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A MASS LUMPING AND DISTRIBUTING FINITE ELEMENT ALGORITHM FOR MODELING FLOW IN VARIABLY SATURATED POROUS MEDIA

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ABSTRACT. The Richards equation for water movement in unsaturated soil is highly nonlinear partial differential equations which are not solvable analytically unless unrealistic and oversimplifying assumptions are made regarding the attributes, dynamics, and properties of the physical systems. Therefore, conventionally, numerical solutions are the only feasible procedures to model flow in partially saturated porous media. The standard Finite element numerical technique is usually coupled with an Euler time discretizations scheme. Except for the fully explicit forward method, any other Euler time-marching algorithm generates nonlinear algebraic equations which should be solved using iterative procedures such as Newton and Picard iterations. In this study, lumped mass and distributed mass in the frame of Picard and Newton iterative techniques were evaluated to determine the most efficient method to solve the Richards equation with finite element model. The accuracy and computational efficiency of the scheme and of the Picard and Newton models are assessed for three test problems simulating one-dimensional flow processes in unsaturated porous media. Results demonstrated that, the conventional mass distributed finite element method suffers from numerical oscillations at the wetting front, especially for very dry initial conditions. Even though small mesh sizes are applied for all the test problems, it is shown that the traditional mass-distributed scheme can still generate an incorrect response due to the highly nonlinear properties of water flow in unsaturated soil and cause numerical oscillation. On the other hand, non oscillatory solutions are obtained and non-physics solutions for these problems are evaded by using the mass-lumped finite element method.

1. INTRODUCTION

The Richards equation is the governing equation for movement of water flow in partially saturated porous media, contains nonlinearities arising from pressure head dependencies in soil moisture and hydraulic conductivity. It is practically impossible to solve the equation analytically in unsaturated soil profiles with complex initial boundary conditions due to its highly nonlinear nature. Therefore numerical approximations are typically used to solve the unsaturated flow equations. The standard approximations that are applied to the spatial domain

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are the finite difference and finite element methods are two popular schemes [1, 2]. Variations of the standard finite difference and finite element methods, such as the sub domain finite element method have also been successfully used to solve these problems [3]. The choice of using either finite difference or finite element method for the solution of variably saturated flow problems is largely personal and promoters of a particular method can easily support the strengths of their preferred solution approach. This can cause problems when a choice has to be made between these approaches, as it is difficult to independently evaluate the methods in a substantial technique.

Several studies have explored the finite element solution [3, 4, 5, 6, 7]. Discussion about efficiency of mass types, element orders, and matrix solution methods, as well as influence of methods of numerical integration for the Richards equation are found few of these studies. The main reason of this is the consistent (distributed) mass scheme with high order elements has been typically accepted to be superior for solving complex nonlinear physical problems. Previous studies established that the numerical solution with consistent mass formulation shows oscillation [3, 5, 4, 6] but they are not explained about the reason of oscillations and methods how to avoid the oscillations. Numerical oscillations is a significant factor to diverge the solution of Richards equation when simulate a sharp wetting front entering a dry soil profile. To overcome this difficulty, it is necessary to reduce the size of the element and time step size. But the reduction in element size could drastically increase the simulation time, as a result the numerical method becomes less attractive.

Numerical solution using pressure based formulation coupled with backward Euler time discretization is shown to produce unacceptably large mass balance errors for many example calculations. It is true for both finite difference and finite element approximation in space, although finite elements are generally inferior to finite differences. Because use of pressure head formulation with a simple time-stepping method is common, these findings appear to have significant practical implications. Almost, all unsaturated flow simulations use either the head-based or the moisture content-based formulation of Richards equation. A variety of finite difference and finite element solution techniques have been used with each of these equation forms [3, 8, 9, 10, 11]. Numerical algorithm based on mixed form of Richards equation have been proposed and mass balance errors occur in standard head-based numerical solution [12].

Fully implicit backward Euler time approximation applied to the mixed form of Richards equation is mass conserving solution procedure for the unsaturated flow equation. Proper expansion of the time derivative produces a simple computational algorithm that is perfectly mass conservative for numerical approximation that preserves spatial symmetry. Thus the finite difference and finite element approximations using this mixed formulation are perfectly mass conservative. This approach is show to be superior to the standard head-based approximations while requiring no more computational effort. However conservation of mass is shown to be inadequate to guarantee good numerical solutions. For infiltration into dry soils, finite element approximations produce oscillatory solutions even while conserving mass. It is shown that diagonalization of the time matrix, which occurs naturally in finite difference approximations, is necessary and sufficient to guarantee oscillations free solutions. Mass conserving solution procedure is presented by the use of modified definition of capacity term to force global mass

balance and illustrated the importance of mass lumping in finite element solutions to unsaturated flow problems [5]. The significance of mass lumping is also shown and report good finite element solutions without mass lumping [3, 4, 10, 11]. Most of the studies are used a one-step Euler time-marching algorithm with the head-based version of Richards equation.

For stability reasons an implicit time discretization requiring evaluation of the nonlinear coefficients at the current time level, is generally used to solve the equation numerically. Newton and Picard iterative schemes are commonly used to linearize the resulting discrete system of equations, with the Picard scheme being the more popular of the two [3, 11, 13, 14, 15]. Picard method is the most intuitive linearization of Richards equation, computationally inexpensive on a per-iteration basis, and preserves symmetry of the discrete system of equations. However, some studies shown experimentally, the method may diverge under certain conditions [7, 10] and verified theoretically [16].

On the other hand, the Newton scheme, which is quadratically convergent [7, 17, 18] yields complex non symmetric system matrices and expensive than Picard linearization, and can be more robust than Picard for certain flow problems. The drawback of Newton's method is that it is only locally convergent and involves the computation of derivatives. Although the use of the solution of the last time step to start the Newton iterations improves considerably the robustness of Newton' method, in the degenerate case (saturated/unsaturated flow) the convergence is ensured only when a regularization step is applied and under additional constraints on the discretization parameters. While the Newton and the Picard schemes are generally robust, these iterative methods entail computational costs associated with having to evaluate and solve the system of equations repeatedly for each time step.

