J. Appl. Math. & Informatics Vol. **34***(2016), No. 5 - 6, pp. 359 - 367*
http://dx.doi.org/10.14317/jami.2016.359

A NOTE ON THE CAUCHY PROBLEM FOR HEAT EQUATIONS WITH COUPLING MOVING REACTIONS OF MIXED TYPE†

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Abstract. This paper deals with the Cauchy problem for heat equations with coupling moving reactions of mixed type. After obtaining the infinite Fujita blow-up exponent, we classify optimally the simultaneous and non-simultaneous blow-up for two components of the solutions. Moreover, blow-up rates and set are determined. By using the analogous procedures, one can fill in the gaps for the other two systems, which are studied in the paper 'Australian and New Zealand Industrial and Applied Mathematics Journal' 48(2006)37–56.

AMS Mathematics Subject Classification : 35K05, 35K60, 35B40, 35B33.
Key words and phrases : Cauchy problem, moving reaction, non-simultaneous blow-up, blow-up rate.

1. Introduction and main results

Recently, Xiang, Chen and Mu in [1] studied the following two types of parabolic equations

$$\begin{cases} u_t = \Delta u + u^m(x_0(t),t)v^p(x_0(t),t) & \text{in } \mathbb{R}^N \times (0,T), \\ v_t = \Delta v + u^q(x_0(t),t)v^n(x_0(t),t) & \text{in } \mathbb{R}^N \times (0,T), \\ u(x,0) = u_0(x),\ v(x,0) = v_0(x) & \text{in } \mathbb{R}^N, \end{cases} \tag{1}$$

and

$$\begin{cases} u_t = \Delta u + e^{mu(x_0(t),t)+pv(x_0(t),t)} & \text{in } \mathbb{R}^N \times (0,T), \\ v_t = \Delta v + e^{qu(x_0(t),t)+nv(x_0(t),t)} & \text{in } \mathbb{R}^N \times (0,T), \\ u(x,0) = u_0(x),\ v(x,0) = v_0(x) & \text{in } \mathbb{R}^N, \end{cases} \tag{2}$$

where $u_0(x), v_0(x) \geq 0, \not\equiv 0$ are continuous bounded functions in $\mathbb{R}^N$; $m,n,p,q \geq 0$ and $pq > 0$ for the coupling; moving site $x_0(t) : \mathbb{R}_+ \to \mathbb{R}^N$ is Hölder continuous. The results for (1) can be summarized below with the help of the figure:

Received January 22, 2013. Revised March 17, 2016. Accepted March 21, 2016. *Corresponding author. †This paper is partially supported by NNSF of China, Shandong Provincial Natural Science Foundation, China, and the Fundamental Research Funds for the Central Universities.

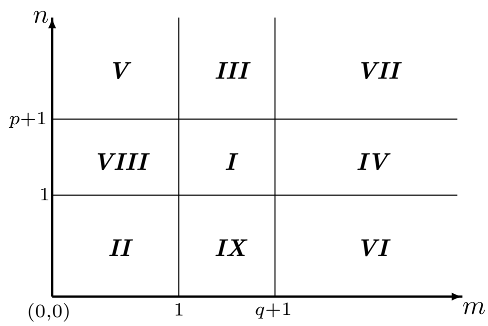

Figure 1.1. Blow-up classifications for system (1)

- $q \geq m-1 > 0$, $p \geq n-1 > 0$, or $q > m-1$, $p > n-1$, $pq-mn+m+n-1 > 0$ (i.e., ***I***, ***II***, ***VIII***, ***IX***): *Simultaneous blow-up* occurs. That is, the components u and v of the solution blow up at the same time (see [2, 3]). Moreover, uniform blow-up profiles are obtained.
- $q < m-1$, $p \geq n-1 > 0$, or $q \geq m-1 > 0$, $p < n-1$ (i.e., ***III***, ***IV***): One component of the solution blows up while another remains bounded. That is, *non-simultaneous blow-up* occurs.

It can be checked that blow-up phenomena are unsettled yet in the exponent regions ***V***, ***VI***, ***VII***, and on the boundary between ***V*** and ***VIII***, between ***IX*** and ***VI***, between ***V*** and ***III***, and between ***IV*** and ***VI***. For system (2), the authors obtained that simultaneous blow up occurs for $q \geq m$ and $p \geq n$; If $q < m$, $p \geq n$, or $q \geq m$, $p < n$, non-simultaneous blow-up occurs; The uniform blow-up profiles are obtained for $q \geq m$, $p \geq n$. Equivalently to ***VII*** in Figure 1.1 (i.e., $m > q$, $n > p$), blow-up phenomena are unknown.

