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ON A HIGHER-ORDER RATIONAL DIFFERENCE EQUATION

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Abstract. In this paper, we investigate the global behavior of the solutions of the difference equation

$$x_{n+1}=\frac{A+Bx_{n-2k-1}}{C+D\prod_{i=l}^{k}x_{n-2i}^{m_i}},\ n=0,1,...,$$

with non-negative initial conditions, the parameters A, B are non-negative real numbers, C, D are positive real numbers, k, l are fixed non-negative integers such that $l\leq k$, and $m_i, i=\overline{l,k}$ are positive integers.

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1. Introduction and Preliminaries

In [3] we investigated the global behavior of the rational third-order difference equation

$$x_{n+1}=\frac{A+Bx_{n-1}}{C+Dx_n^p x_{n-2}^q},\ n=0,1,..., \tag{1}$$

where the initial conditions x_0, x_{-1}, x_{-2} and the parameter B are non-negative real numbers, the parameters A, C, D are positive real numbers and p, q are fixed positive integers. Abo-Zeid [1] discussed the global behavior and boundedness of the solutions of the difference equation

$$x_{n+1}=\frac{A+Bx_{n-2k-1}}{C+D\prod_{i=l}^{k}x_{n-2i}},\ n=0,1,..., \tag{2}$$

where A, B are non-negative real numbers, C, $D>0$ and l,k are non-negative integers such that $l\leq k$. Inspired and motivated by these aforementioned works,

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our aim in this paper is to investigate the global asymptotic behavior of the difference equation

$$x_{n+1} = \frac{A + Bx_{n-2k-1}}{C + D\prod_{i=l}^{k} x_{n-2i}^{m_i}}, \; n = 0, 1, ..., \tag{3}$$

with non-negative initial conditions, the parameters A, B are non-negative real numbers, C, D are positive real numbers, k, l are fixed non-negative integers such that $l \leq k$, and $m_i, i = \overline{l,k}$ are positive integers.

We note that if $m_i = 1$, for all $i = \overline{l,k}$ Eq.(3) is reduced to Eq.(2). Clearly, the results obtained in [1] will follow from the results we shall exhibit here.
In what follows, we present some definitions and results which will be useful in our investigation, for more details we refer to [6], [11], [14] and [15].

Let I be some interval of real numbers and let

$$f : I^{k+1} \longrightarrow I$$

be a continuously differentiable function. Then, for every set of initial conditions $\{x_0, x_{-1}, ..., x_{-k}\} \subset I$, the difference equation

$$x_{n+1} = f(x_n, x_{n-1}, ..., x_{n-k}), \; n = 0, 1, ..., \tag{4}$$

has a unique solution $\{x_n\}_{n=-k}^{\infty}$.

Definition 1.1. A point $\overline{x} \in I$ is called an equilibrium point of Eq.(4) if

$$\overline{x} = f(\overline{x}, \overline{x}, ..., \overline{x}).$$

Definition 1.2. Let $\overline{x}$ be an equilibrium point of Eq.(4).

(i) The equilibrium point $\overline{x}$ is called locally stable if for every $\epsilon > 0$ there exists $\delta > 0$ such that for all $x_0, x_{-1}, ..., x_{-k} \in I$ with $\sum_{i=-k}^{0} |x_i - \overline{x}| < \delta$, we have

$$|x_n - \overline{x}| < \varepsilon \text{ for all } n \geq -k.$$

Otherwise, the equilibrium $\overline{x}$ is called unstable.

(ii) The equilibrium point $\overline{x}$ is called locally asymptotically stable if it is locally stable, and if there exists $\gamma > 0$ such that for all $x_0, x_{-1}, ..., x_{-k} \in I$ with $\sum_{i=-k}^{i=0} |x_i - \overline{x}| < \delta$, we have

$$\lim_{n\to\infty} x_n = \overline{x}.$$

(iii) The equilibrium point $\overline{x}$ is called globally asymptotically stable relative to I^{k+1} if it is locally asymptotically stable, and if for every $x_0, x_{-1}, , x_{-k} \in I$, we have

$$\lim_{n\to\infty} x_n = \overline{x}.$$

Let $p_i = \frac{\partial f}{\partial u_i}(\overline{x}, \overline{x}, ..., \overline{x})$, for $i = \overline{0, k}$ denote the partial derivatives of $f(u_0, u_1, ..., u_k)$ with respect to u_i evaluated at the equilibrium $\overline{x}$ of Eq.(4). Then, the equation

$$z_{n+1} = p_0 z_n + p_1 z_{n-1} + \cdots + p_k z_{n-k}, \ n = 0, 1, ...,$$

is called the linearized equation of Eq.(4) about the equilibrium point $\overline{x}$, and the equation

$$\lambda^{k+1} - p_0 \lambda^k - \cdots - p_{k-1}\lambda - p_k = 0, \tag{5}$$

is called the characteristic equation of Eq.(4) about $\overline{x}$.

