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THE COMPLETE MOMENT CONVERGENCE FOR WEIGHTED SUMS OF STOCHASTICALLY DOMINATED PNQD RANDOM VARIABLES

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Abstract. The purpose of this paper is to establish the complete moment convergence and the integrability of supremum for weighted sums of pairwise negatively quadrant dependent random variables satisfying stochastically dominating condition.

1. Introduction

The concept of complete convergence was introduced by Hsu and Robbins(1947) as follows. A sequence $\{X_n, n \geq 1\}$ of random variables is said to converge completely to a constant c if

$$\sum_{n=1}^{\infty} P(|X_n - c| > \epsilon) < \infty \text{ for all } \epsilon > 0. \tag{1.1}$$

In view of the Borel-Cantelli lemma, this implies that $X_n \to c$ almost surely. Therefore the complete convergence is a very important tool in establishing almost sure convergence of summation of random variables. Hsu and Robbins(1947) proved that the sequence of arithmetic means of independent and identically distributed random variables converges completely to the expected value if the variance of the summands is finite. Erdös(1949) proved the converse. The results of Hsu-Robbin-Erdös have been generalized and extended in several directions. Baum and Katz(1965) proved that if $\{X_n, n \geq 1\}$ is a sequence of i.i.d. random variables with $E|X_1| < \infty$. $E|X_1|^{pr} < \infty (1 \leq p < 2, r \geq 1)$ is equivalent to

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$$(1.2) \qquad \sum_{n=1}^{\infty} n^{r-2} P(|\sum_{i=1}^{n}(X_i - EX_i)| > \epsilon n^{\frac{1}{p}}) < \infty$$

for all $\epsilon > 0$. Chow(1988) generalized the result of Baum and Katz(1965) by showing the following complete moment convergence. If $\{X_n, n \geq 1\}$ is a sequence of i.i.d. random variables with $E|X_1|^{pr} < \infty$ for some $1 \leq p < 2$ and $r \geq 1$, then

$$(1.3) \qquad \sum_{n=1}^{\infty} n^{r-2-\frac{1}{p}} E(|\sum_{i=1}^{n}(X_i - EX_i)| - \epsilon n^{\frac{1}{p}})^{+} < \infty$$

for all $\epsilon > 0$, where $a^{+} = \max(a, 0)$. Note that (1.3) implies (1.2). Since the concept of complete moment convergence was introduced by Chow(1988), many investigations have been found. See for example Sung(2009) obtained a moment inequality of the maximum partial sum of random variables and the complete moment convergence of i.i.d. random variables. Liang(2010) provided necessary and sufficient moment conditions for complete moment convergence of negatively associated random variables. Wu et al.(2012) studied the complete moment convergence of ρ^{*}-mixing random variables. Wang and Hu(2012) established the equivalence of the complete convergence and complete moment convergence for a class of dependent random variables.

A sequence $\{X_n, n \geq 1\}$ of random variables is said to be pairwise negatively quadrant dependent(PNQD) if for every real numbers x_i, x_j and $i \neq j$ $P(X_i \leq x_i, X_j \leq x_j) \leq P(X_i \leq x_i)P(X_j \leq x_j)$ holds.

In this paper, we study the complete moment convergence and the integrability of supremum for weighted sums of pairwise negatively quadrant dependent(PNQD) random variables which are stochastically dominated by a random variable.

2. Some lemmas

The following lemmas will be useful to prove the main results.

Lemma 2.1. (Lehmann, 1966) *Let $\{X_n, n \geq 1\}$ be a sequence of PNQD random variables, and let $\{f_n, n \geq 1\}$ be a sequence of nondecreasing function. Then $\{f_n(X_n), n \geq 1\}$ is still a sequence of PNQD random variables.*

LEMMA 2.2. (Wu, 2006) *Let $\{X_n, n \geq 1\}$ be a sequence of PNQD random variables with mean zero and finite second moment. Then, for all $n \geq 1$*

$$E(\max_{1\leq k\leq n} |\sum_{i=1}^{k} X_i|^2) \leq 4(\log_2 n)^2 \sum_{i=1}^{n} EX_i^2. \tag{2.1}$$

The following lemma is a basic property for stochastic domination.

