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ON EXACT SOLUTIONS FOR IMPULSIVE DIFFERENTIAL EQUATIONS WITH NON-INTEGER ORDERS

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Abstract. This paper deals with linear impulsive differential equations with non-integer orders. We provide the explicit representation of solutions of linear impulsive fractional differential equations with constant coefficient by mean of the Mittag-Leffler functions.

1. Introduction

Fractional calculus is a theory of integrals and derivatives of any arbitrary real (or complex) order. The integrals and derivatives of non-integer order, and the fractional integro-differential equations have found many applications in recent studies in theoretical physics, mechanics and applied mathematics. Fractional-order models are found to be more adequate than integer-order models in some real world problems. The fractional order differential equations play a significant role in modeling the anomalous dynamics of various processes related to complex systems in most areas of science and engineering.

On the other hand, the mathematical investigations of the impulsive differential equations mark their beginning with the work of Mil'man and Myshkis in 1960(see [10]). They gave some general concepts about the systems with impulse effect and obtained the first results on stability of such systems solutions in [10]. Impulsive fractional differential equations are a natural generalization of impulsive ordinary differential equations and of fractional differential equations with fractional derivatives. At the

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present time the qualitative theory of such equations undergoes rapid developments.

Choi et al. [2, 3] studied impulsive integral inequalities with a non-separable kernel and stability of Caputo fractional differential equations. Denton and Vatsala [5] established the explicit representation of the solution of the linear fractional differential equation with variable coefficient and they developed the Gronwall integral inequality for the Riemann-Liouville fractional differential equation.

Fečkan et al. [6] studied a Cauchy problem for a fractional differential equation with linear impulsive conditions and gave a counterexample to illustrate the concepts of piecewise continuous solutions used in current papers are not appropriate. Choi et al. [1] provided an exact solution of linear fractional differential equations with impulse effect by the help of the Mittag-Leffler functions.

In this paper we provide the explicit representation of solutions of homogeneous linear impulsive fractional differential equations involving the Caputo derivatve with constant coefficient by mean of the Mittag-Leffler functions.

2. Preliminaries

The Gamma function and the β-function are the basic functions in fractional calculus. Gamma function Γ given by

$$\Gamma(z) = \int_0^\infty e^{-t} t^{z-1} dt, \ Re(z) > 0$$

satisfies $\Gamma(z+1) = z\Gamma(z)$ and $\Gamma(n+1) = n!$ for $n \in \mathbb{N}$. Also, the β-function is defined by the integral

$$\beta(z, w) = \int_0^1 t^{z-1}(1-t)^{w-1} dt, \ Re(z), Re(w) > 0.$$

The exponential function e^z plays a fundamental role in mathematics and it is really useful in theory of integer order differential equations. We can write it in a form of series:

$$e^z = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(k+1)}.$$

The Mittag-Leffler functions which is the generalizations of exponential function play an important role in the theory of fractional differential equations.

We recall the notion of Mittag-Leffler functions which was originally introduced by G. M. Mittag-Leffler in 1902(see [11]). That is, the Mittag-Leffler function is defined by

$$E_\alpha(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(k\alpha+1)},\ \alpha > 0, z \in \mathbb{C}. \tag{2.1}$$

Note that the particular Mittag-Leffler function e^{at} possesses the semigroup property (i.e., $e^{a(t+s)} = e^{at}e^{as}$ for all $t, s \geq 0$), but the Mittag-Leffler function $E_\alpha(at^\alpha)$ can not satisfy the semigroup property unless $\alpha = 1$ or $a = 0$ (see [12]).

We recall the definition of Caputo fractional derivative of a function $g : [t_0, \infty) \to \mathbb{R}$. For the fractional calculus and the theory of fractional differential equations, we refer the reader to [7, 9, 13].

Definition 2.1. [7] The *Caputo fractional derivative of non-integer order* q of a function g is defined by

$$ {}_{t_0}^{C}D_t^q g(t) = \frac{1}{\Gamma(1-q)} \int_{t_0}^{t} (t-s)^{-q} g'(s)ds,$$

where $g'(t) = \frac{dg(t)}{dt}$.

Let $t_0, T \in [0, \infty)$ and $J = [t_0, T]$. Let q be a positive real number such that $0 < q \leq 1$. We consider the following fractional Cauchy problems

$$\begin{cases} {}_{t_0}^{C}D_t^q u(t) = f(t, u(t)), t \neq t_k, t \in J, \\ u(t_k^+) = u(t_k^-) + I_k(u(t_k^-)),\ k = 1, 2, \cdots, m, \\ u(t_0) = u_0 \in \mathbb{R}, \end{cases} \tag{2.2}$$

where ${}_{t_0}^{C}D_t^q$ is the Caputo fractional derivative of order q with the lower limit t_0, $f : J \times \mathbb{R} \to \mathbb{R}$ is jointly continuous, $I_k : \mathbb{R} \to \mathbb{R}$ and t_k satisfy $0 \leq t_0 < t_1 \cdots < t_m < t_{m+1} = T$, $u(t_k^+) = \lim_{\varepsilon\to 0^+} u(t_k + \varepsilon)$ and $u(t_k^-) = \lim_{\varepsilon\to 0^-} u(t_k + \varepsilon)$ represent the right and left limits of $u(t)$ at $t = t_k$.