Theorem 1.3. *Let $\overline{x}$ be an equilibrium of Eq.(4). Then, the following statements are true*

(i) *If all roots of Eq.(5) lie inside the open unit disk $|\lambda| < 1$, then $\overline{x}$ is locally asymptotically stable.*

(ii) *If at least one root of Eq.(5) has absolute value greater than one, then $\overline{x}$ is unstable.*

Theorem 1.4 (Rouché's Theorem)**.** *Let D be a bounded domain with piecewise smooth boundary ∂D. Let f and g be two analytic functions on $D \cup \partial D$. If $|g(z)| < |f(z)|$ for $z \in \partial D$, then f and $f + g$ have the same number of zeros in D, counting multiplicities.*

Definition 1.5. Let $\overline{x}$ be an equilibrium of Eq.(4) and assume that $\{x_n\}_{n=-k}^{\infty}$ is a solution of the same equation.

(i) A positive semicycle of $\{x_n\}_{n=-k}^{\infty}$ is a "string" of terms $\{x_l, x_{l+1}, ..., x_m\}$ all greater than or equal to the equilibrium $\overline{x}$, with $l \geq -k$ and $m \leq \infty$ such that

$$\text{either } l = -k \ \text{ or } \ l > -k \ \text{ and } \ x_{l-1} < \overline{x},$$

and

$$\text{either } m = \infty \ \text{ or } \ m < \infty \ \text{ and } \ x_{m+1} < \overline{x}.$$

(ii) A negative semicycle of $\{x_n\}_{n=-k}^{\infty}$ is a "string" of terms $\{x_l, x_{l+1}, ..., x_m\}$ all less than $\overline{x}$, with $l \geq -k$ and $m \leq \infty$ such that

$$\text{either } l = -k \text{ or } l > -k \text{ and } x_{l-1} \geq \overline{x},$$

and

$$\text{either } m = \infty \text{ or } m < \infty \text{ and } x_{m+1} \geq \overline{x}.$$

Definition 1.6. A solution $\{x_n\}_{n=-k}^{\infty}$ of Eq.(4) is called non-oscillatory about $\overline{x}$, or simply non-oscillatory, if there exists $N \geq -k$ such that either

$$x_n \geq \overline{x} \text{ for all } n \geq N,$$

or

$$x_n < \overline{x} \text{ for all } n \geq N.$$

Otherwise, the solution $\{x_n\}_{n=-k}^{\infty}$ is called oscillatory about $\overline{x}$, or simply oscillatory.

From now on, we let $p = \sum_{i=l}^{k} m_i$.

Remark 1.7. The change of variables $x_n = (\frac{C}{D})^{\frac{1}{p}} y_n$ reduces Eq.(3) to the difference equation

$$y_{n+1} = \frac{\alpha + \beta y_{n-2k-1}}{1 + \prod_{i=l}^{k} y_{n-2i}^{m_i}}, \ n = 0, 1, ..., \tag{6}$$

where $\alpha = \frac{A}{C}(\frac{D}{C})^{\frac{1}{p}}$ and $\beta = \frac{B}{C}$. It suffices to study Eq.(6) instead of Eq.(3).

2. Main results

2.1. Case $\alpha > 0$.

2.1.1. Local stability. Here, we determine the equilibrium points of Eq.(6) and discuss their local stability.

Lemma 2.1. *The following statements are true.*

(1) *Assume that $\beta \geq 1$. Then Eq.(6) has a unique equilibrium point in $((\frac{\beta-1}{p+1})^{\frac{1}{p}}, \infty)$.*

(2) *Assume that $\beta < 1$. Then*

  (i) *If $\alpha < p(\frac{1-\beta}{p-1})^{\frac{p+1}{p}}$, then Eq.(6) has a unique equilibrium point in $(0, (\frac{1-\beta}{p-1})^{\frac{1}{p}})$.*

  (ii) *If $\alpha > p(\frac{1-\beta}{p-1})^{\frac{p+1}{p}}$, then Eq.(6) has a unique equilibrium point in $((\frac{1-\beta}{p-1})^{\frac{1}{p}}, \infty)$.*

Proof. A point $\overline{y}$ is an equilibrium point of Eq.(6) if and only if $\overline{y}$ is a zero of the function

$$f(x) = x^{p+1} + (1-\beta)x - \alpha.$$

If we consider the above function, we get

$$f(0) = -\alpha < 0 \text{ and } f'(x) = (p+1)x^p + (1-\beta).$$

(1) If $\beta \geq 1$, then f is increasing on $((\frac{\beta-1}{p+1})^{\frac{1}{p}}, \infty)$. But

$$f((\frac{\beta-1}{p+1})^{\frac{1}{p}}) = -p(\frac{\beta-1}{p+1})^{\frac{p+1}{p}} - \alpha < 0.$$

Then, $f(x)$ has a unique zero in $((\frac{\beta-1}{p+1})^{\frac{1}{p}}, \infty)$.