LEMMA 2.3. (Wu, 2006) *Let $\{X_n, n \geq 1\}$ be a sequence of random variables which is stochastically dominated by a random variable X. For any $\alpha > 0$ and $b > 0$, the following two statements hold.*

(i) $E|X_n|^\alpha I(|X_n| \leq b) \leq C_1\{E|X|^\alpha I(|X| \leq b) + b^\alpha P(|X| > b)\}$,

(ii) $E|X_n|^\alpha I(|X_n| > b) \leq C_2 E|X|^\alpha I(|X| > b)$.

Proof. For the proof one can refer to Wu(2006), Shen(2013) or Shen and Wu(2013), where C_1 and C_2 are some positive constants. □

Sung(2009) provided a moment inequality for the maximal partial sum of random variables as follows.

LEMMA 2.4. (Sung, 2009) *Let $\{Y_n, n \geq 1\}$ and $\{Z_n, n \geq 1\}$ be sequences of random variables. Then, for any $q > 1, \epsilon > 0$ and $a > 0$,*

$$E(\max_{1\leq k\leq n} |\sum_{i=1}^{k}(Y_i + Z_i)| - \epsilon a)^+$$
$$\leq (\frac{1}{\epsilon^q} + \frac{1}{q-1})\frac{1}{a^{q-1}} E(\max_{1\leq k\leq n} |\sum_{i=1}^{k} Y_i|^q) + E(\max_{1\leq k\leq n} |\sum_{i=1}^{k} Z_i|).$$

3. Main results

THEOREM 3.1. *Let $r > 1$, $0 < p < 2$ and $1 < pr < 2$. Let $\{X_n, n \geq 1\}$ be a mean zero sequence of PNQD random variables which is stochastically dominated by a random variable X such that $EX^2 < \infty$ and $\{a_{ni}, 1 \leq i \leq n, n \geq 1\}$ be an array of positive real numbers. Assume that*

$$E|X|^{pr}(\log_2(1 + |X|))^2 < \infty \tag{3.1}$$

and

$$\sum_{i=1}^{n} a_{ni}^2 = O(n) \tag{3.2}$$

hold. Then, for every $\epsilon > 0$, $1 < pr < 2$,

$$\sum_{n=1}^{\infty} n^{r-2-\frac{1}{p}} E(\max_{1\le k\le n} |\sum_{i=1}^{k} a_{ni}X_i| - \epsilon n^{\frac{1}{p}})^{+} < \infty. \tag{3.3}$$

Proof. For fixed $n \ge 1$ and $1 \le i \le n$, denote

$$X_{ni} = -n^{\frac{1}{p}} I(X_i < -n^{\frac{1}{p}}) + X_i I(|X_i| \le n^{\frac{1}{p}}) + n^{\frac{1}{p}} I(X_i > n^{\frac{1}{p}}),$$
$$X_{ni}^{'} = X_i - X_{ni} \text{ and } \tilde{X_{ni}} = X_{ni} - EX_{ni}.$$

Note that

$$X_{ni}^{'} = n^{\frac{1}{p}} I(X_i < -n^{\frac{1}{p}}) - n^{\frac{1}{p}} I(X_i > n^{\frac{1}{p}}) + X_i I(|X_i| > n^{\frac{1}{p}})$$

and that $X_i = X_{ni}^{'} + EX_{ni} + \tilde{X_{ni}}$.

Letting $a = n^{\frac{1}{p}}$ and $q = 2$ in Lemma 2.4 we obtain

(3.4)

$$\begin{aligned}
&\sum_{n=1}^{\infty} n^{r-2-\frac{1}{p}} E(\max_{1\le k\le n} |\sum_{i=1}^{k} a_{ni}X_i| - \epsilon n^{\frac{1}{p}})^{+} \\
&\le \sum_{n=1}^{\infty} n^{r-2-\frac{1}{p}} E(\max_{1\le k\le n} |\sum_{i=1}^{k} a_{ni}X_{ni}^{'}|) + \sum_{n=1}^{\infty} n^{r-2-\frac{1}{p}} (\max_{1\le k\le n} |\sum_{i=1}^{k} a_{ni}EX_{ni}|) \\
&\quad + (\frac{1}{\epsilon^2} + 1) \sum_{n=1}^{\infty} n^{r-2-\frac{2}{p}} E(\max_{1\le k\le n} |\sum_{i=1}^{k} a_{ni}\tilde{X_{ni}}|^2) \\
&=: I_1 + I_2 + I_3.
\end{aligned}$$

