JOURNAL OF THE
CHUNGCHEONG MATHEMATICAL SOCIETY
Volume **29**, No. 3, August 2016
http://dx.doi.org/10.14403/jcms.2016.29.3.453

A FIXED POINT APPROACH TO STABILITY OF ADDITIVE FUNCTIONAL INEQUALITIES IN FUZZY NORMED SPACES

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Abstract. In this paper, we investigate the solution of the following functional inequality

$$N(f(x)+f(y)+f(z),t) \geq N(f(x+y+z),mt)$$

for some fixed real number m with $\frac{1}{3} < m \leq 1$ and using the fixed point method, we prove the generalized Hyers-Ulam stability for it in fuzzy Banach spaces.

1. Introduction and preliminaries

The concept of a fuzzy norm on a linear space was introduced by Katsaras [12] in 1984. Later, Cheng and Mordeson [3] gave a new definition of a fuzzy norm in such a manner that the corresponding fuzzy metric is of Kramosil and Michalek type [14]. In this paper, we use the definition of fuzzy normed spaces given in [2], [15], and [17].

Definition 1.1. Let X be a linear space. A function $N : X \times \mathbb{R} \longrightarrow [0,1]$ is called *a fuzzy norm on* X if for any $x, y \in X$ and any $s, t \in \mathbb{R}$,
(N1) $N(x,t) = 0$ for $t \leq 0$;
(N2) $x = 0$ if and only if $N(x,t) = 1$ for all $t > 0$;
(N3) $N(cx,t) = N(x, \frac{t}{|c|})$ if $c \neq 0$;
(N4) $N(x+y, s+t) \geq min\{N(x,s), N(y,t)\}$;
(N5) $N(x,\cdot)$ is a nondecreasing function on $\mathbb{R}$ and $\lim_{t\to\infty} N(x,\ t) = 1$;
(N6) for any $x \neq 0$, $N(x,\cdot)$ is continuous on $\mathbb{R}$.
In this case, the pair (X,N) is called *a fuzzy normed space.*

Received April 18, 2016; Accepted July 15, 2016.
2010 Mathematics Subject Classification: Primary 39B62, 39B72, 54A40, 47H10.
Key words and phrases: Hyers-Ulam stability, additive functional inequality, fuzzy normed space, fixed point theorem.
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Let (X, N) be a fuzzy normed space. A sequence $\{x_n\}$ in X is said to be *convergent* in (X, N) if there exists an $x \in X$ such that $\lim_{n\to\infty} N(x_n - x, t) = 1$ for all $t > 0$. In this case, x is called *the limit of the sequence* $\{x_n\}$ *in* X and one denotes it by $N - \lim_{n\to\infty} x_n = x$. A sequence $\{x_n\}$ in X is said to be *Cauchy* in (X, N) if for any $\epsilon > 0$, there is an $m \in N$ such that for any $n \geq m$ and any positive integer p, $N(x_{n+p} - x_n, t) > 1 - \epsilon$ for all $t > 0$.

It is well known that every convergent sequence in a fuzzy normed space is Cauchy. A fuzzy normed space is said to be *complete* if each Cauchy sequence in it is convergent and a complete fuzzy normed space is called *a fuzzy Banach space*.

In 1940, Ulam proposed the following stability problem (cf. [24]):

"Let G_1 be a group and G_2 a metric group with the metric d. Given a constant $\delta > 0$, does there exist a constant $c > 0$ such that if a mapping $f : G_1 \longrightarrow G_2$ satisfies $d(f(xy), f(x)f(y)) < c$ for all $x, y \in G_1$, then there exists a unique homomorphism $h : G_1 \longrightarrow G_2$ with $d(f(x), h(x)) < \delta$ for all $x \in G_1$?"

In the next year, Hyers [11] gave a partial solution of Ulam's problem for the case of approximate additive mappings. Subsequently, his result was generalized by Aoki ([1]) for additive mappings, and by Rassias [22] for linear mappings, to consider the stability problem with unbounded Cauchy differences. A generalization of the Rassias theorem was obtained by Găvruta [8] by replacing the unbounded Cauchy difference by a general control function in the spirit of the Rassias' approach. During the last decades, the stability problems of functional equations have been extensively investigated by a number of mathematicians (see [4], [5], and [18]).