Motivated by [1], we study the parabolic equations with coupling moving reactions of mixed type,

$$\begin{cases} u_t = \Delta u + u^m(x_0(t),t)e^{pv(x_0(t),t)} & \text{in } \mathbb{R}^N \times (0,T), \\ v_t = \Delta v + u^q(x_0(t),t)e^{nv(x_0(t),t)} & \text{in } \mathbb{R}^N \times (0,T), \end{cases} \tag{3}$$

with smooth initial data $u(x,0) = u_0(x) \geq 0$, $v(x,0) = v_0(x) \geq 0$ in $\mathbb{R}^N$; constants m, n, p, $q \geq 0$; $x_0(t) : \mathbb{R}_+ \to \mathbb{R}^N$ is Hölder continuous; let T be the maximal existence time of local classical solutions. The existence and uniqueness of local positive classical solutions (u,v) of (3) can be obtained by using the standard methods of [4, 5]. Nonlinear parabolic systems like (3) come from population dynamics, chemical reactions, heat transfer, etc., where u and v represent the densities of two biological populations during a migration, the thickness of two kinds of chemical reactants, the temperatures of two different materials during a propagation, etc, in which the nonlinear reactions in such dynamical systems take place only at a single (sometimes several) site(s) and couple through different types of nonlinearities. The interested readers refer to [6, 7, 8] and the papers therein. We obtain the first theorem about the infinite Fujita blow-up exponent and blow-up set for (3).

Theorem 1.1. *Let (u, v) be any positive local classical solution of (3). Then* $\lim_{t\to T}(u+v) = +\infty$, $x \in \mathbb{R}^N$ *if and only if* $\max\{m-1,\ n,\ pq-n(m-1)\} > 0$. *And the solutions blow up everywhere in* $\mathbb{R}^N$.

In the sequel, we discuss the blow-up solutions only. The second result shows the occurring of only simultaneous or non-simultaneous blow-up.

Theorem 1.2.

(i) *Simultaneous blow-up occurs for every initial data if* $m \le q+1$ *and* $n \le p$.
(ii) u *blows up alone for every initial data if* $m > q+1$ *and* $n \le p$.
(iii) v *blows up alone for every initial data if* $m \le q+1$ *and* $n > p$.

Corollary 1.3.

(i) *If there exist initial data such that* u *blows up alone, then* $m > q+1$.
(ii) *If there exist initial data such that* v *blows up alone, then* $n > p$.

Additionally, we assume that $x_0 : \mathbb{R}_+ \to \mathbb{R}$ is smooth and

(H) $x_0'(t)u_0'(x),\ x_0'(t)v_0'(x) \ge 0,\ x \in \mathbb{R},\ t \in [0, T);$ For small $\varepsilon \in (0, 1)$,
$$u_0''(x) + (1-\varepsilon)u^m(x_0(0), 0)e^{pv(x_0(0),0)} \ge 0,$$
$$v_0''(x) + (1-\varepsilon)u^q(x_0(0), 0)e^{nv(x_0(0),0)} \ge 0.$$

Theorem 1.4. *Let assumption (H) be in force.*

(i) *There exist initial data such that* u *blows up alone if* $m > q+1$.
(ii) *There exist initial data such that* v *blows up alone if* $n > p$.
(iii) *Both non-simultaneous blow-up and simultaneous blow-up may occur if* $m > q+1$ *and* $n > p$.

The key clues on non-simultaneous and simultaneous blow-up are the signals of $(m-q-1)$ and $(n-p)$. By using the analogous methods in this paper, one can check that the key clues are the signals of $(m-q-1)$, $(n-p-1)$ and the signals of $(m-q)$, $(n-p)$ for problems (1) and (2), respectively. The following is the detail:

- The exponent region ***V*** and the boundary between **V** and ***III*** for (1): only v blows up.
- The exponent region ***VI*** and the boundary between **IV** and **VI** for (1): only u blows up.
- The exponent region ***VII*** for (1): the coexistence region, in which simultaneous and non-simultaneous blow-up may occur according to the choosing of the initial data.
- The boundary between ***V*** and ***VIII***, and the boundary between ***IX*** and ***VI*** for (1): only simultaneous blow-up occurs.