(2) Assume that $\beta < 1$. Then f is increasing on $(0, \infty)$.

  (i) If $\alpha < p(\frac{1-\beta}{p-1})^{\frac{p+1}{p}}$, then

$$f((\frac{1-\beta}{p-1})^{\frac{1}{p}}) = p(\frac{1-\beta}{p-1})^{\frac{p+1}{p}} - \alpha > 0.$$

  Therefore, $f(x)$ has a unique zero in $(0, (\frac{1-\beta}{p-1})^{\frac{1}{p}})$.

(ii) If $\alpha > p(\frac{1-\beta}{p-1})^{\frac{p+1}{p}}$, then

$$f((\frac{1-\beta}{p-1})^{\frac{1}{p}}) < 0.$$

Therefore, $f(x)$ has a unique zero in $((\frac{1-\beta}{p-1})^{\frac{1}{p}}, \infty)$.

□

Theorem 2.2. *Assume that $\overline{y}$ is the positive equilibrium point of Eq.(6). Then, the following statements are true*

(1) *If $\beta \geq 1$, then $\overline{y}$ is a saddle point.*

(2) *If $\beta < 1$, then*

(i) *$\overline{y}$ is locally asymptotically stable if $\alpha < p(\frac{1-\beta}{p-1})^{\frac{p+1}{p}}$.*

(ii) *$\overline{y}$ is saddle point if $\alpha > p(\frac{1-\beta}{p-1})^{\frac{p+1}{p}}$.*

Proof. The linearized equation associated with Eq.(6) about $\overline{y}$ is

$$z_{n+1} = -\frac{\overline{y}^p}{1+\overline{y}^p}\sum_{i=l}^{k} m_i z_{n-2i} + \frac{\beta}{1+\overline{y}^p} z_{n-2k-1}, \ n = 0, 1, ...$$

The characteristic equation associated with this equation is

$$\lambda^{2k+2} + \frac{\overline{y}^p}{1+\overline{y}^p}\sum_{i=l}^{k} m_i \lambda^{2k-2i+1} - \frac{\beta}{1+\overline{y}^p} = 0.$$

(1) Assume that $\beta \geq 1$ and consider the function

$$g(\lambda) = \lambda^{2k+2} + \frac{\overline{y}^p}{1+\overline{y}^p}\sum_{i=l}^{k} m_i \lambda^{2k-2i+1} - \frac{\beta}{1+\overline{y}^p}.$$

Then

$$\lim_{\lambda\to-\infty} g(\lambda) = \infty \text{ and } g(-1) = 1 - \frac{\beta + p\overline{y}^p}{1+\overline{y}^p} < 1 - \frac{1+\overline{y}^p}{1+\overline{y}^p} = 0.$$

It follows that, the function $g(\lambda)$ has a root λ_1 in $(-\infty, -1)$ with $|\lambda_1| > 1$. We have also $g(0) = -\frac{\beta}{1+\overline{y}^p} < 0$ and $g(1) = 1 + \frac{p\overline{y}^p - \beta}{1+\overline{y}^p} > 0$, then the function $g(\lambda)$ has a root λ_2 in $(0, 1)$ which completes the proof of (1).

(2) Assume that $\beta < 1$, then

(i) If $\alpha < p(\frac{1-\beta}{p-1})^{\frac{p+1}{p}}$, then $\overline{y} < (\frac{1-\beta}{p-1})^{\frac{1}{p}}$. If we consider the functions

$$h_1(\lambda) = \lambda^{2k+2}, \ h_2(\lambda) = -\frac{\overline{y}^p}{1+\overline{y}^p}\sum_{i=0}^{k} m_i \lambda^{2k-2i+1} + \frac{\beta}{1+\overline{y}^p},$$

we get

$$|h_2(\lambda)| \leq \frac{\beta + p\overline{y}^p}{1+\overline{y}^p} < 1 = |h_1(\lambda)|, \ \forall\lambda \in \mathbb{C} : |\lambda| = 1.$$

By Rouché's Theorem all roots of

$$\lambda^{2k+2}+\frac{\overline{y}^p}{1+\overline{y}^p}\sum_{i=0}^{k}m_i\lambda^{2k-2i+1}-\frac{\beta}{1+\overline{y}^p}=0,$$

lie in the open unit disk $|\lambda|<1$ and the result follows from Theorem (1.3).