It follows from (3.2) and Hölder's inequality that, for $1 \le k \le 2$

$$\sum_{i=1}^{n} |a_{ni}|^k \le (\sum_{i=1}^{n} |a_{ni}|^2)^{\frac{k}{2}} (\sum_{i=1}^{n} 1)^{1-\frac{k}{2}} \le Cn. \tag{3.5}$$

By (3.1), (3.5), Lemma 2.3 and the fact that $|X_{ni}^{'}| \le |X_i| I(|X_i| > n^{\frac{1}{p}})$ we have

$$\begin{aligned}
I_1 &\le \sum_{n=1}^{\infty} n^{r-2-\frac{1}{p}} \sum_{i=1}^{n} |a_{ni}| E(|X_i| I(|X_i| > n^{\frac{1}{p}})) \\
&\le C \sum_{n=1}^{\infty} n^{r-2-\frac{1}{p}} \sum_{i=1}^{n} |a_{ni}| E|X| I(|X| > n^{\frac{1}{p}}) \qquad (3.6) \\
&\le C \sum_{n=1}^{\infty} n^{r-1-\frac{1}{p}} \sum_{m=n}^{\infty} E(|X| I(m < |X|^p \le m+1))
\end{aligned}$$

$$= C\sum_{m=1}^{\infty} E(|X|I(m < |X|^p \le m+1))\sum_{n=1}^{m} n^{r-1-\frac{1}{p}}$$
$$\le C\sum_{m=1}^{\infty} m^{r-\frac{1}{p}} E|X|I(m < |X|^p \le m+1)$$
$$\le CE|X|^{rp} < \infty.$$

From the fact $EX_n = 0, n \ge 1$ we have

$$\begin{aligned} &EX_{ni} \\ (3.7)\quad &= -n^{\frac{1}{p}}E(I(X_i < -n^{\frac{1}{p}})) + EX_iI(|X_i| \le n^{\frac{1}{p}}) + n^{\frac{1}{p}}EI(X_i > n^{\frac{1}{p}}) \\ &= -n^{\frac{1}{p}}EI(X_i < -n^{\frac{1}{p}}) - EX_iI(|X_i| > n^{\frac{1}{p}}) + n^{\frac{1}{p}}EI(X_i > n^{\frac{1}{p}}). \end{aligned}$$

Hence, by (3.6) and (3.7) we estimate that

$$(3.8)\quad I_2 \le 3\sum_{n=1}^{\infty} n^{r-2-\frac{1}{p}}\sum_{i=1}^{n}|a_{ni}|E(|X_i|I(|X_i| > n^{\frac{1}{p}})) \le CE|X|^{rp} < \infty.$$

Since $\{a_{ni}\tilde{X_{ni}}, 1 \le i \le n, n \ge 1\}$ is also PNQD by Lemma 2.2 we have

(3.9)

$$\begin{aligned} I_3 &= (\frac{1}{\epsilon^2}+1)\sum_{n=1}^{\infty} n^{r-2-\frac{2}{p}}E(\max_{1\le k\le n}|\sum_{i=1}^{k} a_{ni}\tilde{X_{ni}}|^2) \\ &\le 4C\sum_{n=1}^{\infty} n^{r-2-\frac{2}{p}}(\log_2 n)^2\sum_{i=1}^{n}(|a_{ni}|^2E\tilde{X_{ni}}^2) \text{ by (2.1)} \\ &\le 4C\sum_{n=1}^{\infty} n^{r-2-\frac{2}{p}}(\log_2 n)^2\sum_{i=1}^{n}(|a_{ni}|^2EX_{ni}^2) \text{ by } \tilde{X_{ni}} = X_{ni} - EX_{ni} \\ &\le 4C\sum_{n=1}^{\infty} n^{r-2-\frac{2}{p}}(\log_2 n)^2\sum_{i=1}^{n}(|a_{ni}|^2EX^2) \text{ since } EX^2 < \infty \\ &\le 4C\sum_{n=1}^{\infty} n^{r-1-\frac{2}{p}}(\log_2 n)^2 < \infty \text{ (3.2) and } r-2-\frac{2}{p} < 0. \end{aligned}$$

