# OPTIMALITY AND DUALITY FOR NONDIFFERENTIABLE FRACTIONAL PROGRAMMING WITH GENERALIZED INVEXITY 

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#### Abstract

We establish necessary and sufficient optimality conditions for a class of generalized nondifferentiable fractional optimization programming problems. Moreover, we prove the weak and strong duality theorems under ( $V, \rho$ )-invexity assumption.


## 1. Introduction and preliminaries

Many authors have introduced various concepts of generalized convexity and have obtained optimality and duality results for optimization programming problem ([1]-[4], [6]-[12]). Many practical problems encountered in economics, engineering design, and management science, and so forth can be described by nonsmooth functions. The theory of nonsmooth optimization using locally Lipschitz functions was introduced by Clarke [5].

We consider the following generalized nondifferentiable fractional optimization problem (GFP):

$$
\begin{array}{lll}
(\mathrm{GFP}) & \text { Minimize } & \max \left\{\left.\frac{f_{i}(x)}{g_{i}(x)} \right\rvert\, i=1, \cdots, p\right\} \\
& \text { subject to } & h_{j}(x) \leq 0, \quad j=1, \cdots, m
\end{array}
$$

where $f:=\left(f_{1}, \cdots, f_{p}\right): \mathbb{R}^{n} \rightarrow \mathbb{R}^{p}, g:=\left(g_{1}, \cdots, g_{p}\right): \mathbb{R}^{n} \rightarrow \mathbb{R}^{p}$ and $h:=\left(h_{1}, \cdots, h_{m}\right): \mathbb{R}^{n} \rightarrow \mathbb{R}^{m}$ are locally Lipschitz function. We assume that $f_{i}(x) \geqq 0$ and $g_{i}(x)>0, i=1, \cdots, p$. Let $X_{0}:=\left\{x \in \mathbb{R}^{n} \mid h_{j}(x) \leqq\right.$

[^0]$0, j=1, \cdots, m\}$ be the feasible set of (GFP). Let $J=\{1,2, \cdots, m\}$ and $J\left(x_{0}\right)=\left\{j \in J \mid h_{j}\left(x_{0}\right)=0\right\}$.

We consider the following fractional optimization problem (FP):

$$
\begin{array}{ll}
\text { Minimize } & \max \left\{\left.\frac{f_{i}(x)+s\left(x \mid C_{i}\right)}{g_{i}(x)} \right\rvert\, i=1, \cdots, p\right\}  \tag{FP}\\
\text { subject to } & h_{j}(x) \leq 0, \quad j=1, \cdots, m
\end{array}
$$

where $f:=\left(f_{1}, \cdots, f_{p}\right): \mathbb{R}^{n} \rightarrow \mathbb{R}^{p}, g:=\left(g_{1}, \cdots, g_{p}\right): \mathbb{R}^{n} \rightarrow \mathbb{R}^{p}$ and $h:=\left(h_{1}, \cdots, h_{m}\right): \mathbb{R}^{n} \rightarrow \mathbb{R}^{m}$ are continuously differentiable function. For each $i=1, \cdots, p, C_{i}$ is compact convex set of $\mathbb{R}^{n}$ and $s\left(x \mid C_{i}\right):=$ $\max \left\{\left\langle x, y_{i}\right\rangle \mid y_{i} \in C_{i}\right\}$.

Recently, Kim and Kim [7] consider the nondifferentiable fractional optimization problem (FP), in which each component of the objective function contains a term involving the support function of a compact convex set. They established necessary and sufficient optimality conditions for fractional optimization problem (FP). And they formulated a Mond-Weir type dual problem for (FP) and showed that the weak and strong duality.

In this paper, we apply the approach of Kim and Kim[7] to the generalized nondifferentiable fractional optimization problem (GFP), we establish necessary and sufficient optimality conditions for a nondifferentiable fractional optimization programming involving locally Lipschitz functions. Moreover, we prove the weak and strong duality theorems under ( $V, \rho$ )-invexity assumption.