(ii) The proof is similar to that of 1. and will be omitted.

□

2.1.2. Oscillation and Boundedness of Solutions.

Theorem 2.3. *Let $\overline{y}$ be the positive equilibrium of Eq.(6) and let $\{y_n\}_{n=-2k-1}^{\infty}$ be a solution of the same equation. Then*

(1) *If either*
(a_1) $y_{-2k-1},y_{-2k+1},...,y_{-1}<\overline{y}\le y_{-2k},y_{-2k+2},...,y_0$
or
(a_2) $y_{-2k},y_{-2k+2},...,y_0<\overline{y}\le y_{-2k-1},y_{-2k+1},...,y_{-1}$
is satisfied, the solution $\{y_n\}_{n=-2k-1}^{\infty}$ oscillates about $\overline{y}$ with semicycles of length one.

(2) *Every oscillatory solution of Eq.(6) has semicycles of length at most $2k+1$.*

Proof. (1) Assume that the condition (a_1) is satisfied. Then

$$y_1=\frac{\alpha+\beta y_{-2k-1}}{1+\prod_{i=l}^{k}y_{-2i}^{m_i}}<\frac{\alpha+\beta\overline{y}}{1+\overline{y}^p}=\overline{y},$$

and

$$y_2=\frac{\alpha+\beta y_{-2k}}{1+\prod_{i=l}^{k}y_{1-2i}^{m_i}}\ge\frac{\alpha+\beta\overline{y}}{1+\overline{y}^p}=\overline{y}.$$

By induction we obtain

$$y_{2n}\ge\overline{y}\text{ and }y_{2n+1}<\overline{y}\text{ for all }n\ge 0.$$

For condition (a_2), the proof is similar and will be omitted.

(2) Let $\{y_n\}_{n=-2k-1}^{\infty}$ be an oscillatory solution of Eq.(6). If a semicycle has length greater than or equal $2k+1$, then there is an $N\ge 0$ such that either

$y_{N-(2k+1)}<\bar{y}\le y_{N-(2k)},...,y_{N-1},y_N$ or $y_{N-(2k+1)}\ge\bar{y}>y_{N-(2k)},...,y_{N-1},y_N$.

Consider the first case. Then we get

$$y_{N+1}=\frac{\alpha+\beta y_{N-2k-1}}{1+\prod_{i=l}^{k}y_{N-2i}^{m_i}}<\frac{\alpha+\beta\bar{y}}{1+\bar{y}^p}=\bar{y}.$$

The second case is similar and will be omitted.

□

Lemma 2.4. *Let $\{y_n\}_{n=-2k-1}^{\infty}$ be a solution of Eq.(6). Then*

(1)
$$y_{2(k+1)n} \leq \sum_{j=0}^{n-1} \alpha\beta^j + \beta^n y_0 \textit{ for all } n \geq 0.$$

(2)
$$y_{2(k+1)n+2i} \leq \sum_{j=0}^{n} \alpha\beta^j + \beta^{n+1} y_{-2k+2i-2} \textit{ for all } n \geq 0 \textit{ and } 1 \leq i \leq k.$$

(3)
$$y_{2(k+1)n+2i+1} \leq \sum_{j=0}^{n} \alpha\beta^j + \beta^{n+1} y_{-2k+2i-1} \textit{ for all } n \geq 0 \textit{ and } 0 \leq i \leq k.$$

Proof. Let $\{y_n\}_{n=-2k-1}^{\infty}$ be a solution of Eq.(6). Then

$$y_{n+1} \leq \alpha + \beta y_{n-2k-1} \text{ for all } n \geq 0. \tag{7}$$

(1) For $n = 0$, we have $y_0 \leq \sum_{j=0}^{-1} \alpha\beta^j + \beta^0 y_0$. Now suppose that for a certain n we have
$$y_{2(k+1)n} \leq \sum_{j=0}^{n-1} \alpha\beta^j + \beta^n y_0.$$
Then from (7), we get
$$y_{2(k+1)(n+1)} \leq \alpha + \beta y_{2(k+1)n} \leq \sum_{j=0}^{n} \alpha\beta^j + \beta^{n+1} y_0.$$