Therefore, (3.3) follows from (3.4)-(3.9). □

Corollary 3.2. *Let $r > 1$, $0 < p < 2$ and $1 < pr < 2$. Let $\{X_n, n \ge 1\}$ be a mean zero sequence of PNQD random variables which is stochastically dominated by a random variable X such that $EX^2 < \infty$ and $\{a_n, n \ge 1\}$ be a sequence of positive numbers. Assume that (3.1)* *and*

$$\sum_{i=1}^{n} |a_i|^2 = O(n) \tag{3.10}$$

hold. Then, for every $\epsilon > 0$,

$$\sum_{n=1}^{\infty} n^{r-2-\frac{1}{p}} E(\max_{1\le k\le n} |\sum_{i=1}^{k} a_i X_i| - \epsilon n^{\frac{1}{p}})^+ < \infty, \tag{3.11}$$

and for $1 < r < 2$,

$$\sum_{n=1}^{\infty} n^{r-2} E(\sup_{k\ge n} |\frac{a_k X_k}{k^{\frac{1}{p}}}| - \epsilon)^+ < \infty. \tag{3.12}$$

Proof. We can prove (3.11) by the similar method in the proof of Theorem 3.1. To prove (3.12) we consider that

(3.13)

$$\begin{aligned}
&\sum_{n=1}^{\infty} n^{r-2} E(\sup_{k\ge n} |\frac{\sum_{i=1}^{k} a_i X_i}{k^{\frac{1}{p}}}| - \epsilon)^+ \\
&= \sum_{n=1}^{\infty} n^{r-2} E(\sup_{k\ge n} |\frac{\sum_{i=1}^{k} a_i X_i}{k^{\frac{1}{p}}}| - \epsilon_1 2^{\frac{2}{p}})^+ \text{ letting } \epsilon = \epsilon_1 2^{\frac{2}{p}} \\
&= \sum_{n=1}^{\infty} n^{r-2} \int_0^{\infty} P(\sup_{k\ge n} |\frac{\sum_{i=1}^{k} a_i X_i}{k^{\frac{1}{p}}}| - \epsilon_1 2^{\frac{2}{p}} > t)dt \\
&= \sum_{m=1}^{\infty} \sum_{n=2^{m-1}}^{2^m-1} n^{r-2} \int_0^{\infty} P(\sup_{k\ge n} |\frac{\sum_{i=1}^{k} a_i X_i}{k^{\frac{1}{p}}}| > \epsilon_1 2^{\frac{2}{p}} + t)dt \\
&\le 2^{2-r} \sum_{m=1}^{\infty} \int_0^{\infty} P(\sup_{k\ge 2^{m-1}} |\frac{\sum_{i=1}^{k} a_i X_i}{k^{\frac{1}{p}}}| > \epsilon_1 2^{\frac{2}{p}} + t)dt \sum_{n=2^{m-1}}^{2^m-1} 2^{m(r-2)} \\
&\le 2^{2-r} \sum_{m=1}^{\infty} 2^{m(r-1)} \int_0^{\infty} P(\sup_{k\ge 2^{m-1}} |\frac{\sum_{i=1}^{k} a_i X_i}{k^{\frac{1}{p}}}| > \epsilon_1 2^{\frac{2}{p}} + t)dt \\
&= 2^{2-r} \sum_{m=1}^{\infty} 2^{m(r-1)} \int_0^{\infty} P(\sup_{l\ge m} \max_{2^{l-1}\le k\le 2^l} |\frac{\sum_{i=1}^{k} a_i X_i}{k^{\frac{1}{p}}}| > \epsilon_1 2^{\frac{2}{p}} + t)dt \\
&\le 2^{2-r} \sum_{m=1}^{\infty} 2^{m(r-1)} \sum_{l=m}^{\infty} \int_0^{\infty} P(\max_{1\le k\le 2^l} |\sum_{i=1}^{k} a_i X_i| > (\epsilon_1 2^{\frac{2}{p}} + t) 2^{(l-1)/p})dt
\end{aligned}$$

