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SYMMETRIC BI-(f, g)-DERIVATIONS IN LATTICES

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Abstract. In this paper, as a generalization of symmetric bi-derivations and symmetric bi-f-derivations of a lattice, we introduce the notion of symmetric bi-(f, g)-derivations of a lattice. Also, we define the isotone symmetric bi-(f, g)-derivation and obtain some interesting results about isotone. Using the notion of $Fix_a(L)$ and $KerD$, we give some characterization of symmetric bi-(f, g)-derivations in a lattice.

1. Introduction

Lattices play an important role in many fields such as information theory, information retrieval, information access controls and cryptanalysis ([2], [6], [20]). Recently the properties of lattices were widely researched ([1], [2], [5], [10], [12], [20], [22]). In the theory of rings and near rings, the properties of derivations are an important topic to study ([3], [4], [19]). In [21], G. Szász introduced the notion of derivation on a lattice and discussed some related properties.Y. B. Jun and X. L. Xin [13] applied the notion of derivation in ring, near ring and lattice theory to BCI-algebras. In [24], J. Zhan and Y. L. Liu introduced the notion of left-right (or right-left) f-derivation of a BCI algebra and investigated some properties.

Recently, the notion of f-derivation, symmetric bi-derivations and permuting tri-derivations in lattices are introduced and proved some results([8], [9] and [18]). In this paper, as a generalization of symmetric bi-derivations and symmetric bi-f-derivations of a lattice, we introduce the notion of symmetric bi-(f, g)-derivations of a lattice. Also, we define the isotone symmetric bi-(f, g)-derivation and obtain some interesting

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results about isotone. Using the notion of $Fix_a(L)$ and $KerD$, we give some characterization of symmetric bi-(f, g)-derivations in a lattice.

2. Preliminaries

Definition 2.1. Let L be a nonempty set endowed with operations $\wedge$ and $\vee$. By a *lattice* $(L, \wedge, \vee)$, we mean a set L satisfying the following conditions:

(1) $x \wedge x = x$, $x \vee x = x$,
(2) $x \wedge y = y \wedge x$, $x \vee y = y \vee x$,
(3) $(x \wedge y) \wedge z = x \wedge (y \wedge z)$, $(x \vee y) \vee z = x \vee (y \vee z)$,
(4) $(x \wedge y) \vee x = x$, $(x \vee y) \wedge x = x$, for all $x, y, z \in L$.

Definition 2.2. Let $(L, \wedge, \vee)$ be a lattice. A binary relation $\leq$ is defined by $x \leq y$ if and only if $x \wedge y = x$ and $x \vee y = y$.

Lemma 2.1. *Let $(L, \wedge, \vee)$ be a lattice. Define the binary relation $\leq$ as the Definition 2.2. Then $(L, \leq)$ is a poset and for any $x, y \in L$, $x \wedge y$ is the greatest lower bound of $\{x, y\}$ and $x \vee y$ is the least upper bound of $\{x, y\}$.*

Definition 2.3. A lattice L is *distributive* if the identity (1) or (2) holds:

(1) $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$,
(2) $x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$.

In any lattice, the conditions (1) and (2) are equivalent.

Definition 2.4. A lattice L is *modular* if the following identity holds: If $x \leq z$, then $x \vee (y \wedge z) = (x \vee y) \wedge z$.

Definition 2.5. A non-empty subset I of L is called an *ideal* if the following conditions hold:

(1) If $x \leq y$ and $y \in I$, then $x \in I$ for all $x, y \in L$.
(2) If $x, y \in I$ then $x \vee y \in I$.

Definition 2.6. Let $(L, \wedge, \vee)$ be a lattice. Let $f : L \to M$ be a function from a lattice L to a lattice M.

(1) f is called a *meet-homomorphism* if $f(x \wedge y) = f(x) \wedge f(y)$ for all $x, y \in L$.
(2) f is called a *join-homomorphism* if $f(x \vee y) = f(x) \vee f(y)$ for all $x, y \in L$.

(3) f is called a *lattice-homomorphism* if f is both a join-homomorphism and a meet-homomorphism.

DEFINITION 2.7. Let L be a lattice. A mapping $D(.,.) : L \times L \to L$ is said to be *symmetric* if $D(x, y) = D(y, x)$ holds for all $x, y \in L$.

DEFINITION 2.8. Let L be a lattice. A mapping $d(x) = D(x, x)$ is called a *trace* of $D(.,.)$, where $D(.,.) : L \times L \to L$ is a symmetric mapping.