Now we give some notations for our results in this section;
Let a function $f: \mathbb{R}^{n} \rightarrow \mathbb{R}$ be given. We shall suppose that $f$ is locally Lipschitz, that is, for each $x \in \mathbb{R}^{n}$, there exist an open neighborhood $U$ and a constant $L>0$ such that for all $y$ and $z$ in $U$,

$$
|f(y)-f(z)| \leq L\|y-z\|
$$

Let $g: \mathbb{R}^{n} \rightarrow \mathbb{R} \cup\{+\infty\}$ be a convex function. The subdifferential of $g$ at $a \in \operatorname{dom} g$ is defined by

$$
\partial g(a):=\left\{v \in \mathbb{R}^{n} \mid g(x) \geqq g(a)+\langle v, x-a\rangle \quad \forall x \in \operatorname{dom} g\right\}
$$

where $\langle\cdot, \cdot\rangle$ is the inner product on $\mathbb{R}^{n}$ and $\operatorname{dom} g:=\left\{x \in \mathbb{R}^{n}: g(x)<\right.$ $+\infty\}$.

Definition 1.1. A vector function $f: \mathbb{R}^{n} \rightarrow \mathbb{R}^{p}$ is said to be $(V, \rho)$ invex at $u \in \mathbb{R}^{n}$ with respect to the function $\eta$ and $\theta_{i}: \mathbb{R}^{n} \rightarrow \mathbb{R}^{n}$ if there exists $\alpha_{i}: \mathbb{R}^{n} \times \mathbb{R}^{n} \rightarrow \mathbb{R}_{+} \backslash\{0\}$ and $\rho_{i} \in \mathbb{R}, i=1, \ldots, p$ such that for any
$\xi_{i} \in \partial f_{i}(u), i=1, \ldots, p$ and any $x \in \mathbb{R}^{n}$, and for all $i=1, \ldots, p$,

$$
\alpha_{i}(x, u)\left[f_{i}(x)-f_{i}(u)\right] \geq \xi_{i}^{T} \eta(x, u)+\rho_{i}\left\|\theta_{i}(x, u)\right\|^{2}
$$

Lemma 1.2. [5] Let $f$ and $g$ be Lipschitz near $x$ and suppose that $g(x) \neq 0$. Then $\frac{f}{g}$ is Lipschitz near $x$, and one has

$$
\partial\left(\frac{f}{g}\right)(x) \subset \frac{g(x) \partial f(x)-f(x) \partial g(x)}{\{g(x)\}^{2}}
$$

If in addition $f(x) \geqq 0, g(x)>0$ and if $f$ and $-g$ are regular at $x$, then equality holds and $\frac{\bar{f}}{g}$ is regular at $x$.

Theorem 1.3. Assume that $f$ and $g$ are vector-valued differentiable functions defined on $\mathbb{R}^{n}$ and $f(x) \geq 0, g(x)>0$ for all $x \in \mathbb{R}^{n}$. If $f$ and $-g$ are regular and $(V, \rho)$-invex at $x_{0}$, then $\frac{f}{g}$ is $(V, \rho)$-invex at $x_{0}$, where

$$
\bar{\alpha}_{i}\left(x, x_{0}\right)=\frac{g_{i}(x)}{g_{i}\left(x_{0}\right)} \alpha_{i}\left(x, x_{0}\right), \quad \bar{\theta}_{i}\left(x, x_{0}\right)=\left(\frac{1}{g_{i}\left(x_{0}\right)}\right)^{\frac{1}{2}} \theta_{i}\left(x, x_{0}\right)
$$

Proof. Let $x, x_{0} \in X_{0}$. Then, by the $(V, \rho)$-invexity of $f$ and $-g$, there exists $\alpha_{i}: \mathbb{R}^{n} \times \mathbb{R}^{n} \rightarrow \mathbb{R}_{+} \backslash\{0\}$ and $\rho_{i} \in \mathbb{R}, i=1, \ldots, p$ such that for any $\xi_{i} \in \partial f_{i}\left(x_{0}\right), \zeta_{i} \in \partial g_{i}\left(x_{0}\right), i=1, \ldots, p$ and $x \in \mathbb{R}^{n}$, and for all $i=1, \ldots, p$,

$$
\begin{aligned}
& \alpha_{i}\left(x, x_{0}\right)\left[f_{i}(x)-f_{i}\left(x_{0}\right)\right] \geq \xi_{i}^{T} \eta\left(x, x_{0}\right)+\rho_{i}\left\|\theta_{i}\left(x, x_{0}\right)\right\|^{2} \\
& \alpha_{i}\left(x, x_{0}\right)\left[g_{i}(x)-g_{i}\left(x_{0}\right)\right] \geq \zeta_{i}^{T} \eta\left(x, x_{0}\right)+\rho_{i}\left\|\theta_{i}\left(x, x_{0}\right)\right\|^{2}
\end{aligned}
$$