(2) Let $i = 1$. Then from (7), we have
$$y_2 \leq \alpha + \beta y_{-2k} \text{ and } y_{2(k+1)(n+1)+2} \leq \alpha + \beta y_{2(k+1)n+2}.$$
Now suppose that for a certain i we have
$$y_{2(k+1)n+2i} \leq \sum_{j=0}^{n} \alpha\beta^j + \beta^{n+1} y_{-2k+2i-2} \text{ for all } n \geq 0$$
and prove that
$$y_{2(k+1)n+2(i+1)} \leq \sum_{j=0}^{n} \alpha\beta^j + \beta^{n+1} y_{-2k+2i} \text{ for all } n \geq 0.$$
From (7), we have
$$y_{2i+2} \leq \alpha + \beta y_{-2k+2i}.$$
Suppose
$$y_{2(k+1)n+2(i+1)} \leq \sum_{j=0}^{n} \alpha\beta^j + \beta^{n+1} y_{-2k+2i},$$

then from (7), we get

$$y_{2(k+1)(n+1)+2(i+1)} \leq \alpha + \beta y_{2(k+1)n+2i+2} \leq \sum_{j=0}^{n+1} \alpha\beta^j + \beta^{n+2} y_{-2k+2i}.$$

(3) The proof is similar to that of 2. and will be omitted.

□

Corollary 2.5. *Assume that $\beta < 1$. Then, every solution of Eq.(6) is bounded and persists.*

Lemma 2.6. *Suppose $\beta < 1$ and let $\{y_n\}_{n=-2k-1}^{\infty}$ be a solution of Eq.(6). If $\Lambda = \limsup_{n\to\infty} y_n$ and $\lambda = \liminf_{n\to\infty} y_n$, then Λ and λ satisfy the following inequalities*

$$\frac{\alpha + \beta\lambda}{1 + \Lambda^p} \leq \lambda \leq \Lambda \leq \frac{\alpha + \beta\Lambda}{1 + \lambda^p}.$$

Proof. Let $\beta < 1$. From Corollary 2.5 the solution $\{y_n\}_{n=-2k-1}^{\infty}$ is bounded. Then, for every $\varepsilon \in (0, \lambda)$, there exists $n_0 \in \mathbb{N}$ such that

$$\lambda - \varepsilon \leq y_n \leq \Lambda + \varepsilon \text{ for every } n \geq n_0,$$

so,

$$\frac{\alpha + \beta(\lambda - \varepsilon)}{1 + (\Lambda + \varepsilon)^p} \leq y_{n+1} \leq \frac{\alpha + \beta(\Lambda + \varepsilon)}{1 + (\lambda - \varepsilon)^p} \text{ for every } n \geq n_0 + 2.$$

Therefore,

$$\frac{\alpha + \beta\lambda}{1 + \Lambda^p} \leq \lambda \leq \Lambda \leq \frac{\alpha + \beta\Lambda}{1 + \lambda^p}.$$

□

Lemma 2.7. *Suppose $\beta > 2$. Then, the following statements are true*

(1) *If $x > \sqrt[p]{\beta - 1} + \frac{\alpha}{\sqrt[p]{\beta-1}}$, then $\sqrt[p]{\beta - 1} > \frac{\alpha}{x^p - \beta + 1}$.*
(2) *If $x > \sqrt[p]{\beta - 1}$ and $y > \frac{\alpha}{x^p - \beta + 1}$, then $y > \frac{\alpha + \beta y}{1 + x^p}$.*

Proof. (1) Let $x > \sqrt[p]{\beta - 1} + \frac{\alpha}{\sqrt[p]{\beta-1}}$, then

$$x^p > \sum_{k=0}^{p} u(k, p),$$

where

$$u(k, p) := C_p^k \left(\sqrt[p]{\beta - 1}\right)^k \left(\frac{\alpha}{\sqrt[p]{\beta - 1}}\right)^{p-k}, \; C_p^k := \frac{p!}{k!(p-k)!}.$$

Hence, we obtain

$$x^p > u(p-1, p) + u(p, p) = p\,(\beta - 1)^{\frac{p-1}{p}} \frac{\alpha}{\sqrt[p]{\beta - 1}} + \beta - 1,$$

then,

$$\sqrt[p]{\beta - 1}(x^p - \beta + 1) > p\alpha(\beta - 1)^{\frac{p-1}{p}} > \alpha.$$

(2) Suppose that $x > \sqrt[p]{\beta - 1}$ and $y > \frac{\alpha}{x^p - \beta + 1}$, then

$$(x^p - \beta + 1)y > \alpha,$$

hence,

$$(x^p + 1)y > \alpha + \beta y.$$

□

Theorem 2.8. *Assume that $\beta > 2$. Then, Eq.(6) has solutions which are neither bounded nor persist.*

Proof. Let $\{y_n\}_{n=-2k-1}^{\infty}$ be a solution of Eq.(6) with initial conditions

$$\frac{\alpha}{y_{-2k-1}^p - \beta + 1} < y_0 < y_{-2} < ... < y_{-2k} < \sqrt[p]{\beta - 1}$$

and

$$y_{-1} > ... > y_{-2k+1} > y_{-2k-1} > \sqrt[p]{\beta - 1} + \frac{\alpha}{\sqrt[p]{\beta - 1}}.$$