DEFINITION 2.9. Let L be a lattice and let $D(.,.) : L \times L \to L$ be a symmetric mapping. We call D a *symmetric bi-derivation* on L if it satisfies the following condition

$$D(x \wedge y, z) = (D(x, z) \wedge y) \vee (x \wedge D(y, z))$$

for all $x, y, z \in L$.

Obviously, a symmetric bi-derivation D on L satisfies the relation

$$D(x, y \wedge z) = (D(x, y) \wedge z) \vee (y \wedge D(x, z))$$

for all $x, y, z \in L$.

DEFINITION 2.10. Let L be a lattice and let $D(.,.) : L \times L \to L$ be a symmetric mapping. D is called a *symmetric bi-f-derivation* on L if there exists a function $f : L \to L$ such that

$$D(x \wedge y, z) = (D(x, z) \wedge f(y)) \vee (f(x) \wedge D(y, z))$$

for all $x, y, z \in L$.

3. Symmetric bi-(f, g)-derivations

DEFINITION 3.1. Let L be a lattice and let $D(.,.) : L \times L \to L$ be a symmetric mapping. D is called a *symmetric bi-(f, g)-derivation* on L if there exist two functions $f, g : L \to L$ such that

$$D(x \wedge y, z) = (D(x, z) \wedge f(y)) \vee (g(x) \wedge D(y, z))$$

for all $x, y, z \in L$.

Obviously, a symmetric bi-(f, g)-derivation D on L satisfies the relation

$$D(x, y \wedge z) = (D(x, y) \wedge f(z)) \vee (g(y) \wedge D(x, z))$$

for all $x, y, z \in L$.

Example 3.1. Let $L = \{0, 1, 2\}$ be a lattice of following Figure 1 and define mappings D and f, g on L by

$$D(x,y) = \begin{cases} 1 & \text{if } (x,y) = (0,0) \\ 1 & \text{if } (x,y) = (0,1) \\ 1 & \text{if } (x,y) = (1,0) \\ 0 & \text{if } (x,y) = (0,2) \\ 0 & \text{if } (x,y) = (2,0) \\ 0 & \text{if } (x,y) = (1,1) \\ 0 & \text{if } (x,y) = (2,2) \\ 0 & \text{if } (x,y) = (1,2) \\ 0 & \text{if } (x,y) = (2,1) \end{cases}$$

and

$$f(x) = \begin{cases} 1 & \text{if } x = 0 \\ 2 & \text{if } x = 1 \\ 2 & \text{if } x = 2, \end{cases} \qquad g(x) = \begin{cases} 0 & \text{if } x = 0 \\ 1 & \text{if } x = 1 \\ 1 & \text{if } x = 2 \end{cases}$$

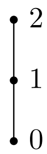

Figure 1

Then it is easily checked that D is a symmetric bi-(f, g)-derivation of a lattice L. But D is not a symmetric bi-derivation since

$$1 = D(0 \wedge 0, 0) \neq (D(0,0) \wedge 0) \vee (0 \wedge D(0,0)) = (1 \wedge 0) \vee (0 \wedge 1) = 0.$$

Proposition 3.1. *Let L be a lattice and d a trace of a symmetric bi-(f, g)-derivation D. Then*

$$d(x) \leq f(x) \vee g(x)$$

for all $x \in L$.

Proof. Since $x \wedge x = x$ for all $x \in L$, we have

$$d(x) = D(x,x) = D(x \wedge x, x) = (D(x,x) \wedge f(x)) \vee (g(x) \wedge D(x,x)).$$

Since $D(x,x) \wedge f(x) \leq f(x)$ and $D(x,x) \wedge g(x) \leq g(x)$, we get $d(x) \leq f(x) \vee g(x)$. □

Proposition 3.2. *Let L be a lattice and let D be a symmetric bi-(f, g)-derivation on L. Then $D(x, y) \leq f(x) \vee g(x)$ and $D(x, y) \leq f(y) \vee g(y)$ for all $x, y \in L$.*

Proof. Since $x \wedge x = x$ for all $x \in L$, we have for all $y \in L$,

$$D(x, y) = D(x \wedge x, y) = (D(x, y) \wedge f(x)) \vee (g(x) \wedge D(x, y).$$

Since $D(x, y) \wedge f(x) \leq f(x)$ and $D(x, y) \wedge g(x) \leq g(x)$, we have $D(x, y) \leq f(x) \vee g(x)$. Similarly, $D(x, y) \leq f(y) \vee g(y)$ for all $x, y \in L$. □