So, we have for any $\xi_{i} \in \partial f_{i}\left(x_{0}\right), \zeta_{i} \in \partial g_{i}\left(x_{0}\right), i=1, \ldots, p$ and $x \in \mathbb{R}^{n}$, and for all $i=1, \ldots, p$,

$$
\begin{aligned}
& \alpha_{i}\left(x, x_{0}\right)\left(\frac{f_{i}(x)}{g_{i}(x)}-\frac{f_{i}\left(x_{0}\right)}{g_{i}\left(x_{0}\right)}\right) \\
& =\alpha_{i}\left(x, x_{0}\right)\left(\frac{f_{i}(x)-f_{i}\left(x_{0}\right)}{g_{i}(x)}-f_{i}\left(x_{0}\right) \frac{g_{i}(x)-g_{i}\left(x_{0}\right)}{g_{i}(x) g_{i}\left(x_{0}\right)}\right) \\
& \geqq \frac{\xi_{i}^{T} \eta\left(x, x_{0}\right)+\rho_{i}\left\|\theta_{i}\left(x, x_{0}\right)\right\|^{2}}{g_{i}(x)}-\frac{f_{i}\left(x_{0}\right)}{g_{i}(x) g_{i}\left(x_{0}\right)}\left(\zeta_{i}^{T} \eta\left(x, x_{0}\right)+\rho_{i}\left\|\theta_{i}\left(x, x_{0}\right)\right\|^{2}\right)
\end{aligned}
$$

Since $g_{i}(x)>0, i=1, \ldots, p$ for all $x \in X_{0}$, we have for any $\xi_{i} \in \partial f_{i}\left(x_{0}\right)$, $\zeta_{i} \in \partial g_{i}\left(x_{0}\right), i=1, \ldots, p$ and $x \in \mathbb{R}^{n}$, and for all $i=1, \ldots, p$,

$$
\begin{aligned}
& \alpha_{i}\left(x, x_{0}\right)\left(\frac{f_{i}(x)}{g_{i}(x)}-\frac{f_{i}\left(x_{0}\right)}{g_{i}\left(x_{0}\right)}\right) \\
& \geqq \frac{g_{i}\left(x_{0}\right)}{g_{i}(x) g_{i}\left(x_{0}\right)}\left[\xi_{i}^{T} \eta\left(x, x_{0}\right)+\rho_{i}\left\|\theta_{i}\left(x, x_{0}\right)\right\|^{2}\right] \\
& \quad-\frac{g_{i}\left(x_{0}\right)}{g_{i}(x)}\left(\frac{f_{i}\left(x_{0}\right) \zeta_{i}^{T} \eta\left(x, x_{0}\right)}{\left(g_{i}\left(x_{0}\right)\right)^{2}}+\rho_{i}\left\|\left(\frac{f_{i}\left(x_{0}\right)}{\left(g_{i}\left(x_{0}\right)\right)^{2}}\right)^{\frac{1}{2}} \theta_{i}\left(x, x_{0}\right)\right\|^{2}\right)
\end{aligned}
$$

Thus, from Lemma 1.2 , for any $\omega_{i} \in \partial\left(\frac{f_{i}}{g_{i}}\right)\left(x_{0}\right), \xi_{i} \in \partial f_{i}\left(x_{0}\right), \zeta_{i} \in$ $\partial g_{i}\left(x_{0}\right), i=1, \ldots, p$ and $x \in \mathbb{R}^{n}$, and for all $i=1, \ldots, p$,