Then,

$$y_1 = \frac{\alpha + \beta y_{-2k-1}}{1 + \prod_{i=l}^{k} y_{-2i}^{m_i}} > \frac{\alpha + \beta y_{-2k-1}}{\beta} = \frac{\alpha}{\beta} + y_{-2k-1},$$

and

$$y_2 = \frac{\alpha + \beta y_{-2k}}{1 + \prod_{i=l}^{k} y_{1-2i}^{m_i}} < \frac{\alpha + \beta y_{-2k}}{1 + y_{-2k-1}^p}.$$

By applying Lemma 2.7, we get $y_2 < y_{-2k}$.
Now consider the subsequences

$$\{y_{2(k+1)n-2k+2j-1}\}_{n=0}^{\infty} \text{ and } \{y_{2(k+1)n-2k+2j}\}_{n=0}^{\infty},$$

where $0 \leq j \leq k$. We will prove that

$$\sqrt[p]{\beta - 1} + \frac{\alpha}{\sqrt[p]{\beta - 1}} < y_{2(k+1)(n-1)-2k+2j-1} < y_{2(k+1)n-2k+2j-1}$$

and

$$\frac{\alpha}{y_{2(k+1)n-2k+2j+1}^p - \beta + 1} < y_{2(k+1)n-2k+2j} < y_{2(k+1)(n-1)-2k+2j} < \sqrt[p]{\beta - 1},$$

for all $n \geq 1$. For $n = 1$, we have

$$y_{2j+1} = \frac{\alpha + \beta y_{-2(k+1)+2j+1}}{1 + \prod_{i=l}^{k} y_{2j-2i}^{m_i}} > \frac{\alpha + \beta y_{-2(k+1)+2j+1}}{\beta} = \frac{\alpha}{\beta} + y_{-2(k+1)+2j+1}$$

and

$$y_{2j+2} = \frac{\alpha + \beta y_{-2(k+1)+2j+2}}{1 + \prod_{i=l}^{k} y_{2j-2i+1}^{m_i}} < \frac{\alpha + \beta y_{-2(k+1)+2j+2}}{1 + y_{2j-2k+1}^p}.$$

By applying Lemma 2.7, we obtain $y_{2j+2} < y_{-2(k+1)+2j+2}$. We also have

$$y_{2(k+1)(n+1)-2k+2j-1} = \frac{\alpha + \beta y_{2(k+1)n-2k+2j-1}}{1 + \prod_{i=l}^{k} y_{2(k+1)n+2j-2i}^{m_i}} > \frac{\alpha}{\beta} + y_{2(k+1)n-2k+2j-1}, \quad (8)$$

and

$$y_{2(k+1)(n+1)-2k+2j} = \frac{\alpha + \beta y_{2(k+1)n-2k+2j}}{1 + \prod_{i=l}^{k} y_{2(k+1)n+2j-2i+1}^{m_i}} < \frac{\alpha + \beta y_{2(k+1)n-2k+2j}}{1 + y_{2(k+1)n+2j-2k+1}^{p}}.$$

By applying Lemma 2.7, we get $y_{2(k+1)(n+1)-2k+2j} < y_{2(k+1)n-2k+2j}$. Now from inequality (8), we have

$$y_{2(k+1)n-2k+2j-1} > \frac{\alpha}{\beta} + y_{2(k+1)(n-1)-2k+2j-1} > ... > n\frac{\alpha}{\beta} + y_{-2k+2j-1}.$$

This implies that

$$\lim_{n\to\infty} y_{2(k+1)n-2k+2j-1} = \infty \ , \ 0 \leq j \leq k$$

and

$$\lim_{n\to\infty} y_{2(k+1)n-2k+2j} = 0, \ \ 0 \leq j \leq k.$$

This completes the proof. □

2.1.3. Global stability.

Theorem 2.9. *Assume that* $\beta < 1$. *If* $\alpha < p(\frac{1-\beta}{p-1})^{\frac{p+1}{p}}$, *then, the positive equilibrium point* $\overline{y} \in (0, (\frac{1-\beta}{p-1})^{\frac{1}{p}})$ *is globally asymptotically stable.*

Proof. Let $\{y_n\}_{n=-2k-1}^{\infty}$ be a solution of Eq.(6). As $\beta < 1$, the solution $\{y_n\}_{n=-2k-1}^{\infty}$ is bounded. Let $\Lambda = \limsup_{n\to\infty} y_n$ and $\lambda = \liminf_{n\to\infty} y_n$. Using Lemma 2.6, we have

$$\frac{\alpha + \beta\lambda}{1 + \Lambda^p} \leq \lambda \leq \Lambda \leq \frac{\alpha + \beta\Lambda}{1 + \lambda^p}.$$