The objective of this study is to investigate the numerical behavior of finite element solution of Richards equation including the suitability of lumped mass and distributed mass. The work is also focused on determining the method which would offer a stable solution without requiring the resizing of the finite element mesh structure. Realistic initial and Dirichlet boundary conditions are imposed in the numerical simulator to the head-based form of Richards equation. To reduce the CPU time and maintain small truncation error, an adaptive time-stepping strategy is implemented. The nonlinear matrix equations are solved using the Picard and Newton iteration schemes. The performance of the algorithm is shown to be superior to the conventional pressure head-based form and can be easily used in layered porous media without any extraordinary treatment.

2. Governing Equation and Discretization

The partial differential equation describing fluid flow in partially saturated porous media, Richards equation, is obtained by combining Darcy's law with the continuity equation. For one-dimensional vertical flow in unsaturated soils, pressure head-based Richards equation is written as

$$C(\psi)\frac{\partial \psi}{\partial t} = \frac{\partial}{\partial z}\left(K(\psi)\left(\frac{\partial \psi}{\partial z} + 1\right)\right) \tag{2.1}$$

where, ψ is the pressure head $[L]$, t is time $[T]$, z denotes the vertical distance from the soil surface assumed positive downward $[L]$, $K(\psi)$ is the hydraulic conductivity $[LT^{-1}]$, $C(\psi) = \frac{d\theta}{d\psi}$ is the specific fluid capacity $[L^{-1}]$, θ is the volumetric water content.

The pressure head-based form can be successfully used in both saturated and unsaturated zones as well as in layered and composite porous materials. However, several studies have shown that it suffers from convergence difficulties and poor mass balance in modeling infiltration into very dry media unless very fine discretizations are used [5, 7, 14, 19, 20, 21, 22, 23] which, in turn, makes the computation very expensive. The time steps required for convergence are several orders of magnitude smaller than is required for reasonable temporal discretization [19]. The reason for these problems is highly nonlinear nature of the saturation-pressure function under dry initial conditions, causing very high fluid pressure gradient near the wetting front and huge computational cost.

Solution of Richards equation requires knowledge of hydraulic conductivity and fluid content versus fluid pressure head. These relationships are known as the hydraulic properties of the porous media. In this study, the most commonly used relationships are the van Genuchten [24] model. This model illustrated in detail as follows

$$\theta(\psi) = \theta_r + \frac{\theta_s - \theta_r}{[1 + |\alpha\psi|^n]^m} \quad \text{if} \quad \psi \leq 0 \tag{2.2}$$

$$\theta(\psi) = \theta_s \quad \text{if} \quad \psi > 0 \tag{2.3}$$

$$K(\psi) = K_s \left[\frac{\theta - \theta_r}{\theta_s - \theta_r}\right]^{\frac{1}{2}} \left\{1 - \left[1 - \left(\frac{\theta - \theta_r}{\theta_s - \theta_r}\right)^{\frac{1}{m}}\right]^m\right\}^2 \quad \text{if} \quad \psi \leq 0 \tag{2.4}$$

$$K(\psi) = K_s \quad \text{if} \quad \psi > 0 \tag{2.5}$$

$$c(\psi) = \alpha m n \frac{\theta_s - \theta_r}{[1 + |\alpha\psi|^n]^{m+1}} |\alpha\psi|^{n-1} \quad \text{if} \quad \psi \leq 0 \tag{2.6}$$

$$c(\psi) = 0 \quad \text{if} \quad \psi > 0 \tag{2.7}$$

Due to the nonlinear nature of the Richards equation (2.1), it should be solve numerically, finite element Galarkin discretization in space and a finite difference discretization of the time derivative term are used. To develop the finite element approximation of the pressure head-based Richards equation, the weak formulation of the dependent variable and the constitutive relations were approximated using interpolating polynomials [6, 25]. It was assumed that the hydraulic conductivity as well as capacitance varies linearly within each element [26]:

$$\psi(z,t) \approx \hat{\psi}(z,t) = \sum_{j=1}^{M} \psi_j(t) N_j(z) \tag{2.8}$$

$$K \approx \hat{K} = \sum_{j=1}^{M} K_j N_j(z) \tag{2.9}$$

$$C \approx \hat{C} = \sum_{j=1}^{M} C_j N_j(z) \tag{2.10}$$

where M is the number of elements, $N_j(z)$ is the selected basis function, and $\psi_j(t)$ is the associated and time-dependent unknown coefficients representing the solution of flow equation at nodes within the domain. The goal of the finite element approximation is to minimize the error and this can be accomplished by introducing the weight function, $N_j(z)$, and setting operator $L(\hat{\psi})$ to be orthogonal to $N_j(z)$:

$$\begin{aligned}\int_{\Omega} L\left(\hat{\psi}\right) N_j(z)\, d\Omega\Omega &= \int_0^{\Delta z} \left(\hat{C}(\psi) \frac{\partial \hat{\psi}(z,t)}{\partial z} - \frac{\partial}{\partial z}\left[\hat{K}(\psi) \frac{\partial \hat{\psi}(z,t)}{\partial z} \right] + \frac{\partial K(\hat{\psi})}{\partial z} \right) N_j(z)\, dz \\ &= 0 \end{aligned} \tag{2.11}$$