$$
\begin{aligned}
& \alpha_{i}\left(x, x_{0}\right)\left(\frac{f_{i}(x)}{g_{i}(x)}-\frac{f_{i}\left(x_{0}\right)}{g_{i}\left(x_{0}\right)}\right) \\
& \geqq \\
& \geqq \frac{g_{i}\left(x_{0}\right)}{g_{i}(x)}\left[\left(\frac{\xi_{i} g_{i}\left(x_{0}\right)-\zeta_{i} f_{i}\left(x_{0}\right)}{\left(g_{i}\left(x_{0}\right)\right)^{2}}\right)^{T} \eta\left(x, x_{0}\right)+\rho_{i}\left\|\left(\frac{1}{g_{i}\left(x_{0}\right)}\right)^{\frac{1}{2}} \theta_{i}\left(x, x_{0}\right)\right\|^{2}\right. \\
& \left.\quad+\rho_{i}\left\|\left(\frac{f_{i}\left(x_{0}\right)}{\left(g_{i}\left(x_{0}\right)\right)^{2}}\right)^{\frac{1}{2}} \theta_{i}\left(x, x_{0}\right)\right\|^{2}\right] \\
& =\frac{g_{i}\left(x_{0}\right)}{g_{i}(x)}\left[\omega_{i}^{T} \eta\left(x, x_{0}\right)+\rho_{i}\left\|\left(\frac{1}{g_{i}\left(x_{0}\right)}\right)^{\frac{1}{2}} \theta_{i}\left(x, x_{0}\right)\right\|^{2}\right. \\
& \left.\quad+\rho_{i}\left\|\left(\frac{f_{i}\left(x_{0}\right)}{\left(g_{i}\left(x_{0}\right)\right)^{2}}\right)^{\frac{1}{2}} \theta_{i}\left(x, x_{0}\right)\right\|^{2}\right] \\
& \geqq \\
& \left.\equiv \frac{g_{i}\left(x_{0}\right)}{g_{i}(x)}\left[\omega_{i}^{T} \eta\left(x, x_{0}\right)+\rho_{i} \|\left(\frac{1}{g_{i}\left(x_{0}\right)}\right)^{\frac{1}{2}}+\left(\frac{\left(f_{i}\left(x_{0}\right)\right)^{\frac{1}{2}}}{g_{i}\left(x_{0}\right)}\right)\right) \theta_{i}\left(x, x_{0}\right) \|^{2}\right] \\
& =\frac{g_{i}\left(x_{0}\right)}{g_{i}(x)}\left[\omega_{i}^{T} \eta\left(x, x_{0}\right)+\rho_{i}\left\|\left(\frac{\left(g_{i}\left(x_{0}\right)\right)^{\frac{1}{2}}+\left(f_{i}\left(x_{0}\right)\right)^{\frac{1}{2}}}{g_{i}\left(x_{0}\right)}\right) \theta_{i}\left(x, x_{0}\right)\right\|^{2}\right] \\
& = \\
& \frac{g_{i}\left(x_{0}\right)}{g_{i}(x)}\left[\omega_{i}^{T} \eta\left(x, x_{0}\right)+\rho_{i}\left\|\left(\frac{1+\left(\frac{f_{i}\left(x_{0}\right)}{g_{i}\left(x_{0}\right)}\right)^{\frac{1}{2}}}{\left(g_{i}\left(x_{0}\right)\right)^{\frac{1}{2}}}\right) \theta_{i}\left(x, x_{0}\right)\right\|^{2}\right] .
\end{aligned}
$$

Since $1+\left(\frac{f_{i}\left(x_{0}\right)}{g_{i}\left(x_{0}\right)}\right)^{\frac{1}{2}} \geqq 1, i=1, \ldots, p$, we have for any $\omega_{i} \in \partial\left(\frac{f_{i}}{g_{i}}\right)\left(x_{0}\right)$,

$$
\alpha_{i}\left(x, x_{0}\right)\left(\frac{f_{i}(x)}{g_{i}(x)}-\frac{f_{i}\left(x_{0}\right)}{g_{i}\left(x_{0}\right)}\right) \geqq \frac{g_{i}\left(x_{0}\right)}{g_{i}(x)}\left[\omega_{i}^{T} \eta\left(x, x_{0}\right)+\rho_{i}\left\|\left(\frac{1}{\left(g_{i}\left(x_{0}\right)\right)^{\frac{1}{2}}}\right) \theta_{i}\left(x, x_{0}\right)\right\|^{2}\right] .
$$

Thus, the function $\frac{f}{g}$ is $(V, \rho)$-invex at $x_{0}$, where

$$
\bar{\alpha}_{i}\left(x, x_{0}\right)=\frac{g_{i}(x)}{g_{i}\left(x_{0}\right)} \alpha_{i}\left(x, x_{0}\right), \quad \bar{\theta}_{i}\left(x, x_{0}\right)=\frac{1}{\left(g_{i}\left(x_{0}\right)\right)^{\frac{1}{2}}} \theta_{i}\left(x, x_{0}\right) .
$$

## 2. Optimality theorems

Now, we establish the Kuhn-Tucker necessary and sufficient conditions for a solution of (GFP).