In 2008, Mirmostafaee and Moslehian [16], [17] used the definition of a fuzzy norm in [2] to obtain a fuzzy version of the stability for the Cauchy functional equation

$$f(x + y) = f(x) + f(y) \tag{1.1}$$

and the quadratic functional equation

$$f(x + y) + f(x - y) = 2f(x) + 2f(y). \tag{1.2}$$

Glányi [9] and Rätz [23] showed that if a mapping $f : X \longrightarrow Y$ satisfies the following functional inequality

$$\|2f(x) + 2f(y) - f(xy^{-1})\| \leq \|f(xy)\|, \tag{1.3}$$

then f satisfies the Jordan-Von Neumann functional equation

$$2f(x) + 2f(y) - f(xy^{-1}) = f(xy).$$

for an abelian group X divisible by 2 into an inner product space Y. Glányi [10] and Fechner [7] proved the Hyers-Ulam stability of (1.3). Park, Cho, and Han [21] proved the generalized Hyers-Ulam stability of the following functional inequality associated with the following Jordan-von Neumann type additive functional equations:

$$\|f(x) + f(y) + f(z)\| \leq \|f(x + y + z)\|. \tag{1.4}$$

and Kim, Jun, and Son [13] proved the generalized Hyers-Ulam stability of the Jensen functional inequality in p-Banach spaces.

Banachs contraction principle is one of the pivotal results of analysis. It is widely considered as the source of the metric fixed point theory. Also, its significance lies in its vast applicability in a number of branches of mathematics. In particular, Diaz and Margolis [6] presented the following definition and the fixed point theorem in a generalized complete metric space.

Definition 1.2. Let X be a non-empty set. Then a mapping $d : X^2 \longrightarrow [0, \infty]$ is called *a generalized metric on* X if d satisfies the following conditions:

(D1) $d(x, y) = 0$ if and only if $x = y$,
(D2) $d(x, y) = d(y, x)$, and
(D3) $d(x, y) \leq d(x, z) + d(z, y)$.

In case, (X, d) is called *a generalized metric space*.

Theorem 1.3. [6] *Let (X, d) be a complete generalized metric space and let $J : X \longrightarrow X$ be a strictly contractive mapping with some Lipschitz constant L with $0 < L < 1$. Then for each given element $x \in X$, either $d(J^n x, J^{n+1} x) = \infty$ for all nonnegative integers n or there exists a positive integer n_0 such that*

(1) *$d(J^n x, J^{n+1} x) < \infty$ for all $n \geq n_0$;*
(2) *the sequence $\{J^n x\}$ converges to a fixed point y^* of J ;*
(3) *y^* is the unique fixed point of J in the set $Y = \{y \in X \mid d(J^{n_0} x, y) < \infty\}$;*
(4) *$d(y, y^*) \leq \dfrac{1}{1 - L}\, d(y, Jy)$ for all $y \in Y$.*

Using the fixed point Theorem, Park [20] proved the generalized Hyers-Ulam stability of the Cauchy additive functional inequality (1.4) in fuzzy Banach spaces if f is an odd mapping.

In this paper, we investigate the solution of the following functional inequality

$$N(f(x)+f(y)+f(z),t)\geq N(f(x+y+z),mt) \tag{1.5}$$

for some fixed positive real number m with $\frac{1}{3}<m\leq 1$ and prove the generalized Hyers-Ulam stability for it in fuzzy Banach spaces in which f need not be odd.

Throughout this paper, we assume that X is a linear space, (Y,N) is a fuzzy Banach space, and (Z,N') is a fuzzy normed space.

2. Solutions and fuzzy stability of (1.5)

In this section, we prove the generalized Hyers-Ulam stability of functional equation (1.5) in fuzzy Banach spaces. We start with the solution of (1.5).

