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BINDING NUMBERS AND FRACTIONAL (g, f, n)-CRITICAL GRAPHS†

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Abstract. Let G be a graph, and let g, f be two nonnegative integer-valued functions defined on $V(G)$ with $g(x) \leq f(x)$ for each $x \in V(G)$. A graph G is called a fractional (g, f, n)-critical graph if after deleting any n vertices of G the remaining graph of G admits a fractional (g, f)-factor. In this paper, we obtain a binding number condition for a graph to be a fractional (g, f, n)-critical graph, which is an extension of Zhou and Shen's previous result (S. Zhou, Q. Shen, On fractional (f, n)-critical graphs, Inform. Process. Lett. 109(2009)811–815). Furthermore, it is shown that the lower bound on the binding number condition is sharp.

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1. Introduction

The graphs considered in this paper are finite, undirected and simple, and see [1] for all notation and terminology not explained here.

Let G be a graph. We denote its vertex set and edge set by $V(G)$ and $E(G)$, respectively. The degree $d_G(v)$ of a vertex $v \in V(G)$ is the number of edges of G incident with v. Set $\delta(G) = \min\{d_G(v) : v \in V(G)\}$. The neighborhood of a vertex v in G is the set $N_G(v) = \{u \in V(G) : vu \in E(G)\}$. For $X \subseteq V(G)$, we write $N_G(X)$ for the union of $N_G(v)$ for each $v \in X$ and denote by $G[X]$ the subgraph of G induced by X. Set $G - X = G[V(G) \setminus X]$. The binding number of a graph G is denoted by $bind(G)$ and it is defined as

$$bind(G) = \min\left\{\frac{|N_G(X)|}{|X|} : \emptyset \neq X \subseteq V(G), N_G(X) \neq V(G)\right\}.$$

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Let g and f be two integer-valued functions defined on $V(G)$ with $0 \leq g(x) \leq f(x)$ for each $x \in V(G)$. A (g,f)-factor of a graph G is a spanning subgraph F of G satisfying $g(x) \leq d_F(x) \leq f(x)$ for each $x \in V(G)$. A fractional (g,f)-factor of a graph G is a function h that assigns to each edge of G a number in $[0,1]$, so that for any $x \in V(G)$ we have $g(x) \leq d_G^h(x) \leq f(x)$, where $d_G^h(x) = \sum_{e \ni x} h(e)$ (the sum is taken over all edges incident to x) is a fractional degree of x in G. A fractional (f,f)-factor is abbreviated to a fractional f-factor. A fractional (g,f)-factor is a fractional $[a,b]$-factor if $g(x) = a$ and $f(x) = b$ for each $x \in V(G)$. If $a = b = k$, then a fractional $[k,k]$-factor is said to be a fractional k-factor. A graph G is called a fractional (g,f,n)-critical graph if after deleting any n vertices of G the remaining graph of G admits a fractional (g,f)-factor. A fractional (f,f,n)-critical graph is abbreviated to a fractional (f,n)-critical graph. If $g(x) = a$ and $f(x) = b$ for each $x \in V(G)$, then a fractional (g,f,n)-critical graph is said to be a fractional (a,b,n)-critical graph. A fractional (f,n)-critical graph is a fractional (k,n)-critical graph if $f(x) = k$ for each $x \in V(G)$.

Many results on factors [2–6,10,14] and fractional factors [7,8,11,13,16] of graphs are known.

Zhou and Shen [15] proved the following theorem, which shows the the relationship between binding number and fractional (f,n)-critical graphs.

Theorem 1 ([15])**.** *Let G be a graph of order p, and let a, b and n be nonnegative integers such that $2 \leq a \leq b$, and let f be an integer-valued function defined on $V(G)$ such that $a \leq f(x) \leq b$ for each $x \in V(G)$. If $bind(G) > \frac{(a+b-1)(p-1)}{ap-(a+b)-bn+2}$ and $p \geq \frac{(a+b)(a+b-3)}{a} + \frac{bn}{a-1}$, then G is fractional (f,n)-critical.*

Liu extended a fractional (f,n)-critical graph to a fractional (g,f,n)-critical graph and obtained a toughness condition for the existence of fractional (g,f,n)-critical graphs in [9].