Corollary 3.1. *Let L be a lattice and let D be a symmetric bi-(f, g)-derivation on L. If $g(x) \leq f(x)$ for all $x \in L$, then $D(x, y) \leq f(x)$ and $D(x, y) \leq f(y)$ for all $x, y \in L$.*

Proposition 3.3. *Let L be a lattice and let D be a symmetric bi-(f, g)-derivation on L. If L has a least element 0 such that $f(0) = 0$ and $g(0) = 0$, we have $D(0, y) = 0$.*

Proof. For all $x, y \in L$, we have $D(x, y) \leq f(x) \vee g(x)$ from Proposition 3.4 Since 0 is the least element of a lattice L, we get

$$0 \leq D(0, y) \leq f(0) \vee g(0) = 0,$$

which implies $D(0, y) = 0$. □

Proposition 3.4. *Let L be a lattice and let D be a symmetric bi-(f, g)-derivation on L where $g(x) \leq f(x)$ for all $x \in L$. Then the following identities hold for all $x, y, w \in L$:*

(1) $D(x, y) \wedge D(w, y) \leq D(x \wedge w, y) \leq D(x, y) \vee D(w, y)$.
(2) $D(x \wedge w, y) \leq f(x) \vee f(w)$.

Proof. (1) For all $x, y, w \in L$, we have

$$D(x \wedge w, y) = (D(x, y) \wedge f(w)) \vee (g(x) \wedge D(w, y)),$$

which implies $D(x, y) \wedge f(w) \leq D(x \wedge w, y)$. Since $D(w, y) \leq f(w)$ for all $y \in L$, we have $D(x, y) \wedge D(w, y) \leq D(x, y) \wedge f(w)$. Hence we get $D(x, y) \wedge D(w, y) \leq D(x \wedge w, y)$. Since $D(x, y) \wedge f(w) \leq D(x, y)$ and $g(x) \wedge D(w, y) \leq D(w, y)$, we have $D(x \wedge w, y) \leq D(x, y) \vee D(w, y)$, which implies $D(x, y) \wedge D(w, y) \leq D(x \wedge w, y) \leq D(x, y) \vee D(w, y)$.

(2) Since $D(x, y) \wedge f(w) \leq f(w)$ and $g(x) \wedge D(w, y) \leq f(x) \wedge D(w, y) \leq f(x)$, we get

$$(D(x, y) \wedge f(w)) \vee (g(x) \wedge D(y, w)) \leq f(x) \vee f(w).$$

□

Proposition 3.5. *Let L be a lattice with a greatest element 1 and let D be a symmetric bi-(f,g)-derivation on L such that $f(1) = g(1) = 1$. Then the following properties hold for all $x, y \in L$:*

(1) *If $f(x) \leq D(1,y)$ and $g(x) \leq D(1,y)$, then $D(x,y) = f(x) \vee g(x)$.*
(2) *If $g(x) \geq D(1,y)$, then $D(x,y) \geq D(1,y)$.*

Proof. (1) For all $x, y \in L$, we have

$$\begin{aligned} D(x,y) &= D(x \wedge 1, y) \\ &= (D(x,y) \wedge f(1)) \vee (g(x) \wedge D(1,y)) \\ &= D(x,y) \vee g(x). \end{aligned}$$

Hence we have $g(x) \leq D(x,y)$.

Similarly, since $x \wedge 1 = x$, we obtain

$$\begin{aligned} D(x,y) &= D(1 \wedge x, y) \\ &= (D(1,y) \wedge f(x)) \vee (g(1) \wedge D(x,y)) \\ &= D(x,y) \vee f(x). \end{aligned}$$

Thus we get $f(x) \leq D(x,y)$.

From (1) and (2), we have

$$f(x) \vee g(x) \leq D(x,y).$$

From Proposition 3.4, we have $D(x,y) \leq f(x) \vee g(x)$. Finally, we have

$$f(x) \vee g(x) \leq D(x,y) \leq f(x) \vee g(x),$$

which implies $D(x,y) = f(x) \vee g(x)$.

(2) For all $x, y \in L$,

$$\begin{aligned} D(x,y) &= D(x \wedge 1, y) \\ &= (D(x,y) \wedge f(1)) \vee (g(x) \wedge D(1,y)) \\ &= D(x,y) \vee D(1,y). \end{aligned}$$

Hence we have $D(x,y) \geq D(1,y)$. □

Theorem 3.1. *Let L be a distribute lattice and let D be a symmetric bi-(f,g)-derivation on L with the trace d. Then*

$$d(x \wedge y) = (d(x) \wedge (f(y)) \vee (g(x) \wedge d(y)) \vee ((g(x) \wedge f(y)) \wedge D(x,y))$$

for all $x, y \in L$.