- For (2), the exponent region $m > q$ and $n > p$ (equivalent to ***VII*** for (1)): a coexistence region for the existence of both simultaneous and non-simultaneous blow-up.

The following results give the blow-up rates. It can be understood that non-simultaneous blow-up rate is equivalent to the one for the scalar equation (see [9]): *If* u (v) *blows up alone, then* $u(x,t) = O\big((T-t)^{-\frac{1}{m-1}}\big)$ $\big(e^{v(x,t)} = O\big((T-t)^{-\frac{1}{n}}\big)\big)$ *for any* $x \in \mathbb{R}^N$. By using the analogous methods of [10], simultaneous blow-up rates are obtained.

Theorem 1.5. *Let* (u,v) *be a blow-up solution of (3).*
(i) *If* $m < q+1$, $n < p$, *then*

$$u(x,t) = O\big((T-t)^{-\frac{p-n}{pq-n(m-1)}}\big),\ e^{v(x,t)} = O\big((T-t)^{-\frac{q+1-m}{pq-n(m-1)}}\big)\ \text{ in } \mathbb{R}^N.$$

(ii) *If* $m < q+1$, $n = p$, *then*

$$u^{q+1-m}(x,t) = O\big(|\log(T-t)|\big),\ e^{nv(x,t)}v^{\frac{q}{q+1-m}}(x,t) = O\big((T-t)^{-1}\big)\ \text{ in } \mathbb{R}^N.$$

(iii) *If* $m = q+1$, $n < p$, *then*

$$u^{m-1}(x,t)(\log u(x,t))^{\frac{p}{p-n}} = O\big((T-t)^{-1}\big),\ e^{(p-n)v(x,t)} = O\big(|\log(T-t)|\big)\ \text{ in } \mathbb{R}^N.$$

(iv) *If* $m = q+1$, $n = p$, *then*

$$\log u(x,t) = O\big(|\log(T-t)|\big),\ v(x,t) = O\big(|\log(T-t)|\big)\ \text{ in } \mathbb{R}^N.$$

(v) *If* $m > q+1$, $n > p$ *and there exist initial data such that simultaneous blow-up occurs, then*

$$u(x,t) = O\big((T-t)^{-\frac{p-n}{pq-n(m-1)}}\big),\ e^{v(x,t)} = O\big((T-t)^{-\frac{q+1-m}{pq-n(m-1)}}\big)\ \text{ in } \mathbb{R}.$$

2. Infinite Fujita blow-up exponent

We prove the blow-up criteria and blow-up set for system (3) by the comparison principle.

Proof. (**Theorem 1.1**) Inspired by Souplet [9], we introduce the following auxiliary functions

$$\underline{u}(x,t) = \underline{u}(t) = \int_0^t u^m(x_0(s),s)e^{pv(x_0(s),s)}ds,$$
$$\underline{v}(x,t) = \underline{v}(t) = \int_0^t u^q(x_0(s),s)e^{nv(x_0(s),s)}ds,$$
$$\overline{u}(x,t) = \overline{u}(t) = \underline{u}(x,t) + \|u_0\|_1, \quad \overline{v}(x,t) = \overline{v}(t) = \underline{v}(x,t) + \|v_0\|_1.$$

It can be checked that

$$\underline{u}_t - \Delta\underline{u} = \overline{u}_t - \Delta\overline{u} = u_t - \Delta u,\ \underline{v}_t - \Delta\underline{v} = \overline{v}_t - \Delta\overline{v} = v_t - \Delta v \text{ in } \mathbb{R}^N \times (0,T), \quad (4)$$

$$\underline{u}(0) \le u_0(x) \le \overline{u}(0), \quad \underline{v}(0) \le v_0(x) \le \overline{v}(0) \quad \text{in } \mathbb{R}^N. \quad (5)$$

By the comparison principle, we have

$$\underline{u}(t) \le u(x,t) \le \overline{u}(t), \quad \underline{v}(t) \le v(x,t) \le \overline{v}(t) \quad \text{in } \mathbb{R}^N \times [0,T). \quad (6)$$