Theorem 2.1. *A mapping $f: X \longrightarrow Y$ saisfies (1.5) if and omly if f is an additive mapping.*

Proof. Suppose that f satisfies (1.5). Setting $x=y=z=0$ in (1.5), by (N3), we have

$$N(f(0),mt)\leq N(3f(0),t)=N\Big(f(0),\frac{t}{3}\Big)$$

for all $t>0$ and since $m>\frac{1}{3}$, by (N5), $N(f(0),\frac{t}{3})\leq N(f(0),mt)$ for all $t>0$. Hence we have

$$N(f(0),mt)=N\Big(f(0),\frac{t}{3}\Big)$$

for all $t>0$. By induction, we have

$$N(f(0),m^nt)=N\Big(f(0),\frac{t}{3^n}\Big) \tag{2.1}$$

for all $t>0$ and all $n\in\mathbb{N}$. Letting $t=3^nt$ in (2.1), we have

$$N(f(0),t)=N(f(0),(3m)^nt) \tag{2.2}$$

for all $t>0$, $n\in\mathbb{N}$ and by(N5), we have

$$N(f(0),t)=\lim_{n\to\infty}N(f(0),(3m)^nt)=1 \tag{2.3}$$

for all $t>0$. Hence by (N2), we have

$$f(0)=0. \tag{2.4}$$

Putting $y=-x$ and $z=0$ in (1.5), by (2.4), we have $N(f(x)+f(-x),t)\geq N(f(0),mt)=1$ for all $t\in Y$ and so by (N2), we have

$$f(-x)=-f(x) \tag{2.5}$$

for all $x \in X$. Replacing z by $-x-y$ in (1.5), by (2.5), we have

$$N(f(x)+f(y)+f(-x,-y),t) = N(f(x)+f(y)-f(x+y),t) \geq N(0,mt) = 1$$

for all $x, y \in X$, $t > 0$ and hence by (N2), we have

$$f(x+y) = f(x) + f(y)$$

for all $x, y \in X$. Thus f is an additive mapping. For the converse, suppose that f is an additive mapping. Then

$$f(x+y+z) = f(x) + f(y) + f(z)$$

for all $x, y, z \in X$ and since $\frac{1}{3} < m \leq 1$, we have (1.5). □

Now, we will prove the generalized Hyers-Ulam stability of (1.5) in fuzzy Banach spaces.

THEOREM 2.2. *Assume that* $\phi : X^3 \longrightarrow [0,\infty)$ *is a function such that*

$$N'(\phi(2x,2y,2z),t) \geq N'(2L\phi(x,y,z),t) \tag{2.6}$$

for all $x, y, z \in X$, $t > 0$ *and some* L *with* $0 < L < 1$. *Let* $f : X \longrightarrow Y$ *be a mapping such that* $f(0) = 0$ *and*

$$N(f(x)+f(y)+f(z),t) \geq \min\{N(f(x+y+z),mt), N'(\phi(x,y,z),(1-m)t)\} \tag{2.7}$$

for all $x, y, z \in X$, $t > 0$ *and some fixed real number* m *with* $\frac{1}{3} < m \leq 1$. *Then there exists a unique additive mapping* $A : X \longrightarrow Y$ *such that*

$$N\Big(f(x) - A(x), \frac{1}{1-L}t\Big) \geq \Psi(x,t) \tag{2.8}$$

for all $x \in X$ *and all* $t > 0$, *where*

$$\Psi(x,t) = \min\Big\{N'(\phi(x,-x,0), \frac{2(1-m)}{3}t), N'(\phi(2x,-x,-x), \frac{2(1-m)}{3}t)\Big\}.$$

Proof. If $m = 1$, then clearly, one has the results. Suppose that $\frac{1}{3} < m < 1$.