This implies that

$$(1-\beta)\lambda^p - \alpha\lambda^{p-1} \geq (1-\beta)\Lambda^p - \alpha\Lambda^{p-1}. \tag{9}$$

Now consider the function

$$h(x) = (1-\beta)x^p - \alpha x^{p-1}.$$

Hence,

$$h'(x) = x^{p-2}(p(1-\beta)x - (p-1)\alpha),$$

and the function $h(x)$ is increasing on $(\frac{\alpha(p-1)}{p(1-\beta)}, \infty)$. As $\alpha < p(\frac{1-\beta}{p-1})^{\frac{p+1}{p}}$, we get $\frac{\alpha(p-1)}{p(1-\beta)} < \overline{y} < (\frac{1-\beta}{p-1})^{\frac{1}{p}}$. In view of inequality (9), we have a contradiction. Therefore, $\lambda = \Lambda = \overline{y}$ and so $\overline{y}$ is a global attractor.

The global asymptotically stability of $\overline{y}$ is obtained by combining the global attractivity and the local asymptotic stability of $\overline{y}$ when $\alpha < p(\frac{1-\beta}{p-1})^{\frac{p+1}{p}}$.

□

In order to confirm our theoretical results, we consider the following numerical example.

Example 2.10. Consider the difference equation Eq.(6) with $l = 2$, $k = 4$, $m_2 = 2$, $m_3 = 3$, $m_4 = 4$, $\alpha = 0.25$ and $\beta = 0.5$, that is:

$$y_{n+1} = \frac{0.25 + 0.5y_{n-9}}{1 + y_{n-4}^2 y_{n-6}^3 y_{n-8}^4}, \; n = 0, 1, \cdots . \tag{10}$$

We have:

$$p = 9, \; \alpha = 0.25 < p\left(\frac{1-\beta}{p-1}\right)^{\frac{p+1}{p}} \simeq 0.41336282$$

Clearly $\overline{y} \simeq 0.49811911$ is the unique equilibrium point of the equation Eq.(10) in $(0, \left(\frac{1-\beta}{p-1}\right)^{\frac{1}{p}}) = (0, (0.06875)^{\frac{1}{9}})$.

If we take $y_0 = 5.85$, $y_{-1} = 0.89$, $y_{-2} = 0.35$, $y_{-3} = 4.55$, $y_{-4} = 0.65$, $y_{-5} = 3.15$, $y_{-6} = 0.75$, $y_{-7} = 2.35$, $y_{-8} = 1.25$, $y_{-9} = 0.25$, then, we get the solution as in figure 1. However, if we take $y_0 = 0.35$, $y_{-1} = 0.19$, $y_{-2} = 0.67$, $y_{-3} = 2.55$, $y_{-4} = 0.5$, $y_{-5} = 1.15$, $y_{-6} = 0.15$, $y_{-7} = 1.35$, $y_{-8} = 0.33$, $y_{-9} = 0.45$, then, the solution will be as in figure 2.

As we can see from figure 1 and figure 2, for two different choices of the initial values we have $\lim_{n\to\infty} y_n = \overline{y} \simeq 0.49811911$.

That is the equilibrium point $\overline{y} \simeq 0.49811911$ is globally asymptotically stable (as expected).

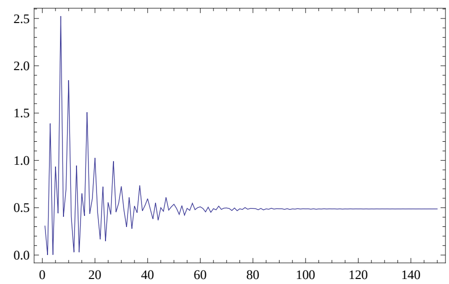

Figure 1. Plot of the solution of Eq.(10) with the initial conditions: $y_0 = 5.85$, $y_{-1} = 0.89$, $y_{-2} = 0.35$, $y_{-3} = 4.55$, $y_{-4} = 0.65$, $y_{-5} = 3.15$, $y_{-6} = 0.75$, $y_{-7} = 2.35$, $y_{-8} = 1.25$, $y_{-9} = 0.25$.

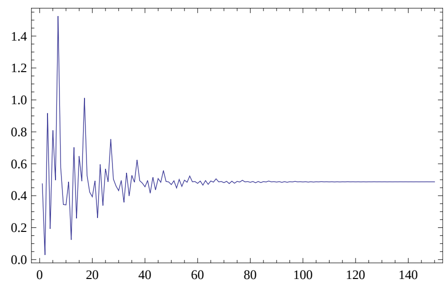

Figure 2. Plot of the solution of Eq.(10) with the initial conditions: $y_0 = 0.35$, $y_{-1} = 0.19$, $y_{-2} = 0.67$, $y_{-3} = 2.55$, $y_{-4} = 0.5$, $y_{-5} = 1.15$, $y_{-6} = 0.15$, $y_{-7} = 1.35$, $y_{-8} = 0.33$, $y_{-9} = 0.45$.