Proof. For all $x, y \in L$, we have

$$\begin{aligned}
d(x \wedge y) &= D(x \wedge y, x \wedge y) \\
&= (D(x, x \wedge y) \wedge f(y)) \vee (g(x) \wedge D(y, x \wedge y)) \\
&= (D(x \wedge y, x) \wedge f(y)) \vee (g(x) \wedge (D(x \wedge y, y))) \\
&= \{[(D(x, x) \wedge f(y)) \vee (g(x) \wedge D(x, y))] \wedge f(y)\} \\
&\vee \{g(x) \wedge [(D(x, y) \wedge f(y)) \vee (g(x) \wedge D(y, y))]\} \\
&= \{((d(x) \wedge f(y)) \wedge f(y)) \vee ((g(x) \wedge f(y)) \wedge D(x, y))\} \\
&\vee \{((g(x) \wedge f(y)) \wedge D(x, y)) \vee ((g(x) \wedge (g(x) \wedge d(y))))\} \\
&= (d(x) \wedge f(y)) \vee (g(x) \wedge d(y)) \vee ((f(y) \wedge g(x)) \wedge D(x, y)).
\end{aligned}$$

□

COROLLARY 3.2. *Let L be a distribute lattice and let D be a symmetric bi-(f, g)-derivation with the trace d. Then for all $x, y \in L$,*

(1) $(g(x) \wedge f(y)) \wedge D(x, y) \leq d(x \wedge y)$.
(2) $g(x) \wedge d(y) \leq d(x \wedge y)$.
(3) $d(x) \wedge f(y) \leq d(x \wedge y)$.

Proof. (1), (2) and (3) are easily seen from the above theorem respectively. □

COROLLARY 3.3. *Let L be a distribute lattice and let D be a symmetric bi-(f, g)-derivation with the trace d. If 1 is the greatest element of L, we have $(g(x) \wedge f(1)) \wedge D(x, 1) \leq d(x \wedge 1) = d(x)$ for all $x \in L$ and $g(x) \wedge d(1) \leq d(x \wedge 1) = d(x)$ for all $x \in L$.*

DEFINITION 3.2. Let L be a lattice and let D be a symmetric bi-(f, g)-derivation on L.

(1) If $x \leq w$ implies $D(x, y) \leq D(w, y)$, then D is called an *isotone symmetric bi-(f, g)-derivation.*
(2) If D is one-to-one, then D is called a *monomorfic symmetric bi-(f, g)-derivation.*
(3) If D is onto, then D is called an *epic symmetric bi-(f, g)-derivation.*

THEOREM 3.2. *Let L be a lattice and let D be a symmetric bi-(f, g)-derivation on L. The following conditions are equivalent.*

(1) *D is an isotone symmetric bi-(f, g)-derivation.*
(2) *$D(x, y) \vee D(w, y) \leq D(x \vee w, y)$ for all $x, y, w \in L$.*

Proof. (1) ⇒ (2). Suppose that D is an isotone symmetric bi-(f, g)-derivation on L. Since $x \leq x \vee w$ and $w \leq x \vee w$, we obtain $D(x, y) \leq$

$D(x \vee w, y)$ and $D(w, y) \leq D(x \vee w, y)$. Therefore, $D(x, y) \vee D(w, y) \leq D(x \vee w, y)$.

(2) $\Rightarrow$ (1). Suppose that $D(x, y) \vee D(w, y) \leq D(x \vee w, y)$ and $x \leq w$. Then we have

$$\begin{aligned} D(x, y) &\leq D(x, y) \vee D(w, y) \leq D(x \vee w, y) \\ &= D(w, y). \end{aligned}$$

Hence D is an isotone symmetric bi-(f, g)-derivation on L. □

Let L be a lattice and let D be a symmetric bi-(f, g)-derivation on L. For each $a \in L$ and define a set $Fix_a(L)$ by

$$Fix_a(L) = \{x \in L \mid D(x, a) = f(x)\}.$$

Proposition 3.6. *Let L be a lattice and let D an isotone symmetric bi-(f, g)-derivation on L. If $f : L \to L$ is a lattice homomorphism and $g(x) \leq f(x)$ for all $x \in L$, then $Fix_a(L)$ is a sublattice of L.*