Especially, $\underline{u}(t) \le u(x_0(t),t) \le \overline{u}(t)$, $\underline{v}(t) \le v(x_0(t),t) \le \overline{v}(t)$. By a simple calculation, we obtain

$$\underline{u}_t(t) \ge \underline{u}^m(t)\underline{v}^p(t), \quad \underline{v}_t(t) \ge \big(\frac{n}{n+1}\big)^{n+1}\underline{u}^q(t)\underline{v}^{n+1}(t). \tag{7}$$

One can check from (7) that $\lim_{t\to T}\big(\underline{u}(t)+\underline{v}(t)\big) = +\imath$ for every initial data if $\max\{m-1,\ n,\ pq-n(m-1)\} > 0$. Hence any positive solution of (3) blows up everywhere in $\mathbb{R}^N$. On the other hand, it is easy to see that every positive solution is global for $m \le 1$ and $n = pq = 0$. □

3. Any blow-up must be simultaneous or non-simultaneous

By (6), $\underline{u}$, u and $\overline{u}$ shall the same blow-up time for all $x \in \mathbb{R}^N$, and so do $\underline{v}$, v and $\overline{v}$. It suffices to discuss blow-up estimates of $\underline{u}$ and $\underline{v}$.

Proof. (**Theorem 1.2**) By Theorem 1.1, every solution blows up. Combining (4) with (5), we obtain that there exists some positive constant C such that

$$\underline{u}^m(t)e^{p\underline{v}(t)} \le \underline{u}_t(t) \le C\underline{u}^m(t)e^{p\underline{v}(t)}, \quad \underline{u}^q(t)e^{n\underline{v}(t)} \le \underline{v}_t(t) \le C\underline{u}^q(t)e^{n\underline{v}(t)}. \tag{8}$$

(i) $m \le q+1$, $n \le p$.

- If $m < q+1$ and $n < p$, then we have $ce^{(p-n)\underline{v}(t)} \le \underline{u}^{q-m+1}(t) \le Ce^{(p-n)\underline{v}(t)}$ by using (8).
- If $m < q+1$ and $n = p$, then $c\underline{v}(t) \le \underline{u}^{q-m+1}(t) \le C\underline{v}(t)$.
- If $m = q+1$ and $n < p$, then $ce^{(p-n)\underline{v}(t)} \le \log \underline{u}(t) \le Ce^{(p-n)\underline{v}(t)}$.
- If $m = q+1$ and $n = p$, then $c\underline{v}(t) \le \log \underline{u}(t) \le C\underline{v}(t)$.

According to the above four subcases, we have, if $m \le q+1$ and $n \le p$, only simultaneous blow-up occurs.

(ii) If $m > q+1$ and $p > n$, then $e^{(p-n)\underline{v}(t)} \le C\underline{u}^{q-m+1}(t)$. Hence there must be the case for u blowing up alone for every initial data. If $m > q+1$ and $p = n$, then $\underline{v}(t) \le C\underline{u}^{q-m+1}(t)$. Then u blows up alone for every initial data.

Case (iii) can be obtained by the similar method of (ii). □

4. Existence of simultaneous and non-simultaneous blow-up

We use two lemmas to prove Theorem 1.4 (i)-(ii). Define the set of the initial data $\mathbb{V}_0 = \{(u_0,v_0) \mid (u_0,v_0) \text{ satisfies (H)}\}$. Without loss of generality, we consider the case: $x_0'(t)$, $u_0'(x)$, $v_0'(x) \ge 0$, $t \in [0,T)$, $x \in \mathbb{R}$.

Lemma 4.1. *For any $(u_0,v_0) \in \mathbb{V}_0$, there are*

$$u_t(x,t),\ v_t(x,t),\ u_x(x,t),\ v_x(x,t) \ge 0 \quad \text{in } \mathbb{R}\times[0,T), \tag{9}$$

$$u_t(x,t) \ge \varepsilon u^m(x_0(t),t)e^{pv(x_0(t),t)},\ v_t(x,t) \ge \varepsilon u^q(x_0(t),t)e^{nv(x_0(t),t)} \quad \text{in } \mathbb{R}\times[0,T). \tag{10}$$