Theorem 2.1. (Kuhn-Tucker Necessary Optimality Theorem) Assume that $f$ and $-g$ are regular. If $x_{0}$ is a solution of (GFP), and assume that $0 \notin \operatorname{co}\left\{\partial h_{j}\left(x_{0}\right) \mid j \in J\left(x_{0}\right)\right\}$, then there exist $\lambda_{i} \geq 0$, $i \in I\left(x_{0}\right):=\left\{i \left\lvert\, \max \left\{\left.\frac{f_{i}\left(x_{0}\right)}{g_{i}\left(x_{0}\right)} \right\rvert\, i=1, \ldots, p\right\}=\frac{f_{i}\left(x_{0}\right)}{g_{i}\left(x_{0}\right)}\right.\right\}, \quad \sum_{i \in I\left(x_{0}\right)} \lambda_{i}=1$ and $\mu_{j} \geq 0, j=1, \ldots, m$ such that

$$
\begin{aligned}
& 0 \in \sum_{i \in I\left(x_{0}\right)} \lambda_{i} \partial\left(\frac{f_{i}}{g_{i}}\right)\left(x_{0}\right)+\sum_{j=1}^{m} \mu_{j} \partial h_{j}\left(x_{0}\right) \\
& \text { and } \quad \sum_{j=1}^{m} \mu_{j} h_{j}\left(x_{0}\right)=0 .
\end{aligned}
$$

Proof. Let $\phi_{i}(x)=\frac{f_{i}(x)}{g_{i}(x)}, i=1, \ldots, p$. Let $x_{0}$ be a solution of (GFP) and let $I\left(x_{0}\right)=\left\{i \mid \max \left\{\phi_{i}\left(x_{0}\right) \mid i=1, \ldots, p\right\}=\phi_{i}\left(x_{0}\right)\right\}$. Then by Proposition 2.3.12 in [5] and Corollary 5.1.8 in [11], there exist $\mu_{j} \geqq 0$, $j=1, \ldots, m$,

$$
\begin{align*}
& 0 \in \operatorname{co}\left\{\partial \phi_{i}\left(x_{0}\right) \mid i \in I\left(x_{0}\right)\right\}+\sum_{j=1}^{m} \mu_{j} \partial h_{j}\left(x_{0}\right)  \tag{2.1}\\
& \text { and } \mu_{j} h_{j}\left(x_{0}\right)=0
\end{align*}
$$

where $\operatorname{co} A$ is the convexhull of the set $A$. By Lemma 1.2,

$$
\begin{aligned}
\partial \phi_{i}\left(x_{0}\right) & =\frac{g_{i}\left(x_{0}\right) \partial f_{i}\left(x_{0}\right)-\partial g_{i}\left(x_{0}\right) f_{i}\left(x_{0}\right)}{\left(g_{i}\left(x_{0}\right)\right)^{2}} \\
& =\partial\left(\frac{f_{i}}{g_{i}}\right)\left(x_{0}\right),
\end{aligned}
$$

and hence from (2.1), there exist $\lambda_{i} \geq 0, i \in I\left(x_{0}\right), \sum_{i \in I\left(x_{0}\right)} \lambda_{i}=1$ and $\mu_{j} \geq 0, j=1, \ldots, m$ such that

$$
\begin{aligned}
& 0 \in \sum_{i \in I\left(x_{0}\right)} \lambda_{i} \partial\left(\frac{f_{i}}{g_{i}}\right)\left(x_{0}\right)+\sum_{j=1}^{m} \mu_{j} \partial h_{j}\left(x_{0}\right) \\
& \text { and } \sum_{j=1}^{m} \mu_{j} h_{j}\left(x_{0}\right)=0 .
\end{aligned}
$$

Corollary 2.2. Let $f:=\left(f_{1}, \cdots, f_{p}\right): \mathbb{R}^{n} \rightarrow \mathbb{R}^{p}, g:=\left(g_{1}, \cdots, g_{p}\right):$ $\mathbb{R}^{n} \rightarrow \mathbb{R}^{p}$ and $h:=\left(h_{1}, \cdots, h_{m}\right): \mathbb{R}^{n} \rightarrow \mathbb{R}^{m}$ are continuously differentiable. If $x_{0}$ is a solution of (GFP), and assume that $0 \notin \operatorname{co}\left\{\nabla h_{j}\left(x_{0}\right) \mid j \in\right.$ $\left.J\left(x_{0}\right)\right\}$, then there exist $\lambda_{i} \geq 0, i \in I\left(x_{0}\right):=\left\{i \left\lvert\, \max \left\{\left.\frac{f_{i}\left(x_{0}\right)}{g_{i}\left(x_{0}\right)} \right\rvert\, i=1, \cdots, p\right\}\right.\right.$ $\left.=\frac{f_{i}\left(x_{0}\right)}{g_{i}\left(x_{0}\right)}\right\}, \sum_{i \in I\left(x_{0}\right)} \lambda_{i}=1$ and $\mu_{j} \geq 0, j=1, \ldots, m$ such that