Theorem 2 ([9])**.** *Let G be a graph and let g, f be two nonnegative integer-valued functions defined on $V(G)$ satisfying $a \leq g(x) \leq f(x) \leq b$ with $1 \leq a \leq b$ and $b \geq 2$ for all $x \in V(G)$, where a, b are positive integers. If $t(G) \geq \frac{(b^2-1)(n+1)}{a}$, then G is a fractional (g,f,n)-critical graph, where n is a positive integer with $|V(G)| \geq n+1$.*

In this paper, we proceed to investigate the fractional (g,f,n)-critical graphs and obtain a binding number condition for the existence of fractional (g,f,n)-critical graphs, which is an extension of Theorem 1. Our main result is the following theorem.

Theorem 3. *Let a, b, r and n be four nonnegative integers with $2 \leq a \leq b - r$, and let G be a graph of order p with $p \geq \frac{(a+b-1)(a+b+1)}{a+r} + \frac{bn}{a+r-1}$, and let g, f be two integer-valued functions defined on $V(G)$ with $a \leq g(x) \leq f(x) - r \leq b - r$ for each $x \in V(G)$. If $bind(G) > \frac{(a+b-1)(p-1)}{(a+r)p-(a+b-2)-bn}$, then G is fractional (g,f,n)-critical.*

If $n = 0$ in Theorem 3, we obtain the following corollary.

Corollary 1. *Let a, b and r be three nonnegative integers with $2 \le a \le b - r$, and let G be a graph of order p with $p \ge \frac{(a+b-1)(a+b+1)}{a+r}$, and let g, f be two integer-valued functions defined on $V(G)$ with $a \le g(x) \le f(x) - r \le b - r$ for each $x \in V(G)$. If $bind(G) > \frac{(a+b-1)(p-1)}{(a+r)p-(a+b-2)}$, then G has a fractional (g, f)-factor.*

If $r = 0$ in Theorem 3, then we have the following corollary.

Corollary 2. *Let a, b and n be three nonnegative integers with $2 \le a \le b$, and let G be a graph of order p with $p \ge \frac{(a+b-1)(a+b+1)}{a} + \frac{bn}{a-1}$, and let g, f be two integer-valued functions defined on $V(G)$ with $a \le g(x) \le f(x) \le b$ for each $x \in V(G)$. If $bind(G) > \frac{(a+b-1)(p-1)}{ap-(a+b-2)-bn}$, then G is fractional (g, f, n)-critical.*

2. The Proof of Theorem 3

The purpose of this section is to prove Theorem 3. For the proof of Theorem 3, we need the following lemmas.

Lemma 2.1 ([9]). *Let G be a graph, and let n be a nonnegative integer, and let g, f be two integer-valued functions defined on $V(G)$ with $0 \le g(x) \le f(x)$ for each $x \in V(G)$. Then G is fractional (g, f, n)-critical if and only if for any subset S of $V(G)$ with $|S| \ge n$*

$$\delta_G(S,T) = f(S) + d_{G-S}(T) - g(T) \ge f_n(S),$$

where $T = \{x : x \in V(G) \setminus S, d_{G-S}(x) \le g(x)\}$, $d_{G-S}(T) = \sum_{x \in T} d_{G-S}(x)$ and $f_n(S) = \max\{f(U) : U \subseteq S, |U| = n\}$.

Lemma 2.2. *Let G be a graph of order p, and let a, b, r and n are four nonnegative integers with $1 \le a \le b - r$, and let g, f be two integer-valued functions defined on $V(G)$ satisfying $a \le g(x) \le f(x) - r \le b - r$ for each $x \in V(G)$. If $p \ge \frac{(a+b-1)(a+b+1)+bn}{a+r}$ and $\delta(G) \ge \frac{(b-r)p+bn}{a+b}$, then G is fractional (g, f, n)-critical.*

Proof. Suppose that G satisfies the hypothesis of Lemma 2.2, but it is not fractional (g, f, n)-critical. Then according to Lemma 2.1, there exists some subset S of $V(G)$ with $|S| \ge n$ satisfying

$$\delta_G(S,T) = f(S) + d_{G-S}(T) - g(T) \le f_n(S) - 1, \tag{1}$$

where $T = \{x : x \in V(G) \setminus S, d_{G-S}(x) \le g(x)\}$, $d_{G-S}(T) = \sum_{x \in T} d_{G-S}(x)$ and $f_n(S) = \max\{f(U) : U \subseteq S, |U| = n\}$.