$$= 2^{2-r}\sum_{l=1}^{\infty}\int_0^\infty P(\max_{1\le k\le 2^l}|\sum_{i=1}^k a_iX_i| > (\epsilon_1 2^{\frac{2}{p}}+t)2^{(l-1)/p})dt\sum_{m=1}^{l}2^{m(r-1)}$$

$$\le 2^{2-r}\sum_{l=1}^{\infty}2^{l(r-1)}\int_0^\infty P(\max_{1\le k\le 2^l}|\sum_{i=1}^k a_iX_i| > (\epsilon_1 2^{\frac{2}{p}}+t)2^{(l-1)/p})dt$$

(let $t_1 = 2^{(l-1)/p}t$)

$$= 2^{2-r+\frac{1}{p}}\sum_{l=1}^{\infty}2^{l(r-1-\frac{1}{p})}\int_0^\infty P(\max_{1\le k\le 2^l}|\sum_{i=1}^k a_iX_i| > \epsilon_1 2^{\frac{l+1}{p}}+t_1)dt_1$$

$$\le C\sum_{l=1}^{\infty}\sum_{n=2^l}^{2^{(l+1)-1}}2^{(l+1)(r-2-\frac{1}{p})}\int_0^\infty P(\max_{1\le k\le n}|\sum_{i=1}^k a_iX_i| > \epsilon_1 2^{\frac{l+1}{p}}+t_1)dt_1$$

since $r < 2$

$$\le C\sum_{l=1}^{\infty}\sum_{n=2^l}^{2^{(l+1)-1}}n^{r-2-\frac{1}{p}}\int_0^\infty P(\max_{1\le k\le n}|\sum_{i=1}^k a_iX_i| > \epsilon_1 n^{\frac{1}{p}}+t_1)dt_1$$

$$\le C\sum_{n=1}^{\infty}n^{r-2-\frac{1}{p}}E(\max_{1\le k\le n}|\sum_{i=1}^k a_iX_i| - \epsilon_1 n^{\frac{1}{p}})^+ < \infty \text{ by (3.11).}$$

Hence, the proof of (3.12) is complete by (3.13). □

THEOREM 3.3. *Let $0 < p < 1$ and $pr = 1$. Let $\{X_n, n \ge 1\}$ be a mean zero sequence of PNQD random variables which is stochastically dominated by a random variable X such that $E|X|(\log_2(1+|X|))^2 < \infty$ and $\{a_{ni}, 1 \le i \le n, n \ge 1\}$ be an array of positive numbers satisfying (3.2). Then, for every $\epsilon > 0$,*

$$\sum_{n=1}^{\infty}n^{-2}E(\max_{1\le k\le n}|\sum_{i=1}^k a_{ni}X_i| - \epsilon n^{\frac{1}{p}})^+ < \infty. \tag{3.14}$$

Proof. First, as in the proof of Theorem 3.1 we define X_{ni}, X'_{ni} and $\tilde{X_{ni}}$, respectively. By letting $a = n^{\frac{1}{p}}$ and $q = 2$ in Lemma 2.4 we obtain that

(3.15)

$$\sum_{n=1}^{\infty}n^{-2}E(\max_{1\le k\le n}|\sum_{i=1}^k a_{ni}X_i| - \epsilon n^{\frac{1}{p}})^+$$

$$\le \sum_{n=1}^{\infty}n^{-2}E(\max_{1\le k\le n}|\sum_{i=1}^k a_{ni}X'_{ni}|) + \sum_{n=1}^{\infty}n^{-2}(\max_{1\le k\le n}|\sum_{i=1}^k a_{ni}EX_{ni}|)$$

$$+ (\frac{1}{\epsilon^2} + 1) \sum_{n=1}^{\infty} n^{-2-\frac{1}{p}} E(\max_{1 \le k \le n} |\sum_{i=1}^{k} a_{ni} \tilde{X_{ni}}|^2)$$
$$=: II_1 + II_2 + II_3.$$