Performing integration by parts to reduce the second derivative and using fully implicit backward Euler time-marching algorithm with the solution is assumed to be known at time level n and unknown at time level $n+1$, one can discretize the time derivative in (2.11) to yield [27]:

$$[A]\{\psi\}_j^{n+1} + [F]\left\{ \frac{\psi_j^{n+1} - \psi_j^n}{\Delta t} \right\} = \{B\} \tag{2.12}$$

where:

$$[A] = \int_0^{\Delta z} \hat{K} \frac{\partial N_i(z)}{\partial z} \frac{\partial N_j(z)}{\partial z} dz \tag{2.13}$$

$$[F] = \int_0^{\Delta z} \hat{C} N_i(z) N_j(z) dz \tag{2.14}$$

$$\{B\} = -\int_0^{\Delta z} \frac{\partial \hat{K}}{\partial z} N_j(z) dz \tag{2.15}$$

One can found the detail evaluations of the above integrals and assembly of the global matrices in the literature [28]. The final results are highlighted herein, first for the stiffness mass matrix $[F]$:

$$[F] = \frac{\Delta z}{12} \begin{pmatrix} 3C_1 + C_2 & C_1 + C_2 & 0 & \cdots & 0 \\ C_1 + C_2 & C_1 + 6C_2 + C_3 & C_2 + C_3 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ \vdots & \vdots & C_{N-2} + C_{N-1} & C_{N-2} + 6C_{N-1} + C_N & C_{N-1} + C_N \\ 0 & 0 & \cdots & C_{N-1} + C_N & C_{N-1} + 3C_N \end{pmatrix}$$

Similarly matrix $[A]$ is assemble to form:

$$[A] = \frac{1}{2\Delta z}\begin{pmatrix} K_1+K_2 & -K_1-K_2 & 0 & \cdots & 0 \\ -K_1-K_2 & K_1+2K_2+K_3 & -K_2-K_3 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ \vdots & \vdots & & & \\ 0 & 0 & -K_{N-2}-K_{N-1} & K_{N-2}+2K_{N-1}+K_N & -K_{N-1}-K_N \\ & & \cdots & -K_{N-1}-K_N & K_{N-1}+K_N \end{pmatrix}$$

Finally the driving force vector $[B]$ can be assembled to produce:

$$[B] = \frac{1}{2}\begin{pmatrix} K_1+K_2 \\ K_3-K_1 \\ \vdots \\ K_{i+1}-K_{i-1} \\ \vdots \\ K_N-K_{N-1} \end{pmatrix} + \begin{pmatrix} q_0(t) \\ 0 \\ \vdots \\ 0 \\ \vdots \\ -q_L(t) \end{pmatrix}$$

where $q_0(t)$ is the net flux at the upper end and $q_L(t)$ is the imposed flux at the lower end of the spatial domain.

Upon substituting these matrices in (2.12) and rearranging the finite element approximation of the pressure head-based form Richards equation can be written in matrix form as [28]:

$$\begin{aligned} &\left[\left(C_{i-1}^{n+1}+C_i^{n+1}\right)\frac{\Delta z}{12\Delta t}+\frac{\left(K_{i-1}^{n+1}+K_i^{n+1}\right)}{2\Delta z}\right]\psi_{i-1}^{n+1} \\ &+\left[\left(C_{i-1}^{n+1}+6C_i^{n+1}+C_{i+1}^{n+1}\right)\frac{\Delta z}{12\Delta t}+\frac{\left(K_{i-1}^{n+1}+2K_i^{n+1}+K_{i+1}^{n+1}\right)}{2\Delta z}\right]\psi_i^{n+1} \\ &+\left[\left(C_i^{n+1}+C_{i+1}^{n+1}\right)\frac{\Delta z}{12\Delta t}+\frac{\left(K_i^{n+1}+K_{i+1}^{n-1}\right)}{2\Delta z}\right]\psi_{i+1}^{n+1} \\ &=\left[\left(C_{i-1}^{n+1}+6C_i^{n+1}+C_{i+1}^{n+1}\right)\frac{\Delta z}{12\Delta t}\right]\psi_i^n-\frac{K_{i+1}^{n+1}-K_{i-1}^{n+1}}{2} \end{aligned} \tag{2.16}$$

This formulation is called the finite element approximation of the pressure head-based form Richards equation with distributed mass matrix.

The finite element formulation of flow equation with lumped mass matrix is as follows [27]:

$$\begin{aligned} &\left[\frac{K_{i-1}^{n+1}+K_i^{n+1}}{2\Delta z}\right]\psi_{i-1}^{n+1}+\left[\frac{\Delta z}{\Delta t}C_i^{n+1}+\frac{\left(K_{i-1}^{n+1}+2K_i^{n+1}+K_{i+1}^{n+1}\right)}{2\Delta z}\right]\psi_i^{n+1} \\ &+\left[\frac{K_i^{n+1}+K_{i+1}^{n+1}}{2\Delta z}\right]\psi_{i+1}^{n+1}=\frac{\Delta z}{\Delta t}C_i^{n+1}\psi_i^n-\frac{K_{i+1}^{n+1}-K_{i-1}^{n+1}}{2} \end{aligned} \tag{2.17}$$

3. Iterative Methods

The Richards equation (2.12) can be written in the following vector form

$$\mathbf{A}\left(\mathbf{\Psi}^{n+1}\right)\mathbf{\Psi}^{n+1}+\mathbf{F}\left(\mathbf{\Psi}^{n+1}\right)\frac{\mathbf{\Psi}^{n+1}-\mathbf{\Psi}^{n}}{\Delta t}=\mathbf{q}\left(t^{n+1}\right)-\mathbf{b}(\mathbf{\Psi}^{n+1}) \tag{3.1}$$

3.1. **Newton scheme.** Let us consider

$$\mathbf{f}\left(\mathbf{\Psi}^{n+1}\right)=\mathbf{A}\left(\mathbf{\Psi}^{n+1}\right)\mathbf{\Psi}^{n+1}+\mathbf{F}\left(\mathbf{\Psi}^{n+1}\right)\frac{\mathbf{\Psi}^{n+1}-\mathbf{\Psi}^{n}}{\Delta t}-\mathbf{q}\left(t^{n+1}\right)-\mathbf{b}\left(\mathbf{\Psi}^{n+1}\right)=0 \tag{3.2}$$