Note that $f(S) \ge f_n(S)$. If $T = \emptyset$, then by (1) we have $f_n(S) - 1 \ge f(S) \ge f_n(S)$, a contradiction. Therefore, $T \ne \emptyset$. In the following, we define $h = \min\{d_{G-S}(x) : x \in T\}$. According to the definition of T, we have $0 \le h \le b - r$.

We choose $x_1 \in T$ with $d_{G-S}(x_1) = h$. Thus, we obtain

$$\delta(G) \le d_G(x_1) \le d_{G-S}(x_1) + |S| = h + |S|.$$

As a consequence,

$$|S| \geq \delta(G) - h. \tag{2}$$

Note that $f_n(S) = \max\{f(U) : U \subseteq S, |U| = n\} \leq bn$. And then using (1), (2) and $|S| + |T| \leq p$, we obtain

$$\begin{aligned} bn - 1 \geq f_n(S) - 1 \geq \delta_G(S,T) &= f(S) + d_{G-S}(T) - g(T) \\ &\geq (a+r)|S| - (b-r-h)|T| \\ &\geq (a+r)|S| - (b-r-h)(p-|S|) \\ &= (a+b-h)|S| - (b-r-h)p \\ &\geq (a+b-h)(\delta(G)-h) - (b-r-h)p \\ &= (a+b-h)\delta(G) - (a+b-h)h - (b-r-h)p. \end{aligned}$$

Solving for $\delta(G)$, we obtain the following

$$\delta(G) \leq \frac{(b-r-h)p + (a+b-h)h + bn - 1}{a+b-h}.$$

Let $F(h) = \frac{(b-r-h)p+(a+b-h)h+bn-1}{a+b-h}$. Taking the derivative of $F(h)$ with respect to h yields

$$\begin{aligned} \frac{dF}{dh} &= \frac{(a+b-h)(-p+a+b-2h) + ((b-r-h)p + (a+b-h)h + bn - 1)}{(a+b-h)^2} \\ &= \frac{-(a+r)p + (a+b-h)^2 + bn - 1}{(a+b-h)^2} \\ &\leq \frac{-(a+r)p + (a+b)^2 + bn - 1}{(a+b-h)^2}. \end{aligned}$$

For $p \geq \frac{(a+b-1)(a+b+1)+bn}{a+r}$, we have $\frac{dF}{dh} \leq 0$, which implies that $F(h)$ attains its maximum value at $h = 0$. Hence,

$$\delta(G) \leq \frac{(b-r)p + bn - 1}{a+b},$$

which contradicts $\delta(G) \geq \frac{(b-r)p+bn}{a+b}$. The proof of Lemma 2.2 is complete. □

Lemma 2.3 ([12]). *Let c be a positive real, and let G be a graph of order p with $bind(G) := \beta > c$. Then $\delta(G) \geq p - \frac{p-1}{\beta} > p - \frac{p-1}{c}$.*

Proof of Theorem 3. Suppose that G satisfies the hypothesis of Theorem 3, but it is not fractional (g, f, n)-critical. Again, we apply Lemma 2.1, with the same notations and sets as defined in the proof of Lemma 2.2. In addition, we use $\beta := bind(G)$ to simplify the notation below.

In the following, we need only to consider $h = 0$; for $h \geq 1$, apply the same argument as in Lemma 2.2. Let $Y = \{x : x \in T, d_{G-S}(x) = 0\}$. Obviously,