Proof. By calculations, we have $u_{tx} - u_{xxx} = v_{tx} - v_{xxx} = 0$ in $\mathbb{R}\times(0,T)$, and $u_0'(x)$, $v_0'(x) \ge 0$ in $\mathbb{R}$. Then u_x, $v_x \ge 0$ in $\mathbb{R}\times[0,T)$ by the comparison principle. One can also check that

$$u_{tt} - u_{txx} = mu^{m-1}(x_0(t),t)e^{pv(x_0(t),t)}\big[u_x(x_0(t),t)x_0'(t) + u_t(x_0(t),t)\big]$$

$$+ u^m(x_0(t),t)e^{pv(x_0(t),t)}p\big[v_x(x_0(t),t)x_0'(t) + v_t(x_0(t),t)\big] \ge 0,$$
$$v_{tt} - v_{txx} \ge 0 \quad \text{in } \mathbb{R}\times(0,T),$$
$$u_0''(x) + u^m(x_0(0),0)e^{pv(x_0(0),0)} \ge 0,\ v_0''(x) + u^q(x_0(0),0)e^{nv(x_0(0),0)} \ge 0 \ \text{ in } \mathbb{R}.$$

By the comparison principle, $u_t(x,t),\ v_t(x,t) \ge 0$ in $\mathbb{R}\times[0,T)$. Then (9) is obtained. In order to prove (10), construct auxiliary functions

$$J(x,t) = u_t(x,t) - \varepsilon u^m(x_0(t),t)e^{pv(x_0(t),t)},$$
$$K(x,t) = v_t(x,t) - \varepsilon u^q(x_0(t),t)e^{nv(x_0(t),t)} \quad \text{in } \mathbb{R}\times[0,T).$$

For constant $\varepsilon \in (0,1)$, one can check

$$J_t - J_{xx} \ge (1-\varepsilon)\big[u^m(x_0(t),t)e^{pv(x_0(t),t)}\big]' \ge 0 \quad \text{in } \mathbb{R}\times(0,T),$$
$$K_t - K_{xx} \ge (1-\varepsilon)\big[u^q(x_0(t),t)e^{nv(x_0(t),t)}\big]' \ge 0 \quad \text{in } \mathbb{R}\times(0,T),$$
$$u_0''(x) + (1-\varepsilon)u^m(x_0(0),0)e^{pv(x_0(0),0)} \ge 0 \quad \text{in } \mathbb{R},$$
$$v_0''(x) + (1-\varepsilon)u^q(x_0(0),0)e^{nv(x_0(0),0)} \ge 0 \quad \text{in } \mathbb{R}.$$

By the comparison principle, $J(x,t),\ K(x,t) \ge 0$ in $\mathbb{R}\times[0,T)$. □

By Lemma 4.1, we obtain the following important estimates

$$u(x_0(t),t) \le C(T-t)^{-\frac{1}{m-1}}, \quad m>1,\ t\in[0,T), \tag{11}$$
$$e^{v(x_0(t),t)} \le C(T-t)^{-\frac{1}{n}}, \qquad n>0,\ t\in[0,T). \tag{12}$$

In fact, by using (10),

$$(u(x_0(t),t))' = u_x(x_0(t),t)x_0'(t) + u_t(x_0(t),t) \ge \varepsilon u^m(x_0(t),t)e^{pv(x_0(t),t)},$$
$$(v(x_0(t),t))' \ge \varepsilon u^q(x_0(t),t)e^{nv(x_0(t),t)}, \quad t\in[0,T).$$

By integrations, estimates (11) and (12) can be obtained.

We will use (11) and (12) to prove the existence of non-simultaneous blow-up.

Lemma 4.2. *Assume the initial data satisfies (H).*

(i) *There exist suitable initial data such that* u *blows up alone if* $m > q+1$.
(ii) *There exist suitable initial data such that* v *blows up alone if* $n > p$.

Proof. At first, we prove the phenomena for u blowing up alone under suitable initial data. Assume that $(\tilde u_0, \tilde v_0)$ be a pair of initial data such that the positive solution of (3) blows up. Fix $v_0(\ge \tilde v_0)$ and take $M_0 > \|v_0\|_\infty$. Let $u_0(\ge \tilde u_0)$ be large such that T satisfies $M_0 \ge \|v_0\|_\infty + \frac{m-1}{m-q-1}C^q e^{nM_0}T^{\frac{m-q-1}{m-1}}$.