Here m stands for iteration index, so the Newton scheme [28] is

$$\mathbf{f}^{'}\left(\mathbf{\Psi}^{n+1,\ m}\right)\mathbf{h}=-\mathbf{f}\left(\mathbf{\Psi}^{n+1,\ m}\right) \tag{3.3}$$

where

$$\mathbf{h}=\mathbf{\Psi}^{n+1,m+1}-\mathbf{\Psi}^{n+1,m} \tag{3.4}$$

and

$$\begin{aligned} f_{ij}^{'}=\lambda A_{ij}&+\frac{1}{\Delta t^{n+1}}F_{ij}+\sum_{s}\frac{\partial A_{is}}{\partial\psi_{j}^{n+1}}\psi_{s}^{n+1}\\ &+\frac{1}{\Delta t^{n+1}}\sum_{s}\frac{\partial F_{is}}{\partial\psi_{j}^{n+1}}\left(\psi_{s}^{n+1}-\psi_{s}^{n}\right)+\frac{\partial b_{i}}{\partial\psi_{j}^{n+1}} \end{aligned} \tag{3.5}$$

is the *ij* th component of the Jacobian matrix $\mathbf{f}^{'}\left(\mathbf{\Psi}^{n+1}\right)$.

3.2. **Picard Scheme.** The simple formulation of Picard scheme [28] can be obtained directly from (2.12) by iterating with all linear occurrences of $\mathbf{\Psi}^{n+1}$ taken at the current iteration level $m+1$ and all nonlinear occurrences at the previous level m. We get,

$$\left[\mathbf{A}\left(\lambda\mathbf{\Psi}^{n+1,m}\right)+\frac{1}{\Delta t}\mathbf{F}\left(\mathbf{\Psi}^{n+1,m}\right)\right]\mathbf{h}=-\mathbf{f}\left(\mathbf{\Psi}^{n+1,m}\right). \tag{3.6}$$

4. Numerical Simulations

All numerical simulations are done by CATHY (CATchment HYdrology) model that features elements of the sequential iterative coupling schemes. CATHY is a physically-based hydrological model where the surface module resolves the one-dimensional (1D) diffusion wave equation and the subsurface module solves the three-dimensional (3D) Richards equation. Coupling between these two equations is based on an extension of the boundary condition switching procedure used in some subsurface models for the handling of atmospheric inputs on the land surface boundary of the catchment. The main objective of this work is to assess, via sensitivity analysis, the accuracy, computational effort and mass balance limitations for the CATHY model over the frame of lumped mass and distributed mass along with the three tests of soil hydraulic parameters which make soil retention functions are highly nonlinear. For the case of convergence criterion, dynamic time stepping control is used to adjust step size of time during simulation according to the convergence behavior of the nonlinear iteration scheme. Nonlinear tolerance ($tol=10^{-3}$) is specified for each time step, along with a maximum number of

iterations, $maxit(=10)$. The simulation begins with a time step size of Δt_0 and proceeds until time T_{max}. The current time step size is increased by a factor of $\Delta t_{mag}\,(=1.20)$ to a maximum size of Δt_{max} if convergence is achieved in fewer than $maxit_1\,(=5)$ iterations, it is remain unchanged if convergence required between $maxit_1$ and $maxit_2\,(=8)$ iterations, and it is decreased by a factor of $\Delta t_{red}(=0.5)$ to a minimum of Δt_{min} if convergence required more than $maxit_2$ iterations. If convergence is not achieved within *maxit*, the solution at the current time level is recomputed using a reduced time step size to the minimum time step size Δt_{min}. For the first time step of simulation, the initial conditions are used as the first solution estimate for the iterative procedure. For subsequent time steps of simulation the pressure head solution from the previous step is used as the first estimate. Thus time step size has a direct effect on convergence behavior, via its influence on the quality of the initial solution estimate. Back-stepping is also triggered if linear solver failed or if the convergence or residual errors become larger than maximum allowable convergence or residual error in the nonlinear solution. In the nonlinear iterative methods, the infinity norm (l_∞) of the convergence error is used as the termination criterion; that is, when $\left\|\boldsymbol{\Psi}^{n+1,m+1}-\boldsymbol{\Psi}^{n+1,m}\right\| \leq tol$ is satisfied, convergence is achieved [29]. The residual error ($\left\|\mathbf{f}\left(\boldsymbol{\Psi}^{n+1,m}\right)\right\|$) is computed using l_∞ and l_2 norms.

One measure of a numerical simulator is its ability to conserve global mass over the domain of interest. Satisfying the mass balance is necessary but not completely adequate prerequisite for a correct solution [7]. To measure the ability of the simulator to conserve mass, one of the most widely used criteria for evaluating the accuracy of a numerical scheme [7] is the mass balance error given by

$$\text{Mass Balance Error} = \left|1-\frac{\text{Total additional mass in the domain}}{\text{Total net flux into the domain}}\right|$$

where the additional mass in the domain is the difference between the mass measured at any instant t and the initial mass in the domain, and the total net flux into the domain is the flux balance integrated in time up to t. For the finite element approximation, this is calculated by the following formula [7]:

$$MB(t)=\frac{\sum_{i=1}^{E-1}\left(\theta_i^{n+1}-\theta_i^0\right)(\Delta z)+\left(\theta_0^{n+1}-\theta_0^0\right)\left(\frac{\Delta z}{2}\right)+\left(\theta_E^{n+1}-\theta_E^0\right)\left(\frac{\Delta z}{2}\right)}{\sum_{j=1}^{n+1}\left\{\left(q_o^j-q_N^j\right)(\Delta t)\right\}} \tag{4.1}$$

with $N=E+1$ nodes $\{z_0,\ z_1,\ z_2,\ \ldots\ldots,\ z_E\}$, and constant nodal spacing Δz is considered and q_0 and q_N being boundary fluxes calculated from the finite element equations associated with the boundary modes z_0 and z_N.

