2n+1)}.\end{aligned}$$

Putting $x = 1$ in Corollary 2.4 yields an identity

$$\begin{aligned}T_{n+1} + T_n = n \sum_{l=0}^{n+1} \binom{n+1}{l} T_{n+1-l} B(n, l+1) \\ - \sum_{l=2}^{n} \frac{n(n-1)\cdots(n-l+2)}{(n+2)\cdots(n+l)} T_{n-l+1} - \frac{n!}{(n+2)\cdots(2n)(2n+1)}.\end{aligned}$$

Now we observe that

$$\begin{aligned}
\int_0^2 T_n(x)T_m(x)dx &= \int_0^2 \sum_{l=0}^{n} \binom{n}{l} T_l x^{n-l}(-1)^m T_m(2-x)dx \\
&= \int_0^2 \sum_{l=0}^{n} \binom{n}{l} T_l x^{n-l}(-1)^m \sum_{k=0}^{m} \binom{m}{k} T_k (2-x)^{m-k}dx \\
&= \sum_{l=0}^{n}\sum_{k=0}^{m} \binom{n}{l}\binom{m}{k} T_l T_k (-1)^m 2^{n+m-l-k+1} \int_0^1 x^{n-l}(1-x)^{m-k}dx \\
&= \sum_{l=0}^{n}\sum_{k=0}^{m} \binom{n}{l}\binom{m}{k} T_l T_k (-1)^m 2^{n+m-l-k+1} B(n-l+1, m-k+1) \\
&= \sum_{l=0}^{n}\sum_{k=0}^{m} \binom{n}{l}\binom{m}{k} T_l T_k (-1)^m 2^{n+m-l-k+1} \frac{\Gamma(n-l+1)\Gamma(m-k+1)}{\Gamma(n+m-l-k+2)} \\
&= \sum_{l=0}^{n}\sum_{k=0}^{m} \frac{\binom{n}{l}\binom{m}{k}(-1)^m 2^{n+m-l-k+1}}{\binom{m+n-l-k}{n-l}} \frac{T_l T_k}{(n+m-l-k+1)}
\end{aligned} \tag{2.6}$$

For $m, n \in \mathbb{N}$ with $m, n \geq 2$, we have

$$\begin{aligned}
\int_0^2 T_n(x)T_m(x)dx &= T_m(x)\frac{T_{n+1}(x)}{n+1}\Big|_0^2 - \int_0^2 mT_{m-1}(x)\frac{T_{n+1}(x)}{n+1}dx \\
&= -\frac{m}{n+1}\int_0^2 T_{m-1}(x)T_{n+1}(x)dx \\
&= (-1)^2 \frac{m(m-1)}{(n+1)(n+2)}\int_0^2 T_{m-2}(x)T_{n+2}(x)dx
\end{aligned}$$

Continuing this process, we get

$$\begin{aligned}
&\int_0^2 T_n(x)T_m(x)dx \\
&= (-1)^m \frac{m(m-1)\cdots 3\cdot 2\cdot 1}{(n+1)(n+2)\cdots(n+m)}\int_0^2 T_{n+m}(x)T_0(x)dx \\
&= (-1)^{m+1} \frac{m(m-1)\cdots 3\cdot 2\cdot 1}{(n+1)(n+2)\cdots(n+m)}\frac{2T_{n+m+1}}{n+m+1}
\end{aligned} \tag{2.7}$$

By (2.6) and (2.7), we have the following theorem.

Theorem 2.6. *For $m, n \in \mathbb{N}$, we have*

$$\begin{aligned}
&\sum_{l=0}^{n}\sum_{k=0}^{m} \frac{\binom{n}{l}\binom{m}{k} 2^{n+m-l-k}}{\binom{m+n-l-k}{n-l}} \frac{T_l T_k}{(n+m-l-k+1)} \\
&= -\frac{m! T_{n+m+1}}{(n+1)(n+2)\cdots(n+m)}.
\end{aligned}$$

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