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# STUDY ON THE ARITHMETIC OF MODULAR FORMS

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Abstract. By constructing a canonical basis for the space $M_k^\sharp(\Gamma_0(N))$ explicitly, we find a basis of the space of cusp forms for $\Gamma_0(N)$ consisting of Poincaré series.

## 1. Introduction and statement of results

We assume that $k$ is an even integer and $N > 1$ is a positive integer not a prime for which the genus of a Hecke group $\Gamma_0(N)$ is zero, that is,

$$N \in \{4, 6, 8, 9, 10, 12, 16, 18, 25\}.$$

Let $M_k(\Gamma_0(N))$ (resp. $S_k(\Gamma_0(N))$ be the vector space of holomorphic modular forms (resp. cusp forms) for $\Gamma_0(N)$ and $M_k^\sharp(\Gamma_0(N))$ be the space of weakly holomorphic modular forms of weight $k$ for $\Gamma_0(N)$ that are holomorphic away from the cusp at infinity.

The classical Poincaré series at $\infty$, $P(m, k, N; z)$ are defined by

$$P(m, k, N; z) := \sum_{\gamma \in \Gamma_\infty \backslash \Gamma_0(N)} \frac{e^{2\pi i m \gamma z}}{(cz+d)^k},$$

for $m \in \mathbb{N}$, $k \in \mathbb{Z}$ with $k > 2$. Here $\Gamma_\infty = \{\pm \left(\begin{smallmatrix} 1 & n \\ 0 & 1 \end{smallmatrix}\right) \mid n \in \mathbb{Z}\}$. For $m \geq 1$, we know that $P(m, k, N; z) \in S_k(\Gamma_0(N))$ (see [4]). Moreover it is well known in [4] that the set $\{P(m, k, N; z) \mid m \geq 1\}$ spans the space $S_k(\Gamma_0(N))$. Beyond this, little is known about such Poincaré series. For example, Iwaniec in [4] gave two open problems about Poincaré series such as: Since the space $S_k(\Gamma_0(N))$ is finite dimensional, there are many relations among the Poincaré series. Find all the linear relations among Poincaré series and find a basis of $S_k(\Gamma_0(N))$ consisting of the Poincaré series. Recently Rhoades [6] gave a partial answer to the first question.

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In this paper we give an answer to the second question as follows. Let the dimension of $S_k(\Gamma_0(N))$ be $t$.

THEOREM 1.1. *Let $k$ be an even integer with $k > 2$. We have that $\{P(m,k,N;z) \mid 1 \le m \le t\}$ is a basis for the space $S_k(\Gamma_0(N))$.*

REMARK 1.2. For a prime $N$ for which the genus of $\Gamma_0(N)$ is zero, Ahn and Choi in [1] showed that for $k \in \mathbb{Z}$ with $k > 2$, $\{P(m,k,N;z) \mid 1 \le m \le t\}$ is a basis for the space $S_k(\Gamma_0(N))$. Moreover, It is well known that $\{P(m,k,1;z) \mid 1 \le m \le t\}$ is a basis for the space $S_k(\Gamma_0(1))$.

## 2. A basis for the space $M_k^\sharp(\Gamma_0(N))$

Let $\Delta_{N,k}(z)$ be the unique normalized modular form of weight $k$ on $\Gamma_0(N)$ with zero of maximum order at $\infty$. We denote the order of the zero of $\Delta_{N,k}(z)$ at $\infty$ by $\xi_{N,k}$. Since the genus of $\Gamma_0(N)$ is zero, we have that $\dim M_k(\Gamma_0(N)) = \xi_{N,k} + 1$ if $k \ge 2$.

In particular, we need only the following $\Delta_{N,k}(z)$ (see[3]):

$$\begin{aligned}
\Delta_{4,2}(z) &= \frac{\eta^8(4z)}{\eta^4(2z)} = q + O(q^2),\\
\Delta_{6,2}(z) &= \frac{\eta^{12}(6z)\eta^2(z)}{\eta^4(2z)\eta^6(3z)} = q^2 + O(q^3),\\
\Delta_{8,2}(z) &= \frac{\eta^8(8z)}{\eta^4(4z)} = q^2 + O(q^3),\\
\Delta_{9,2}(z) &= \frac{\eta^6(9z)}{\eta^2(3z)} = q^2 + O(q^3),\\
\Delta_{12,2}(z) &= \frac{\eta^{12}(12z)\eta^2(2z)}{\eta^6(6z)\eta^4(4z)} = q^4 + O(q^5),\\
\Delta_{16,2}(z) &= \frac{\eta^8(16z)}{\eta^4(8z)} = q^4 + O(q^5),\\
\Delta_{18,2}(z) &= \frac{\eta^{12}(18z)\eta^2(3z)}{\eta^6(9z)\eta^4(6z)} = q^6 + O(q^7),\\
\Delta_{10,4}(z) &= \frac{\eta^2(z)\eta^{20}(10z)}{\eta^4(2z)\eta^{10}(5z)} = q^6 + O(q^7),\\
\Delta_{25,4}(z) &= \frac{\eta^{10}(25z)}{\eta^2(5z)} = q^{10} + O(q^{11}).
\end{aligned}$$