5. Results and Discussions

To assure the purposes of this work, sum up three one-dimensional test problems on which these investigations were based, examine methods for quantifying the efficiency, accuracy and applicability of the resultant solutions, and draw methods to evaluate the computational work

required to achieve the results. A set of numerical experiments along with lumped and consistent mass was performed, to assess the robustness of the approach, to investigate methods for improving the efficiency of solutions to Richards equation and to evaluate the advantage of using the technique.

5.1. **Test Problem 1.** It is a benchmark test problem that has been previously examined [22, 23, 31, 32, 33]. The domain of this test problem is short and saturated conditions are not developed. To run CATHY model, two sets of nodal spacing are used (that is, 126 and 251 nodes are used). To assess the oscillation of the solution profile between mass lumping and distributed mass, very small time step ($\Delta t_{min} = 10^{-8}\, s$) is used. In this test case, constant pressure head boundary conditions are imposed, at the bottom and top of the soil column are $-10m$ and $-0.75m$ respectively. The initial pressure head is $-10m$. Since the initial and boundary conditions are not consistent, so a steep gradient in the pressure head is setup. A $0.3m$ column of soil with van Genuchten parameters $\theta_s = 0.368$, $\theta_r = 0.102$, $\alpha = 3.35/m$, $n = 2.0$ and $K_s = 7.970m/day$.

The pressure head profiles of the finite element results for lumped and distributed mass matrix for 126 and 251 nodes are shown in the FIGURE 1 and FIGURE 2 respectively. Note that Pic=Picard & New=Newton in FIGURES. The agreement is quite perfect if using the lumped mass for both the Picard and Newton iteration. Few numerical oscillations are produces at the bottom of the soil column based on the consistent mass scheme and it does not improve even with larger grid spacing. This dissimilarity occurs only for the handling of time (mass) matrices in the two solution procedures.

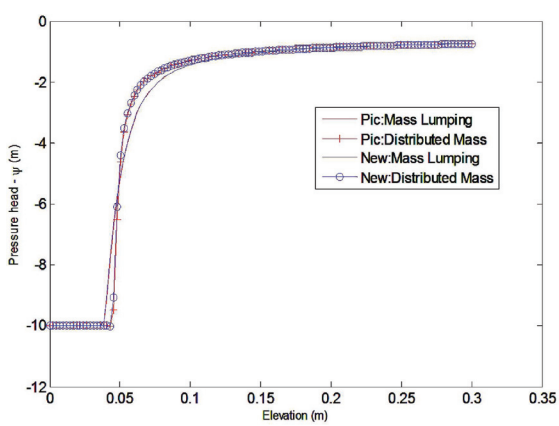

FIGURE 1. Pressure head profiles at $21600s$ for the Picard and Newton iterative schemes for the lumped and distributed mass of 125 layers.

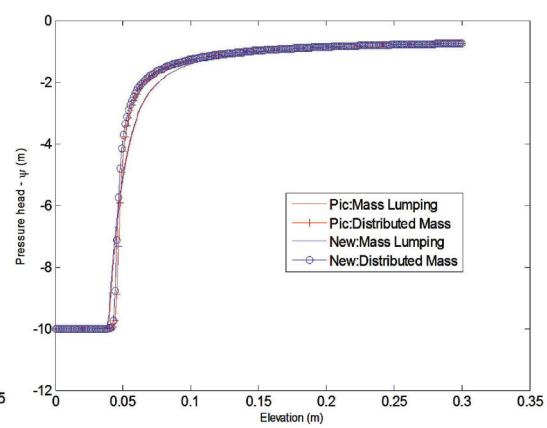

FIGURE 2. Pressure head profiles at $21600s$ for the Picard and Newton iterative schemes for the lumped and distributed mass of 250 layers.

Satisfactory cumulative mass balance errors and nonlinear iterations per time step for the Picard scheme are shown in FIGURE 3 and FIGURE 4 respectively. Similar behaviors are

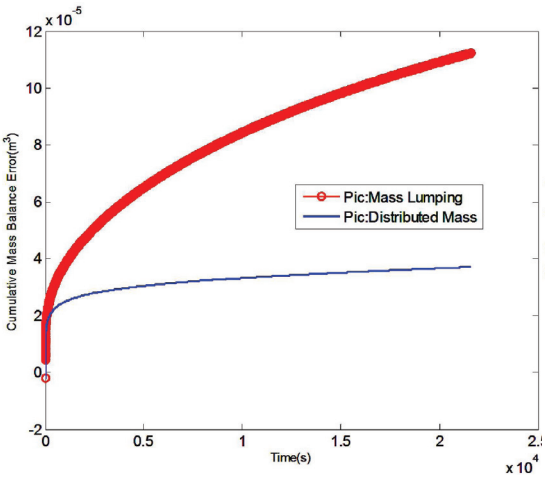

FIGURE 3. Mass balance result for the Picard iterative scheme for the lumped and distributed mass of 250 layers.

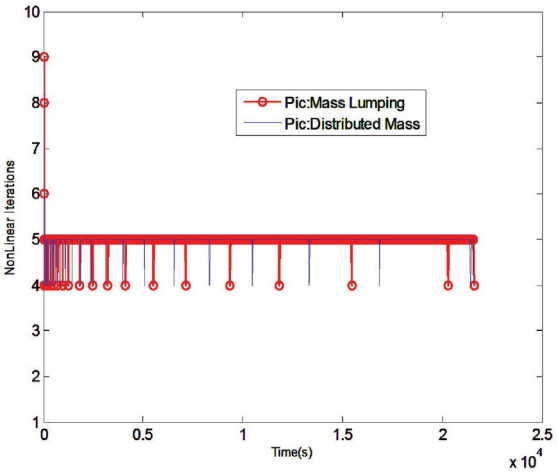

FIGURE 4. Convergence result for the Picard iterative scheme for the lumped and distributed mass of 250 layers.

observed for the Newton iteration technique. The evaluated methods clearly outperform the predictable algorithm and these methods seem to handle this test case without any significant problems. It is also noted that the performance of the schemes in all cases is very similar to that of the published reports [23, 31, 32, 33]. The computational statistics of the methods under these conditions is shown in TABLE 1.