Consider the auxiliary problem

$$w_t = w_{xx} + C^q e^{nM_0}(T-t)^{-\frac{q}{m-1}} \quad \text{in } \mathbb{R}\times(0,T),\ \text{and } w(x,0) = v_0(x) \quad \text{in } \mathbb{R}.$$

For $m > q+1$ and by Green's identity [4, 11], we have

$$w(x,t) = \int_{-\infty}^{+\infty}\Gamma(x,y,t,0)v_0(y)dy + \int_0^t\int_{-\infty}^{+\infty}\Gamma(x,y,t,\tau)C^q e^{nM_0}(T-\tau)^{-\frac{q}{m-1}}dyd\tau$$

$$\leq \|v_0\|_\infty + \frac{m-1}{m-q-1} C^q e^{nM_0} T^{\frac{m-q-1}{m-1}} \leq M_0,$$

where Γ is the fundamental solution of the heat equation. Hence $w(x_0(t),t) \leq M_0$. So w satisfies

$$w_t \geq w_{xx} + C^q (T-t)^{-\frac{q}{m-1}} e^{nw(x_0(t),t)} \quad \text{in } \mathbb{R}\times(0,T),\ w(x,0)=v_0(x) \quad \text{in } \mathbb{R}.$$

It follows from (11) that v satisfies

$$v_t \leq v_{xx} + C^q (T-t)^{-\frac{q}{m-1}} e^{nv(x_0(t),t)} \quad \text{in } \mathbb{R}\times(0,T),\ v(x,0)=v_0(x) \quad \text{in } \mathbb{R}.$$

By the comparison principle, $v \leq w \leq M_0$ in $\mathbb{R}\times(0,T)$. Since $(u_0,v_0) \geq (\tilde{u}_0,\tilde{v}_0)$, (u,v) blows up. And hence only u blows up at time T.

Secondly, we prove the phenomena for v blowing up alone under suitable initial data. Assume that $(\tilde{u}_0,\tilde{v}_0)$ be a pair of initial data such that the positive solution of (3) blows up. Fix $u_0(\geq \tilde{u}_0)$ and take $M_1 > \|u_0\|_\infty$. Let $v_0(\geq \tilde{v}_0)$ be large such that T satisfies $M_1 \geq \|u_0\|_\infty + \frac{n}{n-p} C^p M_1^m T^{\frac{n-p}{n}}$.

Consider the auxiliary problem

$$z_t = z_{xx} + C^p M_1^m (T-t)^{-\frac{p}{n}} \quad \text{in } \mathbb{R}\times(0,T), \text{ and } z(x,0)=u_0(x) \quad \text{in } \mathbb{R}.$$

For $n>p$ and by Green's identity, we have $z(x,t) \leq \|u_0\|_\infty + \frac{n}{n-p} C^p M_1^m T^{\frac{n-p}{n}} \leq M_1$. So z satisfies $z_t \geq z_{xx} + C^p z^m(x_0(t),t)(T-t)^{-\frac{p}{n}}$ in $\mathbb{R}\times(0,T)$. It follows from (12) that $u_t \leq u_{xx} + C^p u^m(x_0(t),t)(T-t)^{-\frac{p}{n}}$ in $\mathbb{R}\times(0,T)$. By the comparison principle, $u(x,t) \leq z(x,t) \leq M_1$ in $\mathbb{R}\times(0,T)$. Since $(u_0,v_0) \geq (\tilde{u}_0,\tilde{v}_0)$, (u,v) blows up. And hence only v blows up. □

We use two lemmas to prove Theorem 1.4 (iii).

Lemma 4.3. *The set of* (u_0,v_0) *in* $\mathbb{V}_0$ *such that* v *(*u*) blows up alone is open in* L^∞*-norm.*

Proof. Without loss of generality, we only prove the case for v blowing up with u remaining bounded. Let (u,v) be a solution of (3) with initial data $(u_0,v_0) \in \mathbb{V}_0$ such that v blows up while u remains bounded up to blow-up time T, say $0 < u(x,t) \leq M$. By Corollary 1.3, there must be $n > p$. It suffices to find a L^1-neighborhood of (u_0,v_0) in $\mathbb{V}_0$ such that any solution $(\hat{u},\hat{v})$ of (3) coming from this neighborhood maintains the property that $\hat{v}$ blows up while $\hat{u}$ remains bounded. Take $M_2 > M + \|u_0\|_\infty/2$. Let $(\tilde{u},\tilde{v})$ be the solution of the following problem