$$
\begin{aligned}
& \sum_{i \in I\left(x_{0}\right)} \lambda_{i} \nabla\left(\frac{f_{i}\left(x_{0}\right)}{g_{i}\left(x_{0}\right)}\right)+\sum_{j=1}^{m} \mu_{j} \nabla h_{j}\left(x_{0}\right)=0 \\
& \sum_{j=1}^{m} \mu_{j} h_{j}\left(x_{0}\right)=0
\end{aligned}
$$

Theorem 2.3. (Kuhn-Tucker Sufficient Optimality Theorem) Assume that $f$ and $-g$ are regular. Let $x_{0}$ be a feasible solution of (GFP). Suppose that there exist $\lambda_{i} \geqq 0, i \in I\left(x_{0}\right), \sum_{i \in I\left(x_{0}\right)} \lambda_{i}=1$ and $\mu_{j} \geqq 0, j=1, \ldots, m$ such that

$$
\begin{align*}
& 0 \in \sum_{i \in I\left(x_{0}\right)} \lambda_{i} \partial\left(\frac{f_{i}}{g_{i}}\right)\left(x_{0}\right)+\sum_{j=1}^{m} \mu_{j} \partial h_{j}\left(x_{0}\right)  \tag{2.2}\\
& \text { and } \sum_{j=1}^{m} \mu_{j} h_{j}\left(x_{0}\right)=0 .
\end{align*}
$$

If $f(\cdot)$ and $-g(\cdot)$ are $(V, \rho)$-invex at $x_{0}$, and $h$ is $\eta$-invex at $x_{0}$ with respect to the same $\eta$, and $\sum_{i \in I\left(x_{0}\right)} \lambda_{i} \rho_{i}\left\|\bar{\theta}_{i}\left(x, x_{0}\right)\right\|^{2} \geqq 0$, then $x_{0}$ is a solution of (GFP).

Proof. Suppose that $x_{0}$ is not a solution of (GFP). Then there exist a feasible solution $x$ of (GFP) such that

$$
\max _{1 \leqq i \leqq p} \frac{f_{i}\left(x_{0}\right)}{g_{i}\left(x_{0}\right)}>\max _{1 \leqq i \leqq p} \frac{f_{i}(x)}{g_{i}(x)}
$$

Then

$$
\frac{f_{i}\left(x_{0}\right)}{g_{i}\left(x_{0}\right)}>\frac{f_{i}(x)}{g_{i}(x)}, \text { for all } i \in I\left(x_{0}\right)
$$

and hence $\bar{\alpha}_{i}\left(x, x_{0}\right)>0$,

$$
\bar{\alpha}_{i}\left(x, x_{0}\right)\left[\frac{f_{i}(x)}{g_{i}(x)}-\frac{f_{i}\left(x_{0}\right)}{g_{i}\left(x_{0}\right)}\right]<0
$$

Since $f(\cdot)$ and $-g(\cdot)$ are $(V, \rho)$-invex and regular at $x_{0}$, by Theorem 1.3, we have for any $w_{i} \in \partial\left(\frac{f_{i}}{g_{i}}\right)\left(x_{0}\right), i \in I\left(x_{0}\right)$

$$
w_{i} \eta\left(x, x_{0}\right)+\rho_{i}\left\|\bar{\theta}\left(x, x_{0}\right)\right\|^{2}<0
$$

Hence, there exist $\lambda_{i} \geqq 0, i \in I\left(x_{0}\right), \sum_{i \in I\left(x_{0}\right)} \lambda_{i}=1$ such that

$$
\sum_{i \in I\left(x_{0}\right)} \lambda_{i} w_{i} \eta\left(x, x_{0}\right)+\sum_{i \in I\left(x_{0}\right)} \lambda_{i} \rho_{i}\left\|\bar{\theta}\left(x, x_{0}\right)\right\|^{2}<0
$$