TABLE 1. Computational statistics of the Test Problem 1.
Note: Dist=Distributed,ite=iteration,CMBE=Cumulative mass balance error.

No. of layers	Technique	Mass	No. of time steps	NL ite/ Time step	CMBE
125	Picard	Lump	2622	5.02	1.2954e-4
		Dist	3783	5.02	5.9864e-5
	Newton	Lump	6130	5.02	1.2940e-4
		Dist	9620	5.01	5.9181e-5
250	Picard	Lump	1372	5.02	1.1228e-4
		Dist	1360	5.02	3.7021e-5
	Newton	Lump	3008	5.14	1.1114e-4
		Dist	4281	5.01	3.3256e-5

Graphical results and statistics of the simulation clearly indicate that the mass lumping technique is robust than the mass distributing technique for the numerical solution of Richards equation.

5.2. **Test Problem 2.** It is a very difficult vertical infiltration one-dimensional problem of $10m$ high soil column for the van Genuchten model with the soil parameters $\theta_s = 0.301$, $\theta_r = 0.093$, $\alpha = 5.47/m$, $n = 4.264$ and $K_s = 5.040m/days$. In order to evaluate the influence of the algorithm, two sets of grid spacing, one of them is very dense (e.g., $\Delta z = 0.05m$ and $0.025m$) and very small time step size (i.e., for $\Delta z = 0.05m$, $\Delta t_{min} = 10^{-5}s$ and for $\Delta z = 0.025\,m$, $\Delta t_{min} = 10^{-15}\,s$) are considered and has been already analyzed in details [23, 30, 33]. It has constant head boundary conditions at both top ($\psi\,(10,t) = 0.1$) and bottom ($\psi\,(0,t) = 0.0$) boundaries and a hydrostatic equilibrium initial condition ($\psi\,(z,0) = -z$). The set of conditions investigated represents range of medium and auxiliary conditions that are representative of a difficult class of infiltration problem frequently solved using Richards equation.

The pressure head profile from the solution for Test Problem 2 of 200 layers is shown in FIGURE 5. It is shown that mass distributed finite element method suffers from numerical

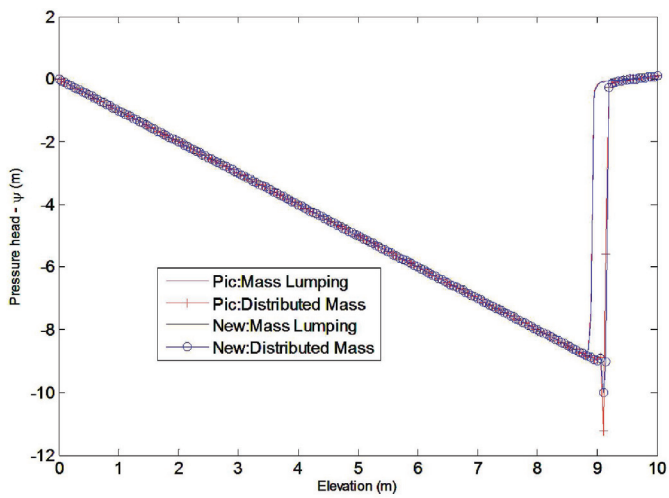

FIGURE 5. Pressure head profiles at $17280s$ for the Picard and Newton iterative schemes for the lumped and distributed mass of 200 layers.

oscillations at the wetting front but the lumped mass scheme offered oscillations free stable solution for dry initial conditions. The results hold for the iterative solution to the discretized version of the pressure head Richards equation including Picard and Newton techniques. To avoid the divergence, mesh and step size are reduced and FIGURE 6 represents pressure head solution of 400 layers with very small time step size $\Delta t_{min} = 10^{-15}\,s$ for mass lumped and distributed mass. Still, the consistent mass scheme of Picard and Newton methods exhibits

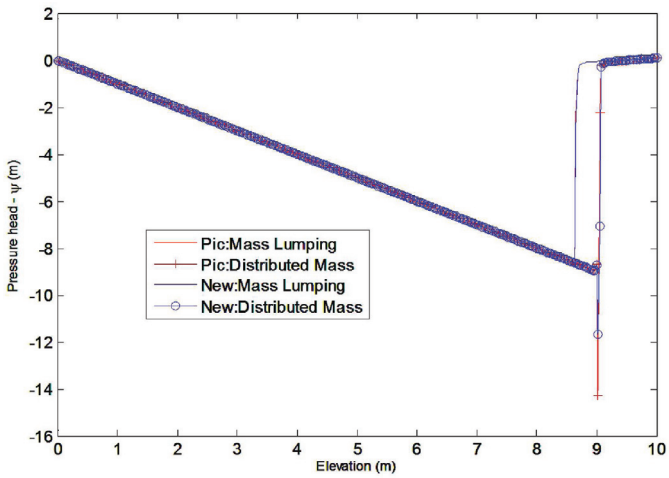

FIGURE 6. Pressure head profiles at $17280s$ for the Picard and Newton iterative schemes for the lumped and distributed mass of 400 layers.

significant numerical errors ahead of the infiltration front but such oscillations are not present in the mass lumping formulation. Because the only difference between two solution procedures is the treatment of the time derivative term, these results imply that diagonalized time (mass) matrices are to be preferred. The cumulative mass balance errors plots are shown in

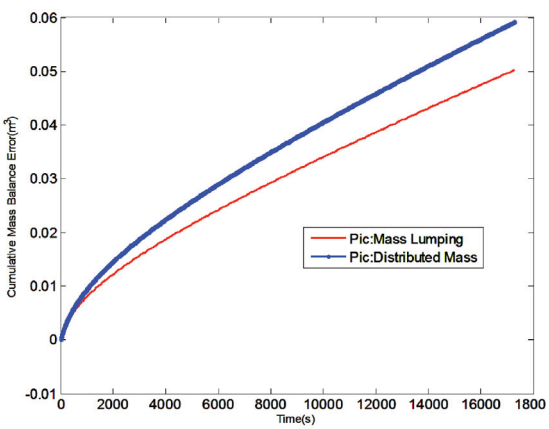

FIGURE 7. Mass balance result for the Picard iterative scheme for the lumped and distributed mass of 400 layers.