For the notion of solution and the existence of solutions for Equation (2.2), see [6, 14].

Denote by $C(J, \mathbb{R})$ the set of all continuous functions from J into $\mathbb{R}$. Also, let $PC(J, \mathbb{R})$ be the set of all functions from J into $\mathbb{R}$ as follows:

$$\begin{aligned} PC(J, \mathbb{R}) = \{&u : J \to \mathbb{R} | u \in C((t_k, t_{k+1}], \mathbb{R}), k = 0, 1, \cdots, m, \text{ and} \\ &\text{there exist } u(t_k^-) \text{ and } u(t_k^+), k = 1, \cdots, m, \text{ with } u(t_k^-) = u(t_k)\}. \end{aligned}$$

3. Main results

In this section we deal with linear impulsive Caputo fractional differential equations with constant coefficient. We establish a explicit formula of solutions of homogeneous linear impulsive fractional differential equations by the help of the Mittag-Leffler functions. For the general theory and applications of impulsive differential equations, we refer the reader to [8, 13].

To prove our main theorem, we need some lemmas for the fractional differential equations and impulsive fractional differential equations.

LEMMA 3.1. [14] *A function $u \in C(J, \mathbb{R})$ is a solution of the fractional integral equation*

$$u(t) = u_a - \frac{1}{\Gamma(q)} \int_{t_0}^{a} (a-s)^{q-1} f(s, u(s))ds + \frac{1}{\Gamma(q)} \int_{t_0}^{t} (t-s)^{q-1} f(s, u(s))ds,$$

if and only if u is a solution of the following fractional Cauchy problems

$$\begin{cases} {}_{t_0}^{C}D_t^q u(t) = f(t, u(t)), t \in J, \\ u(a) = u_a, \ a > t_0. \end{cases} \tag{3.1}$$

LEMMA 3.2. [9] *Let λ be a real constant. A function $u \in C(J, \mathbb{R})$ is a solution of the following linear fractional differential equation with initial condition*

$$\begin{cases} {}_{t_0}^{C}D_t^q u(t) = \lambda u(t), t \in J, \\ u(t_0) = u_{t_0}, \ t \geq t_0 \geq 0 \end{cases} \tag{3.2}$$

if and only if the solution u of Equation (3.2) is given by

$$u(t) \quad = \quad u(t_0) E_q(\lambda (t - t_0)^q), \ t \geq t_0 \geq 0.$$

LEMMA 3.3. [14] *A function $u \in PC(J, \mathbb{R})$ is a solution of the impulsive fractional integral equation*

$$u(t) = \begin{cases} u(t_0) + \frac{1}{\Gamma(q)} \int_{t_0}^{t} (t-s)^{q-1} f(s, u(s))ds, t \in [t_0, t_1], \\ \vdots \\ u(t_0) + \displaystyle\sum_{t_0 < t_k < t} I_k(u(t_k^-)) + \frac{1}{\Gamma(q)} \int_{t_0}^{t} (t-s)^{q-1} f(s, u(s))ds, \\ \hfill t \in (t_k, t_{k+1}], \end{cases}$$

for $k = 1, \cdots, m$ if and only if the function u is a solution of Equation (2.2).

The following our main result is an improvement of Theorem 2.4 in [4].

THEOREM 3.4. *Assume that $f(t,u) = \lambda u$ and $I_k(u(t_k^-)) = \beta_k u(t_k^-)$ with constants λ and β_k in Equation (2.2) for $k = 1, 2, \cdots, m$. Then the solution u of Equation (2.2) is given by*

$$u(t) = \begin{cases} u(t_0)E_q(\lambda(t-t_0)^q), \ t \in [t_0, t_1], \\ u(t_0)(1+\beta_1)E_q(\lambda(t_1-t_0)^q)E_q(\lambda(t-t_1)^q), t \in (t_1, t_2] \\ \vdots \\ u(t_0)\displaystyle\prod_{i=1}^{k}(1+\beta_i)E_q(\lambda(t_1-t_0)^q)\cdots E_q(\lambda(t_k-t_{k-1})^q)E_q(\lambda(t-t_k)^q), \\ \hfill t \in (t_k, t_{k+1}], \end{cases}$$

for $k = 2, \cdots, m$.