$Y \neq \emptyset$ and $N_G(V(G) \setminus S) \cap Y = \emptyset$. Note that $|N_G(V(G) \setminus S)| \leq p - |Y|$. According to the definition of $bind(G)$, we have

$$bind(G) = \beta \leq \frac{|N_G(V(G) \setminus S)|}{|V(G) \setminus S|} \leq \frac{p - |Y|}{p - |S|},$$

that is,

$$|S| \geq (1 - \frac{1}{\beta})p + \frac{1}{\beta}|Y|. \tag{3}$$

It follows from (1), (3), $f_n(S) \leq bn$ and $|S| + |T| \leq p$ that

$$\begin{aligned} bn - 1 &\geq f_n(S) - 1 \geq \delta_G(S, T) = f(S) + d_{G-S}(T) - g(T) \\ &\geq (a + r)|S| - (b - r - 1)|T| - |Y| \\ &\geq (a + r)|S| - (b - r - 1)(p - |S|) - |Y| \\ &= (a + b - 1)|S| - (b - r - 1)p - |Y| \\ &\geq (a + b - 1)((1 - \frac{1}{\beta})p + \frac{1}{\beta}|Y|) - (b - r - 1)p - |Y| \\ &= (a + r)p - \frac{a + b - 1}{\beta}p + (\frac{a + b - 1}{\beta} - 1)|Y|, \end{aligned}$$

that is,

$$bn - 1 \geq (a + r)p - \frac{a + b - 1}{\beta}p + (\frac{a + b - 1}{\beta} - 1)|Y|. \tag{4}$$

We may assume that $\beta \leq a + b - 1$. Otherwise, by Lemma 2.3 and $p \geq \frac{(a+b-1)(a+b+1)}{a+r} + \frac{bn}{a+r-1}$, we have $\delta(G) \geq p - \frac{p-1}{\beta} > p - \frac{p-1}{a+b-1} \geq \frac{(b-r)p+bn}{a+b}$, and Lemma 2.2 can be applied. Furthermore, we obtain by (4)

$$\begin{aligned} bn - 1 &\geq (a + r)p - \frac{a + b - 1}{\beta}p + (\frac{a + b - 1}{\beta} - 1)|Y| \\ &\geq (a + r)p - \frac{a + b - 1}{\beta}p + (\frac{a + b - 1}{\beta} - 1) \\ &= (a + r)p - \frac{(a + b - 1)(p - 1)}{\beta} - 1 \\ &\geq (a + r)p - \frac{(a + b - 1)(p - 1)}{\beta} - (a + b - 1), \end{aligned}$$

which implies

$$\beta \leq \frac{(a + b - 1)(p - 1)}{(a + r)p - (a + b - 2) - bn},$$

which contradicts $bind(G) > \frac{(a+b-1)(p-1)}{(a+r)p-(a+b-2)-bn}$. This completes the proof of Theorem 3. □

3. Remark

In this section, we show that the condition $bind(G) > \frac{(a+b-1)(p-1)}{(a+r)p-(a+b-2)-bn}$ in Theorem 3 is best possible.

Let a, b, r and n be four nonnegative integers such that $2 \le a = b - r$, $a + b + 2 + n$ is even and $p = \frac{(a+b-1)(a+b+2)+(a+2b-1)n-1}{a+r}$ is an integer. We write $2l = a + b + 2 + n$ and $m = p - 2l = \frac{(a+b+2)(b-r-1)+(2b-r-1)n-1}{a+r}$. Set $G = K_m \bigvee (lK_2)$. Let $g(x)$ and $f(x)$ be two integer-valued functions defined on $V(G)$ with $g(x) \equiv a$ and $f(x) \equiv b = a + r$. We choose $X = V(lK_2)$. Then $|N_G(X \setminus x)| = p - 1$ for each $x \in X$. Obviously, $bind(G) = \frac{|N_G(X\setminus x)|}{|X\setminus x|} = \frac{p-1}{2l-1} = \frac{(a+b-1)(p-1)}{(a+b-1)(2l-1)} = \frac{(a+b-1)(p-1)}{(a+r)p-(a+b-2)-bn}$. For $S = V(K_m)$ and $T = V(lK_2)$, we obtain

$$\begin{aligned}\delta_G(S,T) &= f(S) + d_{G-S}(T) - g(T)\\ &= b|S| - (a-1)|T|\\ &= (a+r)\cdot m - (b-r-1)\cdot(2l)\\ &= (a+b+2)(b-r-1) + (2b-r-1)n - 1 - (b-r-1)(a+b+2+n)\\ &= bn - 1 < bn = f_n(S).\end{aligned}$$

So by Lemma 2.1, G is not fractional (g, f, n)-critical.

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