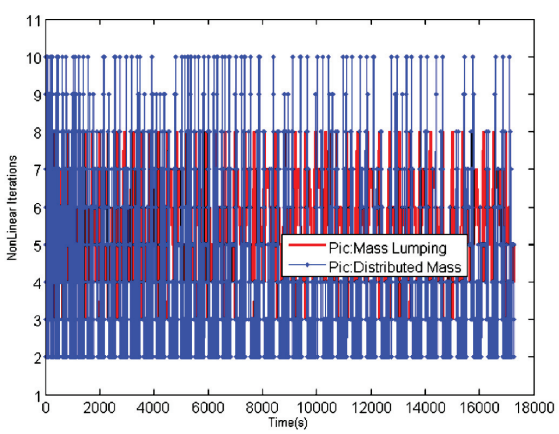

FIGURE 8. Convergence result for the Picard iterative scheme for the lumped and distributed mass of 400 layers.

FIGURE 7 for lumped and distributed mass of Picard scheme of 400 layers case. It is evident

that excellent mass balance errors are shown for mass lumping case and this implies the numerical results are strictly maintained accuracy. Same error profile is obtained by the Newton method. FIGURE 8 is the graphical representation of the convergence behavior of the lumped and distributed mass of Picard scheme in terms of the number of nonlinear iterations required at each time step. It is found most striking here the very different behavior between the lumped and distributed schemes during the simulation period. Where the consistent mass scheme is forced to take very small step sizes from the beginning to the end of the simulation, that is, constants convergence oscillations are shown in the simulation of distributed mass procedure. A comparison of computational statistics, such as the cumulative mass balance errors, the total number of time steps, and the nonlinear iterations per time steps for the various runs for the two time matrices approaches, are tabulated in TABLE 2.

TABLE 2. Computational statistics of the Test Problem 2.
Note: Dist=Distributed, ite=iteration, CMBE=Cumulative mass balance error.

No. of layers	Technique	Mass	No. of time steps	NL ite/ Time step	CMBE
200	Picard	Lump	2109	5.49	5.3998e-2
		Dist	4679	5.06	6.0757e-2
	Newton	Lump	1137	5.73	5.4068e-2
		Dist	764	5.49	6.1042e-2
400	Picard	Lump	4028	5.60	5.0243e-2
		Dist	9760	5.20	5.9236e-2
	Newton	Lump	2539	5.56	5.0267e-2
		Dist	2072	5.42	5.9502e-2

From these statistics in TABLE 2 it can be concluded that all runs have adequate and comparable accuracy for the case of mass lumping. Thus the flow equation is one that benefits from mass lumping in finite element approximation.

5.3. **Test Problem 3.** This test study consists of vertical infiltration with redistribution [30, 35]. Current problem considers a one-dimensional soil column of $5m$ deep discretized for two sets of vertical resolution $\Delta z = 0.0250m$ and $0.0125m$. Constant head boundary condition $\psi(0,t) = 0.0$ at the bottom of the domain and a time dependent boundary condition $\psi(10,t) = -10(1.0 - 1.01e^{-t})$ at the top of the domain with hydrostatic equilibrium initial conditions $\psi(z,0) = -z$ are applied. The time varying boundary condition yields a difficult two-front problem. This soil column is parameterized using the van Genuchten relationships with $\theta_s = 0.301$, $\theta_r = 0.093$, $\alpha = 5.47/m$, $n = 4.264.0$ and $K_s = 5.040m/days$.

FIGURE 9 and FIGURE 10 show the comparison of pressure head solution profiles for the cases of lumped and consistent mass of 200 and 400 layers respectively and the solution obtained with lumped mass is very similar to the previous studies [30, 36]. Picard and Newton iterative techniques with lumped and distributed mass schemes are achieved convergence with the minimum time step size $\Delta t_{min} = 10^{-18} days$. Run of Newton iteration of distributed

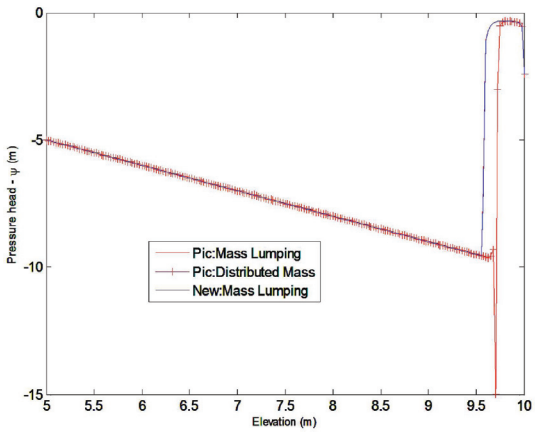

FIGURE 9. Pressure head profiles at $0.3days$ for the Picard and Newton iterative schemes for the lumped and distributed mass of 200 layers.