Proof. Let $x, y \in Fix_a(L)$. Then $D(x, a) = f(x)$ and $D(y, a) = f(y)$. Then $f(x \wedge y) = f(x) \wedge f(y) = D(x, a) \wedge D(y, a) \leq D(x \wedge y, a)$. Hence $D(x \wedge y, a) = f(x \wedge y)$, that is, $x \wedge y \in Fix_a(L)$. Moreover, we have $f(x \vee y) = f(x) \vee f(y) = D(x, a) \vee D(y, a) \leq D(x \vee y, a)$ by Theorem 3.2. Thus $D(x \vee y, a) = f(x \vee y)$, which implies $x \vee y \in Fix_a(L)$. □

Proposition 3.7. *Let L be a lattice and let D be a symmetric bi-(f, g)-derivation on L where $g(x) \leq f(x)$ for all $x \in L$. If f is an increasing function, $x \leq y$ and $y \in Fix_a(L)$ imply $D(x, a) = D(x, a) \vee g(x)$.*

Proof. Let $x \leq y$ and $y \in Fix_a(L)$. Then we have $D(x, a) \leq f(x) \leq f(y)$ and $g(x) \leq f(x) \leq f(y)$. Hence we obtain

$$\begin{aligned} D(x, a) &= D(x \wedge y, a) \\ &= (D(x, a) \wedge f(y)) \vee (g(x) \wedge D(y, a)) \\ &= (D(x, a) \wedge f(y)) \vee (g(x) \wedge f(y)) \\ &= D(x, a) \vee g(x). \end{aligned}$$

This completes the proof. □

Proposition 3.8. *Let L be a distributive lattice and let D be a symmetric bi-(f, g)-derivation of L where $g(x) \leq f(x)$ for all $x, y \in L$. If f is a meet-homomorphism and $x, y \in Fix_a(L)$, we have $x \wedge y \in Fix_a(L)$ for all $x, y \in L$.*

Proof. Let $x, y \in Fix_a(L)$. Then $f(x) = D(x, a)$ and $f(y) = D(y, a)$. Hence we have

$$\begin{aligned} D(x \wedge y, a) &= (D(x, a) \wedge f(y)) \vee (g(x) \wedge D(y, a)) \\ &= (f(x) \wedge f(y)) \vee (g(x) \wedge f(y)) \\ &= (f(x) \vee g(x)) \wedge f(y) \\ &= f(x) \wedge f(y) \\ &= f(x \wedge y), \end{aligned}$$

which implies $x \wedge y \in Fix_a(L)$. □

PROPOSITION 3.9. *Let L be a lattice and let D be an isotone symmetric bi-(f, g)-derivation on L where $g(x) \leq f(x)$ for all $x \in L$. If $x, y \in Fix_a(L)$ and f is a increasing function, then $x \vee y \in Fix_a(L)$.*

Proof. Since $x \leq x \vee y$ and $y \leq x \vee y$, we have $f(x \vee y) \leq f(x)$ and $f(x \vee y) \leq f(y)$ respectively. Hence we obtain $f(x \vee y) \leq f(x) \vee f(y) = D(x, a) \vee D(y, a) \leq D(x \vee y, a)$ since D is an isotone symmetric bi-(f, g)-derivation. From Proposition 3.4 (2), we have $D(x \vee y, a) \leq f(x \vee y)$, which implies $D(x \vee y, a) = f(x \vee y)$. Hence $x \vee y \in Fix_a(L)$. □

PROPOSITION 3.10. *Let L be a lattice, D a symmetric bi-(f, g)-derivation on L where $f(x) \leq g(x)$ and 1 the greatest element of L. Then the following identities hold.*

(1) *If $g(x) \leq D(1, y)$ and $f(1) = 1$, then $D(x, y) = g(x)$.*
(2) *If $g(x) \geq D(1, y)$ and $f(1) = 1$, then $D(x, y) \geq D(1, y)$.*

Proof. (1) Let $g(x) \leq D(1, y)$. Then we have $D(x, y) \leq f(x) \vee g(x) = g(x)$, and so

$$\begin{aligned} D(x, y) &= D(x \wedge 1, y) \\ &= (D(x, y) \wedge f(1)) \vee (g(x) \wedge D(1, y) \\ &= D(x, y) \vee g(x) \\ &= g(x). \end{aligned}$$