Since $\sum_{i \in I\left(x_{0}\right)} \lambda_{i} \rho_{i}\left\|\bar{\theta}\left(x, x_{0}\right)\right\|^{2} \geqq 0$,

$$
\sum_{i \in I\left(x_{0}\right)} \lambda_{i} w_{i} \eta\left(x, x_{0}\right)<0
$$

and so, it follows from (2.2) that there exist $\nu_{j} \in \partial h_{j}\left(x_{0}\right), j=1, \ldots, m$ such that

$$
\sum_{j=1}^{m} \mu_{j} \nu_{j} \eta\left(x, x_{0}\right)>0
$$

Then, by the $\eta$-invexity of $h$, we have

$$
\sum_{j=1}^{m} \mu_{j} h_{j}(x)>\sum_{j=1}^{m} \mu_{j} h_{j}\left(x_{0}\right)
$$

Since $\sum_{j=1}^{m} \mu_{j} h_{j}\left(x_{0}\right)=0$, we have $\sum_{j=1}^{m} \mu_{j} h_{j}(x)>0$, which is a contradiction since $\mu_{j} \geqq 0, j=1, \ldots, m$ and $x$ is a feasible solution of (GFP). Consequently, $x_{0}$ is a solution of (GFP).

## 3. Duality theorems

Now, we propose the following Mond-Weir type dual problem (DGFP):
(DGFP) Maximize $\max \left\{\left.\frac{f_{i}(u)}{g_{i}(u)} \right\rvert\, i=1, \ldots, p\right\}$
(3.1) subject to

$$
\begin{aligned}
& 0 \in \sum_{i \in I(u)} \lambda_{i} \partial\left(\frac{f_{i}}{g_{i}}\right)(u)+\sum_{j=1}^{m} \mu_{j} \partial h_{j}(u) \\
& \sum_{j=1}^{m} \mu_{j} h_{j}(u)=0 \\
& \lambda_{i} \geqq 0, \quad i \in I(u), \quad \sum_{i \in I(u)} \lambda_{i}=1, \quad \mu_{j} \geqq 0, j=1, \ldots, m
\end{aligned}
$$

Now we show that the following weak duality theorem holds between (GFP) and (DGFP).

Theorem 3.1. (Weak Duality) Assume that $f$ and $-g$ are regular. Let $x$ be a feasible for (GFP) and let $(u, \lambda, \mu)$ be feasible for (DGFP). Assume that $f(\cdot)$ and $-g(\cdot)$ are $(V, \rho)$-invex at $u$, and let $h$ is $\eta$-invex at $u$ with respect to the same $\eta$, and $\sum_{i \in I(u)} \lambda_{i} \rho_{i}\left\|\bar{\theta}_{i}(x, u)\right\|^{2}>0$. Then the following holds:

$$
\max \left\{\left.\frac{f_{i}(x)}{g_{i}(x)} \right\rvert\, i=1, \ldots, p\right\} \geqq \max \left\{\left.\frac{f_{i}(u)}{g_{i}(u)} \right\rvert\, i=1, \ldots, p\right\}
$$

Proof. Let $x$ be any feasible for (GFP) and let $(u, \lambda, \mu)$ be any feasible for (DGFP). Then we have

$$
\sum_{j=1}^{m} \mu_{j} h_{j}(x) \leqq 0 \leqq \sum_{j=1}^{m} \mu_{j} h_{j}(u)
$$

By the $\eta$-invexity of $h_{j}(u), j=1, \ldots, m$, there exists $\nu_{j}^{*} \in \partial h_{j}(u), j=$ $1, \cdots, m$ such that

$$
\sum_{j=1}^{m} \mu_{j} \nu_{j}^{*} \eta(x, u) \leqq 0
$$

Using (3.1), we have there exists $w_{i}^{*} \in \partial\left(\frac{f_{i}}{g_{i}}\right)(u), i \in I(u)$,

$$
\begin{equation*}
\sum_{i \in I(u)} \lambda_{i} w_{i}^{*} \eta(x, u) \geqq 0 \tag{3.2}
\end{equation*}
$$

Now suppose that

$$
\max \left\{\left.\frac{f_{i}(x)}{g_{i}(x)} \right\rvert\, i=1, \ldots, p\right\}<\max \left\{\left.\frac{f_{i}(u)}{g_{i}(u)} \right\rvert\, i=1, \ldots, p\right\} .
$$