2.2. Case $\alpha = 0$. When $\alpha = 0$, Eq.(6) becomes

$$y_{n+1} = \frac{\beta y_{n-2k-1}}{1 + \prod_{i=l}^{k} y_{n-2i}^{m_i}}, \; n = 0, 1, ... \tag{11}$$

Clearly $\overline{y} = 0$ is always an equilibrium point of Eq.(11). When $\beta > 1$, Eq.(11) also possesses the positive equilibrium point $\overline{y} = \sqrt[p]{\beta - 1}$.
Some very close equations and systems of difference equations to Eq.(11) have been studied, for example, [2], [4], [5], [10] and [17].

Following the above mentioned papers, we summarize the main results for this particular equation.

Lemma 2.11. *Assume that $\beta < 1$. Then, every solution of Eq.(11) is bounded.*

Proof. Let $\{y_n\}_{n=-2k-1}^{\infty}$ be a solution of Eq.(11). Then

(1)
$$y_{2(k+1)n} \leq \beta^n y_0 \text{ for all } n \geq 0.$$

(2)
$$y_{2(k+1)n+2i} \leq \beta^{n+1} y_{-2k+2i-2} \text{ for all } n \geq 0 \text{ and } 1 \leq i \leq k.$$

(3)
$$y_{2(k+1)n+2i+1} \leq \beta^{n+1} y_{-2k+2i-1} \text{ for all } n \geq 0 \text{ and } 0 \leq i \leq k.$$

The proof follows from the above inequalities. □

Theorem 2.12. *Assume that $\beta > 1$. Let $\{y_n\}_{n=-2k-1}^{\infty}$ be a solution of the Eq.(11) and $\overline{y} = \sqrt[p]{(\beta - 1)}$. Then if either*
(b_1) $y_{-2k-1}, y_{-2k+1}, ..., y_{-1} < \overline{y} \leq y_{-2k}, y_{-2k+2}, ..., y_0$
or
(b_2) $y_{-2k}, y_{-2k+2}, ..., y_0 < \overline{y} \leq y_{-2k-1}, y_{-2k+1}, ..., y_{-1}$
is satisfied, the solution $\{y_n\}_{n=-2k-1}^{\infty}$ oscillates about $\overline{y}$ with semicycles of length one.

Proof. The proof is similar to that of Theorem 2.3 and will be omitted. □

Theorem 2.13. *The following statements are true*

(1) *The zero equilibrium point of Eq.(11) is locally asymptotically stable if $\beta < 1$ and it is unstable if $\beta > 1$.*
(2) *In addition, when $\beta < 1$ then, the zero equilibrium point of Eq.(11) is globally asymptotically stable.*
(3) *If $\beta > 1$, then the equilibrium point $\overline{y} = \sqrt[p]{\beta - 1}$ of Eq.(11) is a saddle point.*

Proof. (1) The linearized equation associated with Eq.(11) about the equilibrium point $\overline{y} = 0$ is

$$z_{n+1} = \beta z_{n-2k-1}, \; n = 0, 1, ...$$

Its characteristic equation is

$$\lambda^{2k+2} - \beta = 0.$$

Hence, $|\lambda| < 1$ for all roots if $\beta < 1$ and $|\lambda| > 1$ for all roots if $\beta > 1$. Therefore, the point $\overline{y} = 0$ is locally asymptotically stable if $\beta < 1$ and it is unstable if $\beta > 1$.

(2) The proof is a direct consequence of Lemma 2.11.

(3) The linearized equation associated with Eq.(11) about the equilibrium point $\overline{y} = \sqrt[p]{\beta - 1}$ is

$$z_{n+1} = \frac{1-\beta}{\beta} \sum_{i=l}^{k} m_i z_{n-2i} + z_{n-2k-1} \; n = 0, 1, ...$$

Its characteristic equation is

$$\lambda^{2k+2} + \frac{\beta - 1}{\beta} \sum_{i=l}^{k} m_i \lambda^{2k-2i+1} - 1 = 0.$$

Consider the function

$$g(\lambda) = \lambda^{2k+2} + \frac{\beta - 1}{\beta} \sum_{i=l}^{k} m_i \lambda^{2k-2i+1} - 1,$$

we can see that $g(\lambda)$ has a real root in $(-\infty, -1)$ and a root with modulus less than one. Therefore, the point $\overline{y} = \sqrt[p]{\beta - 1}$ is a saddle point.

□

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