By the assumption $E|X|(\log_2(1+|X|))^2 < \infty$ and the fact that

$$|X_{ni}^{'}| \le |X_i| I(|X_i| > n^{\frac{1}{p}})$$

we obtain that

$$\begin{aligned} II_1 &\le C \sum_{n=1}^{\infty} n^{-1} E(|X| I(|X| > n^{\frac{1}{p}})) \\ &= C \sum_{n=1}^{\infty} n^{-1} \sum_{m=n}^{\infty} E(|X| I(m < |X|^p \le m+1)) \\ &= C \sum_{m=1}^{\infty} E(|X| I(m < |X|^p \le m+1)) \sum_{n=1}^{m} n^{-1} \\ &\le C \sum_{m=1}^{\infty} \log(1+m) E(|X| I(m < |X|^p \le m+1)) \\ &\le C E|X| \log(1+|X|) < \infty. \end{aligned} \tag{3.16}$$

For II_2, it is easy to see that

$$\begin{aligned} II_2 &\le C \sum_{n=1}^{\infty} n^{-2} \sum_{i=1}^{n} |a_{ni}| E(|X_i| I(|X_i| > n^{\frac{1}{p}})) \\ &\le C \sum_{n=1}^{\infty} n^{-1} E(|X| I(|X| > n^{\frac{1}{p}})) \\ &\le C E(|X| \log(1+|X|)) < \infty \text{ by (3.16).} \end{aligned} \tag{3.17}$$

For II_3, we have

$$\begin{aligned} II_3 &\le 4(\frac{1}{\epsilon^2} + 1) \sum_{n=1}^{\infty} n^{-2-\frac{1}{p}} (\log_2 n)^2 \sum_{i=1}^{n} |a_{ni}|^2 E|\tilde{X_{ni}}|^2 \\ &\le C \sum_{n=1}^{\infty} n^{-2-\frac{1}{p}} (\log_2 n)^2 \sum_{i=1}^{n} |a_{ni}|^2 E|X_{ni}|^2 \end{aligned} \tag{3.18}$$

$$\leq C\sum_{n=1}^{\infty} n^{-2-\frac{1}{p}}(\log_2 n)^2 \sum_{i=1}^{n} |a_{ni}|^2\{E|X_i|^2 I(|X_i| \leq n^{\frac{1}{p}}) + n^{\frac{2}{p}} EI(|X| > n^{\frac{1}{p}})\}$$

$$\leq C\sum_{n=1}^{\infty} n^{-2-\frac{1}{p}}(\log_2 n)^2 \sum_{i=1}^{n} |a_{ni}|^2 E(|X_i|^2 I(|X_i| \leq n^{\frac{1}{p}}))$$

$$+ C\sum_{n=1}^{\infty} n^{-1+\frac{1}{p}}(\log_2 n)^2 P(|X| > n^{\frac{1}{p}})$$

$$=: II_{31} + II_{32}.$$

For II_{31}, by $E|X|(\log_2(1+|X|))^2 < \infty$ we have

(3.19)

$$\begin{aligned} II_{31} &= C\sum_{n=1}^{\infty} n^{-1-\frac{1}{p}}(\log_2 n)^2 \sum_{m=1}^{n}(E|X|^2 I((m-1)^{\frac{1}{p}} < |X| \leq m^{\frac{1}{p}})) \\ &= C\sum_{m=1}^{\infty} E(|X|^2 I((m-1)^{\frac{1}{p}} < |X| \leq m^{\frac{1}{p}})) \sum_{n=m}^{\infty} n^{-1-\frac{1}{p}}(\log_2 n)^2 \\ &\leq C\sum_{m=1}^{\infty} m^{-\frac{1}{p}}(\log_2(1+m))^2 E(|X|^2 I((m-1)^{\frac{1}{p}} < |X| \leq m^{\frac{1}{p}})) \\ &\leq CE(|X|(\log_2((1+|X|)))^2) < \infty. \end{aligned}$$