(2) Let $g(x) \geq D(1, y)$. Then we have

$$\begin{aligned} D(x, y) &= D(x \wedge 1, y) \\ &= (D(x, y) \wedge f(1)) \vee (g(x) \wedge D(1, y) \\ &= D(x, y) \vee D(1, y). \end{aligned}$$

Hence we obtain $D(1, y) \leq D(x, y)$ for all $x, y \in L$. □

THEOREM 3.3. *Let L be a lattice with the greatest element 1 and let D be an isotone symmetric bi-(f, g)-derivation on L. Let $f(1) = g(1) = 1$ and either $f(x) \geq g(x)$ or $f(x) \leq g(x)$ for all $x \in L$. Then*

$$D(x, y) = (f(x) \vee g(x)) \wedge D(1, y)$$

for all $x, y, z \in L$.

Proof. Suppose that D is an isotone symmetric bi-(f, g)-derivation on L. Then $D(x, y) \leq D(1, y)$ for all $x, y \in L$. Now let $g(x) \leq f(x)$ for $x \in L$. Then we have $D(x, y) \leq g(x) \vee f(x) = f(x)$. From this, we get $D(x, y) \leq f(x) \wedge D(1, y)$. Also, we obtain

$$\begin{aligned} D(x, y) &= D((x \vee 1) \wedge x, y) \\ &= [(D(x \vee 1), y) \wedge f(x)] \vee [g(x \vee 1) \wedge D(x, y)] \\ &= [D(1, y) \wedge f(x)] \vee [g(1) \wedge D(x, y)] \\ &= [D(D(1, y) \wedge f(x)] \vee [1 \wedge D(x, y)] \\ &= [D(1, y) \wedge f(x)] \vee D(x, y) \\ &= D(1, y) \wedge f(x). \end{aligned}$$

Since $f(x) \vee g(x) = f(x)$, we have

$$D(x, y) = (f(x) \vee g(x)) \wedge D(1, y).$$

Now suppose that $f(x) \leq g(x)$ for $x \in L$. Similarly, we have $D(x, y) \leq f(x) \vee g(x) = g(x)$. From this, we have $D(x, y) \leq g(x) \wedge D(1, y)$. Also, we obtain

$$\begin{aligned} D(x, y) &= D(x \wedge (x \vee 1), y) \\ &= [(D(x, y) \wedge f(x \vee 1)] \vee [g(x) \wedge D((x \vee 1), y)] \\ &= [D(x, y) \wedge f(1)] \vee [g(x) \wedge D(1, y)] \\ &= [D(D(x, y) \wedge 1)] \vee [g(x) \wedge D(1, y)] \\ &= D(x, y) \vee [g(x) \wedge D(1, y)] \\ &= g(x) \wedge D(1, y). \end{aligned}$$

Since $f(x) \vee g(x) = g(x)$, we have

$$D(x, y) = (f(x) \vee g(x)) \wedge D(1, y).$$

This completes the proof. □

Let D be a symmetric bi-(f, g)-derivation of L and let 0 be a least element of L. Define a set $KerD$ by

$$KerD = \{x \in L \mid D(x, 0) = 0\}.$$

Proposition 3.11. *Let L be a lattice with a least element 0 and let D be a symmetric bi-(f, g)-derivation on L. If $x, y \in KerD$, then $x \wedge y \in KerD$.*

Proof. Let $x, y \in KerD$. Then $D(x, 0) = D(y, 0) = 0$. Hence we have

$$\begin{aligned} D(x \wedge y, 0) &= (D(x, 0) \wedge f(y)) \vee (g(x) \wedge D(y, 0)) \\ &= (0 \wedge f(x)) \vee (g(x) \wedge 0) \\ &= 0 \vee 0 = 0, \end{aligned}$$

which implies $x \wedge y \in KerD$. □

Proposition 3.12. *Let L be a lattice with a least element 0 and let D be an isotone symmetric bi-(f, g)-derivation on L. If $x \leq y$ and $y \in KerD$, then $x \in KerD$.*

Proof. Let $y \in KerD$. Then $D(y, 0) = 0$ and $D(x, 0) \leq D(y, 0) = 0$ since D is isotone. Hence we have $D(x, 0) = 0$, and so

$$\begin{aligned} D(x, 0) = D(x \wedge y, 0) &= (D(x, 0) \wedge f(y)) \vee (g(x) \wedge D(y, 0)) \\ &= (0 \wedge f(x)) \vee (g(x) \wedge 0)) \\ &= 0 \vee 0 = 0, \end{aligned}$$

which implies $x \in KerD$. □

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