$$\begin{cases} \tilde{u}_t = \tilde{u}_{xx} + \tilde{u}^m(x_0(t),t) e^{p\tilde{v}(x_0(t),t)}, \\ \tilde{v}_t = \tilde{v}_{xx} + \tilde{u}^q(x_0(t),t) e^{n\tilde{v}(x_0(t),t)} \quad \text{in } \mathbb{R}\times(T-\varepsilon_0, T-\varepsilon_0+T_0), \\ \tilde{u}(x,T-\varepsilon_0) = \tilde{u}_0(x),\ \tilde{v}(x,T-\varepsilon_0)=\tilde{v}_0(x) \quad \text{in } \mathbb{R}, \end{cases}$$

where $(\tilde{u}_0,\tilde{v}_0)$ and ε_0 are to be determined. Denote

$$\mathbb{N}(u_0,v_0) = \Big\{ (\tilde{u}_0,\tilde{v}_0) \in \mathbb{V}_0 \mid \|\tilde{u}_0(x) - u(x,T-\varepsilon_0)\|_\infty,$$

$$\|\tilde{v}_0(x) - v(x, T - \varepsilon_0)\|_\infty < \|u_0\|_\infty/2\Big\}.$$

Since v blows up at time T, there exists some small constant $\varepsilon_0 > 0$ such that $(\tilde{u}, \tilde{v})$ blows up and T_0 satisfies $M_2 > M + \frac{\|u_0\|_\infty}{2} + \frac{n}{n-p} C^p T_0^{\frac{n-p}{n}} M_2^m$, provided $(\tilde{u}_0, \tilde{v}_0) \in \mathbb{N}(u_0, v_0)$. Consider the auxiliary system

$$\begin{cases} U_t = U_{xx} + C^p (T - \varepsilon_0 + T_0 - t)^{-\frac{p}{n}} M_2^m & \text{in } \mathbb{R} \times (T - \varepsilon_0, T - \varepsilon_0 + T_0), \\ U(x, T - \varepsilon_0) = \tilde{u}_0(x) & \text{in } \mathbb{R}. \end{cases}$$

By Green's identity, we have $U(x,t) \le M + \frac{\|u_0\|_\infty}{2} + \frac{n}{n-p} C^p T_0^{\frac{n-p}{n}} M_2^m < M_2$. Hence there is

$$\begin{cases} U_t \ge U_{xx} + C^p (T - \varepsilon_0 + T_0 - t)^{-\frac{p}{n}} U^m(x_0(t), t) & \text{in } \mathbb{R} \times (T - \varepsilon_0, T - \varepsilon_0 + T_0), \\ U(x, T - \varepsilon_0) = \tilde{u}_0(x) & \text{in } \mathbb{R}. \end{cases}$$

On the other hand, we have

$$\begin{cases} \tilde{u}_t \le \tilde{u}_{xx} + C^p (T - \varepsilon_0 + T_0 - t)^{-\frac{p}{n}} \tilde{u}^m(x_0(t), t) & \text{in } \mathbb{R} \times (T - \varepsilon_0, T - \varepsilon_0 + T_0), \\ \tilde{u}_0(x, T - \varepsilon_0) = \tilde{u}_0(x) & \text{in } \mathbb{R}. \end{cases}$$

By the comparison principle, $\tilde{u} \le U \le M_2$ in $\mathbb{R} \times (T - \varepsilon_0, T - \varepsilon_0 + T_0)$, then $\tilde{v}$ blow up.

According to the continuity about initial data for bounded solutions, there exists a neighborhood of (u_0, v_0) in $\mathbb{V}_0$ such that every solution $(\hat{u}, \hat{v})$ starting from the neighborhood will enter $\mathbb{N}(u_0, v_0)$ at time $T - \varepsilon_0$, and keeps the property that $\hat{v}$ blows up while $\hat{u}$ remains bounded. □

Lemma 4.4. *Assume $m > q + 1$ and $n > p$. Then both non-simultaneous blow-up and simultaneous blow-up may occur.*

Proof. Assume $(u_0, v_0) \in \mathbb{V}_0$ such that the solution of (3) blows up. Then the positive solution with initial data $(u_0/l, v_0/(1-l)) \in \mathbb{V}_0$ for any $l \in (0,1)$ also blows up. By Lemma 4.2, we know there exist some l_1 near 0 such that u blows up while v remains bounded if $l = l_1$, and some l_2 near 1 such that v blows up while u remains bounded if $l = l_2$, respectively. By Lemma 4.3, such sets of initial data are open and connected. Then there exists some $l \in (l_1, l_2)$ such that simultaneous blow-up happens. □

Till now, the proof of Theorem 1.4 is finished.

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