Here $\eta(z) = q^{\frac{1}{24}} \prod_{n=1}^{\infty}(1-q^n)$. The above modular forms $\Delta_{N,k}(z)$ have no zeros on $\mathbb{H}$ and at all cusps except for $\infty$. Indeed by easy calculation we obtain

$$\frac{k[\Gamma_0(1):\Gamma_0(N)]}{12} = \xi_{N,k}$$

and hence from the valence formula we have that $\Delta_{N,k}(z)$ has no zero on $\mathbb{H}$ and at all cusps away form the cusp at infinity.

We now find an upper bound of $\mathrm{ord}_\infty f$ for nonzero $f \in M_k^\sharp(\Gamma_0(N))$.

Case $I$. $N = 4, 6, 8, 12, 16$ and $18$. In this case we let $k = 2l_k$. Then we have an isomorphism from $M_k^\sharp(\Gamma_0(N))$ onto $M_0^\sharp(\Gamma_0(N))$ by $f \mapsto f/\Delta_{N,2}^{l_k}$. This implies that for any nonzero $f \in M_k^\sharp(\Gamma_0(N))$ we have $\mathrm{ord}_\infty f \le \xi_{N,2} l_k$. We denote $\xi_{N,2} l_k$ by $m_{N,k}$.

Case $II$. $N = 10$ and $25$. In this case we let $k = 4l_k + r_k$ with $r_k \in \{0, 2\}$. Then we have an isomorphism from $M_k^\sharp(\Gamma_0(N))$ onto $M_{r_k}^\sharp(\Gamma_0(N))$ by $f \mapsto f/\Delta_{N,4}^{l_k}$. This implies that for any nonzero $f \in M_k^\sharp(\Gamma_0(N))$ we obtain $\mathrm{ord}_\infty f \le \xi_{N,4} l_k + \xi_{N,r_k}$. We denote $\xi_{N,4} l_k + \xi_{N,r_k}$ by $m_{N,k}$.

Under these notations we have the following theorem.

THEOREM 2.1. *For each integer $m$ such that $-m \le m_{N,k}$, there exists a unique weakly holomorphic modular form $f_{k,m} \in M_k^\sharp(\Gamma_0(N))$ with a $q$-expansion of the form*

$$f_{k,m} = q^{-m} + O(q^{m_{N,k}+1}).$$

*Explicitly,*

$$f_{k,m} = (\Delta_N)^{l_k} \Delta_{N,r_k} F_{k,m+m_{N,k}}(j_N),$$

*where $F_{k,D}(x)$ is a monic polynomial in $x$ of degree $D$ and $j_N(z)$ is the Hauptmodul for $\Gamma_0(N)$. Here in the case $N = 4, 6, 8, 12, 16$ and $18$, we define $\Delta_{N,r_k} = 1$.*

*Proof.* For convenience let

$$\Delta_N := \begin{cases} \Delta_{N,2}, & \text{if } N = 4, 6, 8, 9, 12, 16, 18 \\ \Delta_{N,4} & \text{if } N = 10, 25. \end{cases}$$

We observe that

$$\begin{aligned} (\Delta_N)^{l_k} \Delta_{N,r_k} (j_N)^{m+m_{N,k}} &= q^{-m} + \cdots \\ (\Delta_N)^{l_k} \Delta_{N,r_k} (j_N)^{m+m_{N,k}-1} &= q^{-m+1} + \cdots \\ &\vdots \\ (\Delta_N)^{l_k} \Delta_{N,r_k} &= q^{m_{N,k}} + \cdots \ . \end{aligned}$$

Now $f_{k,m}$ is constructed by taking a suitable linear combination of the above forms. Moreover, since $\mathrm{ord}_\infty f_{k,m} \le m_{N,k}$, a weakly holomorphic modular form $f_{k,m}$ is unique. □