Proof. Let $t \in [t_0, t_1]$. Then it follows from Lemma 3.2 that

$$u(t) = u(t_0)E_q(\lambda(t-t_0)^q).$$

Let $t \in (t_1, t_2]$. Then it follows from Lemma 3.3 that the solution u of Equation (2.2) satisfies

$$u(t) = u(t_0) + \beta_1 u(t_1^-) + \frac{1}{\Gamma(q)}\int_{t_0}^{t}(t-s)^{q-1}\lambda u(s)ds.$$

Thus the solution u on $[t_0, t_2]$ of Equation (2.2) is given by

$$u(t) = \begin{cases} u(t_0)E_q(\lambda(t-t_0)^q), t \in [t_0, t_1], \\ u(t_0)(1+\beta_1)E_q(\lambda(t_1-t_0)^q)E_q(\lambda(t-t_1)^q), t \in (t_1, t_2], \end{cases}$$

since it satisfies the following initial condition

$$\begin{aligned} u(t_1^+) &= u(t_0) + \beta_1 u(t_1^-) + \frac{1}{\Gamma(q)}\int_{t_0}^{t_1}(t_1-s)^{q-1}\lambda u(t_0)E_q(\lambda(s-t_0)^q)ds \\ &= u(t_0) + \beta_1 u(t_1^-) + \frac{u(t_0)}{\Gamma(q)}\sum_{k=0}^{\infty}\frac{\lambda^{k+1}}{\Gamma(qk+1)}\int_{t_0}^{t_1}(t_1-s)^{q-1}(s-t_0)^{qk}ds \\ &= u(t_0) + \beta_1 u(t_1^-) + \frac{u(t_0)}{\Gamma(q)}\sum_{k=0}^{\infty}\frac{\lambda^{k+1}(t_1-t_0)^{qk+q}}{\Gamma(qk+1)}\int_0^1(1-\xi)^{q-1}\xi^{qk}d\xi \\ &= u(t_0) + \beta_1 u(t_1^-) + \frac{u(t_0)}{\Gamma(q)}\sum_{k=0}^{\infty}\frac{\lambda^{k+1}(t_1-t_0)^{q(k+1)}}{\Gamma(qk+1)}\frac{\Gamma(q)\Gamma(qk+1)}{\Gamma(q(k+1)+1)} \end{aligned}$$

$$
\begin{aligned}
&= u(t_0) + \beta_1 u(t_1^-) + u(t_0)\sum_{k=1}^{\infty}\frac{\lambda^k (t_1 - t_0)^{qk}}{\Gamma(qk+1)} \\
&= u(t_0) + \beta_1 u(t_1^-) + u(t_0)(E_q(\lambda(t_1 - t_0)^q) - 1) \\
&= (1+\beta_1)u(t_1^-)
\end{aligned}
$$

and one obtain

$$
\begin{aligned}
& u(t) \\
= \;& u(t_0) + \beta_1 u(t_1^-) + \frac{1}{\Gamma(q)}\int_{t_0}^{t}(t-s)^{q-1}\lambda u(s)ds \\
= \;& u(t_0) + \beta_1 u(t_1^-) + \frac{1}{\Gamma(q)}\int_{t_0}^{t_1}(t_1-s)^{q-1}\lambda u(t_0)E_q(\lambda(s-t_0)^q)ds \\
& + \frac{1}{\Gamma(q)}\int_{t_1}^{t}(t-s)^{q-1}\lambda u(t_0)(1+\beta_1)E_q(\lambda(t_1-t_0)^q)E_q(\lambda(s-t_1)^q)ds \\
= \;& (1+\beta_1)u(t_1^-) + \frac{(1+\beta_1)u(t_1^-)\sum_{k=0}^{\infty}\frac{\lambda^{k+1}}{\Gamma(qk+1)}\int_{t_1}^{t}(t-s)^{q-1}(s-t_1)^{qk}ds}{\Gamma(q)} \\
= \;& (1+\beta_1)u(t_1^-) + (1+\beta_1)u(t_1^-)\sum_{k=0}^{\infty}\frac{\lambda^{k+1}(t-t_1)^{qk+q}}{\Gamma(q(k+1)+1)} \\
= \;& (1+\beta_1)u(t_1^-) + (1+\beta_1)u(t_1^-)(E_q(\lambda(t-t_1)^q) - 1) \\
= \;& (1+\beta_1)u(t_1^-)E_q(\lambda(t-t_1)^q) \\
= \;& u(t_0)(1+\beta_1)E_q(\lambda(t_1-t_0)^q)E_q(\lambda(t-t_1)^q),\ t \in (t_1, t_2].
\end{aligned}
$$

Let $t \in (t_k, t_{k+1}]$. By above similar argument, then one obtain

$$
u(t) = u(t_0)\prod_{i=1}^{k}(1+\beta_i)E_q(\lambda(t_1-t_0)^q)\cdots E_q(\lambda(t_k-t_{k-1})^q)E_q(\lambda(t-t_k)^q),
$$

for $k = 2, 3, \cdots, m$. This completes the proof. □

Remark 3.5. Let $q = 1$ in Equation (3.2). Then the solution u of the impulsive differential equation (3.2) of integer order in Theorem 3.4 reduces to

$$
u(t) = u(t_0)\prod_{i=1}^{k}(1+\beta_i)e^{\lambda(t-t_0)}, t \in (t_k, t_{k+1}], \quad \text{for k} = 1, 2\cdots, \text{m}.
$$

Also, in case when $q = 1$ and $\beta_k = 0$ for $k = 1, \cdots, m$ in Equation (3.2), then the solution u of Equation (3.2) reduces to the exponential function

$$
u(t) \;=\; u(t_0)e^{\lambda(t-t_0)},\ t \in [t_0, T].
$$

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