For II_{32} we have

$$\begin{aligned} II_{32} &= C\sum_{n=1}^{\infty} n^{-1}(\log_2 n)^2 E(|X| I(|X| > n^{\frac{1}{p}})) \\ &= C\sum_{n=1}^{\infty} n^{-1}(\log_2 n)^2 \sum_{m=n}^{\infty} E(|X| I(m^{\frac{1}{p}} < |X| < (m+1)^{\frac{1}{p}})) \\ &= C\sum_{m=1}^{\infty} E(|X| I(m^{\frac{1}{p}} < |X| < (m+1)^{\frac{1}{p}})) \sum_{n=1}^{m} n^{-1}(\log_2 n)^2 \quad (3.20) \\ &= C\sum_{m=1}^{\infty} (\log_2(1+m))^2 E(|X| I(m < |X|^p < (m+1))) \\ &\leq CE(|X|(\log_2(1+|X|))^2) < \infty. \end{aligned}$$

□

Corollary 3.4. *Let $0 < p < 1$ and let $\{X_n, n \geq 1\}$ be a mean zero sequence of PNQD random variables which is stochastically dominated by a random variable X such that $E|X|(\log(1+|X|))^2 < \infty$. Let $\{a_n, n \geq$*

1} be a sequence of positive real numbers satisfying (3.10). Then for every $\epsilon > 0$ *and* $0 < p < 1$,

(i) $\sum_{n=1}^{\infty} n^{-2} E(\max_{1\le k\le n} |\sum_{i=1}^{k} a_i X_i| - \epsilon n^{\frac{1}{p}})^{+} < \infty$

(ii) *for every* $\epsilon > 0$ *and* $\frac{1}{2} < p < 1$,

$$\sum_{n=1}^{\infty} n^{\frac{1}{p}-2} E(\sup_{k\ge n} |\frac{a_k X_k}{k^{\frac{1}{p}}}| - \epsilon)^{+} < \infty.$$

(iii) *In particular, for any* $\epsilon > 0$,

$$\sum_{n=1}^{\infty} n^{\frac{1}{p}-2} P(\max_{1\le k\le n} |\sum_{i=1}^{k} a_i X_i| > \epsilon n^{\frac{1}{p}}) < \infty.$$

Remark 3.5. Under the conditions of Corollary 2.2 it is true that for every $\epsilon > 0$

$$\begin{aligned} &\sum_{n=1}^{\infty} n^{r-2-\frac{1}{p}} E(\max_{1\le k\le n} |\sum_{i=1}^{k} a_i X_i| - \epsilon n^{\frac{1}{p}})^{+} \\ \ge\ & \epsilon \sum_{n=1}^{\infty} n^{r-2} P(\max_{1\le k\le n} |\sum_{i=1}^{k} a_i X_i| > 2\epsilon n^{\frac{1}{p}}). \end{aligned}$$

Corollary 3.6. *Assume that* $\{X_n, n \ge 1\}$ *is a mean zero sequence of PNQD random variables which is stochastically dominated by a random variable* X *such that* $EX^2 < \infty$. *If the conditions of Corollary 3.2 hold then for every* $\epsilon > 0$,

$$\sum_{n=1}^{\infty} n^{r-2} P(\max_{1\le k\le n} |\sum_{i=1}^{k} a_i X_i| > \epsilon n^{\frac{1}{p}}) < \infty \tag{3.21}$$

and for $1 < r < 2$,

$$\sum_{n=1}^{\infty} n^{r-2} P(\sup_{k\ge n} |\frac{\sum_{i=1}^{k} a_i X_i}{k^{\frac{1}{p}}}| > \epsilon) < \infty. \tag{3.22}$$

Proof. Obviously, (3.11) implies (3.21)(see above Remark). Let $\epsilon = 2^{\frac{2}{p}}\epsilon'$. Inspired by the proof of 12.1 of Gut(2005), we obtain that

$$\sum_{n=1}^{\infty} n^{r-2} P(\sup_{k\ge n} |\frac{\sum_{i=1}^{k} a_i X_i}{k^{\frac{1}{p}}}| > \epsilon) \tag{3.23}$$