REMARK 2.2. Since the Hauptmoduln $j_N$ has a integral Fourier coefficients at $\infty$ (see [3]) and we have

$$\begin{aligned}
\Delta_{10,2}(z) &= \frac{1}{24}((5E_2(10z) - E_2(2z)) - 4(2E_2(10z) - E_2(5z))),\\
\Delta_{25,2}(z) &= \frac{1}{50}(5E_2(5z) - E_2(z))(\frac{1}{2} + \frac{3}{10}\frac{\eta^2(z)}{\eta^2(25z)} + \frac{1}{5}\frac{\eta^2(z)}{\eta^2(25z)} + \frac{1}{25}\frac{\eta^3(z)}{\eta^3(25z)})\\
&\quad -\frac{1}{50}(25E_2(25z) - E_2(z))(\frac{1}{3} + \frac{1}{4}\frac{\eta(z)}{\eta(25z)} + \frac{1}{15}\frac{\eta^2(z)}{\eta^2(25z)} + \frac{1}{150}\frac{\eta^3(z)}{\eta^3(25z)}),
\end{aligned}$$

we come up with that $f_{k,m}$ have integral Fourier coefficients at $\infty$ except for the case $N = 25$. On the other hand, $f_{k,m}$ have a rational Fourier coefficients at $\infty$ in the case $N = 25$. Here $E_2(z) = 1 - 24\sum_{n=1}^{\infty}\sigma(n)q^n$ and $\sigma(n) = \sum_{0<d|n} d$.

## 3. Proof of Theorem 1.1

*Proof of Theorem 1.1* We now show that the set $\{P(m, k, N; z) \mid 1 \le m \le t\}$ is a basis for the space $S_k(\Gamma_0(N))$. To do it we need the following property.

PROPOSITION 3.1. *Let $k \in \mathbb{Z}$ with $k \ge 2$ and $I$ be a finite set of positive integers. Then*

$$\sum_{m\in I}\alpha_m P(m, k, N; z) \equiv 0$$

*if and only if there exists a weakly holomorphic modular form $f \in M^\sharp_{2-k}(\Gamma_0(N))$ with principal part at $\infty$ equal to*

$$\sum_{m\in I}\frac{\alpha_m}{m^{k-1}}q^{-m}.$$

*Proof.* See [6, Theroem 1.1.] □

Let $v_\infty(N)$ be the number of $\Gamma_0(N)$-inequivalent cusp. Then we have (see [5, Theorem 4.2.7 and Theorem 2.5.2 ])

$$v_\infty(N) = \sum_{0<d|N}\phi((d, N/d)) \tag{3.1}$$

and

$$\text{(3.2)} \qquad \dim M_k(\Gamma_0(N)) = \dim S_k(\Gamma_0(N)) + v_\infty(N) \qquad \text{if } k > 2.$$

Here $\phi$ is the Euler function.

LEMMA 3.2. $m_{N,2-k} = -t-1$.

*Proof.* Case $I$. $N = 4, 6, 8, 9, 12, 16$ and $18$. From (3.1) we see that $v_\infty(N) - 2 = \xi_{N,2}$. We note that

$$\dim M_k(\Gamma_0(N)) = 1 + \frac{k}{2}(v_\infty(N) - 2).$$

This and (3.2) mean that $t := \dim S_k(\Gamma_0(N)) = (1-l_k)(2-v_\infty(N))-1 = -\xi_{N,2}(1-l_k)-1$. On the other hand, we see that $2-k = 2-2l_k = 2(1-l_k)$ which means that $l_{2-k} = 1-l_k$ and hence $m_{N,2-k} = \xi_{N,2}(1-l_k) = -t-1$.

Case $II$. $N = 10$ and $25$. We note that

$$\dim M_k(\Gamma_0(N)) = -(k-1) + \frac{k}{2}v_\infty(N) + 2[\frac{k}{4}].$$

This and (3.2) mean that

$$t := \dim S_k(\Gamma_0(N)) = \begin{cases} 6l_k + r_k - 3, & N = 10 \\ 10l_k + 2r_k - 5, & N = 25. \end{cases}$$

Because

$$v_\infty(N) = \begin{cases} 4, & N = 10 \\ 6, & N = 25 \end{cases}$$

On the other hand, we see that $2-k = -4l_k + 2 - r_k$ which means that $l_{2-k} = -l_k$ and $r_{2-k} = 2 - r_k$. Hence we obtain that

$$m_{N,2-k} = \xi_{N,4}(-l_k) + \xi_{N,2-r_k} = \begin{cases} -6l_k + 2, & N = 10, r_k = 0 \\ -6l_k, & N = 10, r_k = 2 \\ -10l_k + 4, & N = 25, r_k = 0 \\ -10l_k, & N = 25, r_k = 2, \end{cases}$$

which implies that $m_{N,2-k} = -t-1$. □

We are ready to prove Theorem 1.1. We assume $\alpha_1 P(1,k,N;z) + \alpha_2 P(2,k,N;z) + \cdots + \alpha_t P(t,k,N;z) \equiv 0$. Then by Proposition 3.1 there exists a weakly holomorphic modular form $f \in M_k^\sharp(\Gamma_0(N))$ with principal part at $\infty$ equal to

$$\sum_{1 \le m \le t} \frac{\alpha_m}{m^{k-1}} q^{-m}.$$

This is a contradiction to the fact that $\mathrm{ord}_\infty f \le m_{N,2-k}$ if $f$ is not zero. Thus $\alpha_1 = \alpha_2 = \cdots = \alpha_t = 0$ which implies Theorem 1.1.

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