Then

$$
\frac{f_{i}(x)}{g_{i}(x)}<\frac{f_{i}(u)}{g_{i}(u)}, \text { for all } i \in I(u)
$$

By Theorem 1.3, we have there exists $w_{i}^{*} \in \partial\left(\frac{f_{i}}{g_{i}}\right)(u), i \in I(u)$ such that

$$
\begin{aligned}
0 & >\bar{\alpha}_{i}(x, u)\left[\frac{f_{i}(x)}{g_{i}(x)}-\frac{f_{i}(u)}{g_{i}(u)}\right] \\
& \geqq w_{i}^{*} \eta(x, u)+\rho_{i}\left\|\bar{\theta}_{i}(x, u)\right\|^{2}
\end{aligned}
$$

By using $\lambda_{i} \geqq 0, i \in I(u)$, we have,

$$
\sum_{i \in I(u)} \lambda_{i} w_{i}^{*} \eta(x, u)+\sum_{i \in I(u)} \lambda_{i} \rho_{i}\left\|\bar{\theta}_{i}(x, u)\right\|^{2}<0
$$

Since $\sum_{i \in I(u)} \lambda_{i} \rho_{i}\left\|\bar{\theta}_{i}(x, u)\right\|^{2} \geqq 0$, we have

$$
\sum_{i \in I(u)} \lambda_{i} w_{i}^{*} \eta(x, u)<0
$$

which contradicts (3.2). Hence the result holds.
Now we give a strong duality theorem which holds between (GFP) and (DGFP).

Theorem 3.2. (Strong Duality) If $\bar{x}$ is a solution of (GFP) and suppose that $0 \notin \operatorname{co}\left\{\partial h_{j}\left(x_{0}\right) \mid j \in J\left(x_{0}\right)\right\}$. Then there exist $\bar{\lambda} \in \mathbb{R}^{p}$ and $\bar{\mu} \in \mathbb{R}^{m}$ such that $(\bar{x}, \bar{\lambda}, \bar{\mu})$ is feasible for (DGFP). Moreover if the weak duality holds, then $(\bar{x}, \bar{\lambda}, \bar{\mu})$ is a solution of (DGFP).

Proof. By Theorem 2.1, there exist $\bar{\lambda}_{i} \geq 0, i \in I(\bar{x}):=\left\{i \left\lvert\, \max \left\{\left.\frac{\left.f_{( } \bar{x}\right)}{g_{i}(\bar{x})} \right\rvert\, i=\right.\right.\right.$ $\left.1, \ldots, p\}=\frac{\left.f_{( } \bar{x}\right)}{g_{i}(\bar{x})}\right\}, \sum_{i \in I(\bar{x})} \bar{\lambda}_{i}=1$ and $\bar{\mu}_{j} \geq 0, j=1, \ldots, m$ such that

$$
\begin{aligned}
& 0 \in \sum_{i \in I(\bar{x})} \bar{\lambda}_{i} \partial\left(\frac{f_{i}}{g_{i}}\right)(\bar{x})+\sum_{j=1}^{m} \bar{\mu}_{j} \partial h_{j}(\bar{x}) \\
& \text { and } \quad \sum_{j=1}^{m} \bar{\mu}_{j} h_{j}(\bar{x})=0 .
\end{aligned}
$$

Thus $(\bar{x}, \bar{\lambda}, \bar{\mu})$ is a feasible for (DGFP). On the other hand, by weak duality (Theorem 3.1),

$$
\max \left\{\left.\frac{f_{i}(\bar{x})}{g_{i}(\bar{x})} \right\rvert\, i=1, \cdots, p\right\} \geq \max \left\{\left.\frac{f_{i}(u)}{g_{i}(u)} \right\rvert\, i=1, \cdots, p\right\}
$$

for any (DGFP) feasible solution $(u, \lambda, \mu)$. Hence $(\bar{x}, \bar{\lambda}, \bar{\mu})$ is a solution of (DGFP).

## 4. Conclusions

This paper is concerned with optimality conditions and duality theorems for fractional optimization problems involving locally Lipschitz functions. Using Clarke's generalized subdifferential, we gave necessary and sufficient optimality theorems for the problems. The sufficient optimality conditions were verified under generalized invexity conditions on involved functions. The Mond-Weir dual problems were formulated, and then duality theorems were established, that is, weak and strong duality theorems for the nondifferentiable fractional optimization problems.

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