$$
\begin{aligned}
&= \sum_{n=1}^{\infty} n^{r-2} P(\sup_{k\geq n} |\frac{\sum_{i=1}^{k} a_i X_i}{k^{\frac{1}{p}}}| > 2^{\frac{2}{p}} \epsilon^{'}) \\
&= \sum_{m=1}^{\infty} \sum_{n=2^{m-1}}^{2^m-1} n^{r-2} P(\sup_{k\geq n} |\frac{\sum_{i=1}^{k} a_i X_i}{k^{\frac{1}{p}}}| > 2^{\frac{2}{p}} \epsilon^{'}) \\
&\leq \sum_{m=1}^{\infty} P(\sup_{k\geq 2^{m-1}} |\frac{\sum_{i=1}^{k} a_i X_i}{k^{\frac{1}{p}}}| > 2^{\frac{2}{p}} \epsilon^{'}) \sum_{n=2^{m-1}}^{2^m-1} 2^{m(r-2)} \\
&\leq \sum_{m=1}^{\infty} 2^{m(r-1)} P(\sup_{k\geq 2^{m-1}} |\frac{\sum_{i=1}^{k} a_i X_i}{k^{\frac{1}{p}}}| > 2^{\frac{2}{p}} \epsilon^{'}) \\
&= \sum_{m=1}^{\infty} 2^{m(r-1)} P(\sup_{l\geq m} \max_{2^{l-1}\leq k\leq 2^l} |\frac{\sum_{i=1}^{k} a_i X_i}{k^{\frac{1}{p}}}| > 2^{\frac{2}{p}} \epsilon^{'}) \\
&\leq \sum_{m=1}^{\infty} 2^{m(r-1)} \sum_{l=m}^{\infty} P(\max_{1\leq k\leq 2^l} |\sum_{i=1}^{k} a_i X_i| > \epsilon^{'} 2^{\frac{l+1}{p}}) \\
&= \sum_{l=1}^{m} P(\max_{1\leq k\leq 2^l} |\sum_{i=1}^{k} a_i X_i| > \epsilon^{'} 2^{\frac{l+1}{p}}) \sum_{m=1}^{l} 2^{m(r-1)} \\
&\leq C \sum_{l=1}^{\infty} 2^{l(r-1)} P(\max_{1\leq k\leq 2^l} |\sum_{i=1}^{k} a_i X_i| > \epsilon^{'} 2^{\frac{l+1}{p}}) \\
&= 2^{2-r} C \sum_{l=1}^{\infty} \sum_{n=2^l}^{2^{l+1}-1} 2^{(l+1)(r-2)} P(\max_{1\leq k\leq 2^l} |\sum_{i=1}^{k} a_i X_i| > \epsilon^{'} 2^{\frac{l+1}{p}}) \\
&\leq C \sum_{l=1}^{\infty} \sum_{n=2^l}^{2^{l+1}-1} n^{r-2} P(\max_{1\leq k\leq n} |\sum_{i=1}^{k} a_i X_i| > \epsilon^{'} n^{\frac{1}{p}}) \text{ since } 1 < r < 2 \\
&\leq C \sum_{n=1}^{\infty} n^{r-2} P(\max_{1\leq k\leq n} |\sum_{i=1}^{k} a_i X_i| > \epsilon^{'} n^{\frac{1}{p}}) < \infty.
\end{aligned}
$$

Hence, by (3.21) and (3.23) we obtain (3.22). □

Next, we consider the Marcinkiewicz-Zygmund-type strong law of large numbers for PNQD sequence satisfying stochastically dominating condition.

COROLLARY 3.7. *Let $1 < p < 2$. Let $\{X_n, n \geq 1\}$ be a sequence of zero mean PNQD random variables which is stochastically dominated by* *a random variable* X *with*$E|X|^p < \infty$. *Under the conditions of Corollary 2.2* $n^{-\frac{1}{p}} \sum_{i=1}^{n} a_i X_i \to 0$ *a.s. holds.*

Proof. Since $r > 1$ it can be seen by (3.21) that

$$\sum_{n=1}^{\infty} n^{-1} P(\max_{1\le k\le n} |\sum_{i=1}^{k} a_i X_i| > \epsilon n^{\frac{1}{p}}) \le \sum_{n=1}^{\infty} n^{r-2} P(\max_{1\le k\le n} |\sum_{i=1}^{k} a_i X_i| > \epsilon n^{\frac{1}{p}}).$$

It is easy to obtain $n^{-\frac{1}{p}} \sum_{i=1}^{n} a_i X_i \to 0$ $a.s.,\ n \to \infty$. □

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