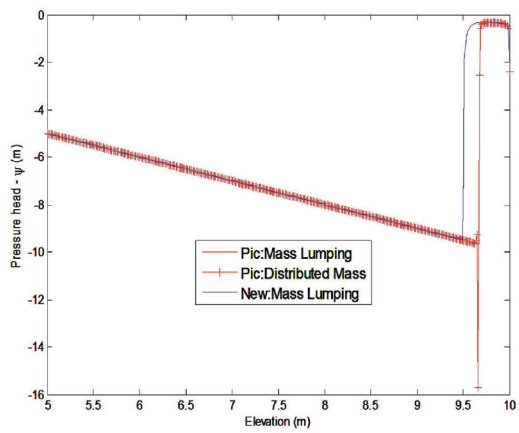

FIGURE 10. Pressure head profiles at $0.3days$ for the Picard and Newton iterative schemes for the lumped and distributed mass of 400 layers.

mass method had to forced stopped due to the time of simulation is significantly increased for both the grid spacing and it is true even for infinitesimal minimum time step $\Delta t_{min} = 10^{-20} days$. It is evident from these figures (FIGURE 9 and FIGURE 10) that there is a rapid infiltration of water from the surface, followed by a period of redistribution of the water due to the dynamic boundary condition at the top of the domain based on the mass lumping process. Mass-distributed scheme can still generate numerical oscillations due to the highly nonlinear properties of water flow in unsaturated soil. Similar numerical results hold for the larger grid spacing.

Figure 11 shows the cumulative mass balance errors profile of the Picard iterative scheme for the mass lumping and distributed mass of 400 layers. The magnitude of the errors is remarkably small for the lumped mass scheme throughout the entire domain of integration, attesting to the consistency of the mass lumping approximation. Similar results are obtained with respect to 201 nodes of spatial discretization, again confirming the robustness of the mass lumping mechanism. The number of iterations required for convergence per time step with the mass lumping scheme is compared to the distributed mass scheme in FIGURE 12. Lumped finite element solution converge rapidly, conversely the number of iterations for the distributed mass solution increases at the beginning of the simulation. The cumulative mass balance errors, number of time steps, nonlinear iterations per time step for different runs of the mass lumping method and their comparison with the mass distributed method are presented in TABLE 3 and it is clear that good performances are exhibited by the mass lumping algorithm. These results

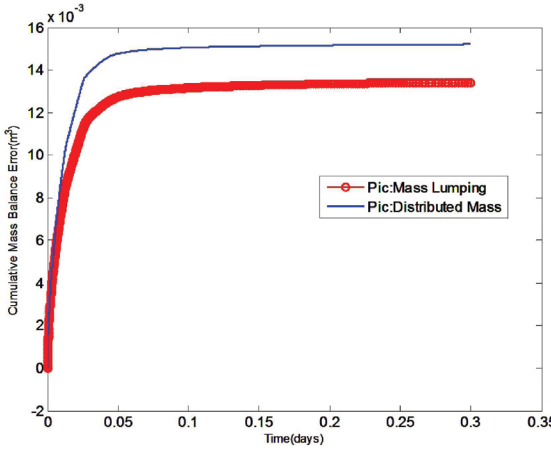

FIGURE 11. Mass balance result for the Picard iterative scheme for the lumped and distributed mass of 400 layers.

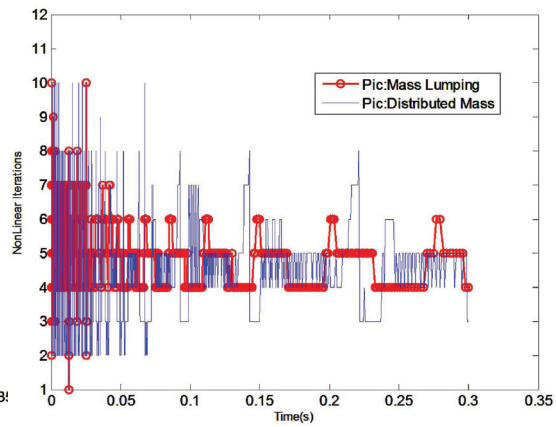

FIGURE 12. Convergence result for the Picard iterative scheme for the lumped and distributed mass of 400 layers

are illustrative of numerical comparisons and indicate clearly that diagonal time matrices are superior to the distributed matrices.

TABLE 3. Computational statistics of the Test Problem 3.
Note: Dist=Distributed, ite=iteration, CMBE=Cumulative mass balance error, NF=Simulation not finished.

No. of layers	Technique	Mass	No. of time steps	NL ite/ Time step	CMBE
200	Picard	Lump	2010	3.77	1.3358e-2
		Dist	3402	3.44	1.5059e-2
	Newton	Lump	1736	3.61	1.3354e-2
		Dist	NF	NF	NF
400	Picard	Lump	3579	4.37	1.3419e-2
		Dist	6896	3.49	1.5226e-2
	Newton	Lump	2865	4.13	1.3416e-2
		Dist	NF	NF	NF

6. CONCLUSION

A simple one-dimensional finite element model for flow equation into homogeneous soils was presented and the algorithm was implemented by applying the traditional iterative formulations, Picard and Newton based on lumped and distributed mass. Lumping scheme ensures

mass conservation and allows the handling of highly nonlinear phenomena. The similarity of number of time steps to complete the simulation and rate of convergence between lumped and distributed cases suggests that lumping formulation is generally superior to the distributing approximation. Presented results indicate that non oscillatory numerical solution becomes very critical in solving problems of infiltration into dry soils with distributing mass. By the choice of mass lumping numerical method performs very well for unsaturated flow problems involving vertical infiltration as well as redistribution. According to the experiences learned from this study, very small time step size or very dense spatial discretization does not eliminate numerical oscillations when solving the highly nonlinear Richards equation based on distributed mass. The numerical evidence is complemented and explained by the fact that, based on the pressure head form of Richards equation, coupled with a lumped matrix, yields consistently reliable and robust for unsaturated flow problems, even for very dry initial conditions. Numerical evaluations are presented by this study confirmed that the suitable methodology for unsaturated flow problems is one that is based on the pressure head form of Richards equation, and uses a lumped form of the time matrix and can easily be extended to multidimensional problem in saturated-unsaturated regions in heterogeneous and layered porous media.

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