Letting $y = -x$ and $z = 0$ in (2.7), by (N2), we have

$$\begin{aligned} &N(f(x) + f(-x), t) \\ &\geq \min\{N(0,mt), N'(\phi(x,-x,0),(1-m)t)\} \\ &= N'(\phi(x,-x,0),(1-m)t) \end{aligned} \tag{2.9}$$

for all $x \in X$ and all $t > 0$. Letting $x = 2x$ and $y = z = -x$ in (2.7), we have

$$N(f(2x) + 2f(-x), t) \geq N'(\phi(2x,-x,-x),(1-m)t). \tag{2.10}$$

By (2.9) and (2.10), we have

$$
\begin{aligned}
&N\Big(f(x) - \frac{f(2x)}{2}, t\Big) \\
&\geq \min\Big\{N(f(x) + f(-x), \frac{2}{3}t), N(f(2x) + 2f(-x), \frac{2}{3}t)\Big\} \\
&\geq \Psi(x,t)
\end{aligned} \tag{2.11}
$$

for all $x \in X$ and all $t > 0$.

Consider the set $S = \{g \mid g : X \longrightarrow Y\}$ and the generalized metric d on S defined by

$$d(g,h) = \inf\{c \in [0,\infty) \mid N(g(x) - h(x), ct) \geq \Psi(x,t), \forall x \in X, \forall t > 0\}.$$

Then (S,d) is a complete metric space(see [19]). Define a mapping $J : S \longrightarrow S$ by $Jg(x) = \frac{1}{2}g(2x)$ for all $x \in X$ and all $g \in S$. Let $g, h \in S$ and $d(g,h) \leq c$ for some $c \in [0,\infty)$. Then by (2.6), we have

$$
\begin{aligned}
&N(Jg(x) - Jh(x), cLt) \\
&= N\Big(\frac{1}{2}g(2x) - \frac{1}{2}h(2x), cLt\Big) = N(g(2x) - h(2x), 2cLt) \\
&\geq \min\Big\{N'(\phi(2x,-2x,0), \frac{4(1-m)L}{3}t), N'(\phi(4x,-2x,-2x), \frac{4(1-m)L}{3}t)\Big\} \\
&\geq \Psi(x,t)
\end{aligned}
$$

for all $x \in X$ and all $t > 0$. Hence $N(Jg(x) - Jh(x), cLt) \geq \Psi(x,t)$ for all $x \in X$, $t > 0$ and thus $d(Jg, Jh) \leq Ld(g,h)$. Moreover, by (2.11), we have $d(f, Jf) \leq 1 < \infty$. By Theorem 1.3, there exists a mapping $A : X \longrightarrow Y$ which is a fixed point of J such that $d(J^n f, A) \to 0$ as $n \to \infty$. That is,

$$A(x) = N - \lim_{n\to\infty} \frac{f(2^n x)}{2^n} \tag{2.12}$$

for all $x \in X$. Replacing x, y, z by $2^n x$, $2^n y$, $2^n z$ in (2.7), respectively, by (2.6) and (N4), we have

$$
\begin{aligned}
&N(f(2^n x) + f(2^n y) + f(2^n z), 2^n t) \\
&\geq \min\{N(f(2^n(x+y+z)), 2^n mt), N'(L^n \phi(x,y,z), (1-m)t)\}
\end{aligned} \tag{2.13}
$$

for all $x, y, z \in X$ and all $n \in \mathbb{N}$. Letting $n \to \infty$ in (2.13), A is a solution of (1.5) and so by Theorem 2.1, A is an additive mapping. Since $d(f, Jf) \leq 1$, by Theorem 1.3, we have (2.8).

Now, we show the uniqueness of A. Let A_0 be another additive mapping with (2.8). Then for any positive integer n,

$$A(2^n x) = \frac{A(2^n x)}{2^n}, \ A_0(2^n x) = \frac{A_0(2^n x)}{2^n}$$

for all $x \in X$. Hence by (2.8), (N3) and (N4), we have

$$\begin{aligned}
&N(A(x) - A_0(x), t) \\
&= N(A(2^n x) - A_0(2^n x), 2^n t) \\
&\geq \min\{N(A(2^n x) - f(2^n x), 2^{n-1} t), N(A_0(2^n x) - f(2^n x), 2^{n-1} t)\} \\
&\geq \min\Big\{N'(L^n \phi(x, -x, 0), \frac{(1-L)(1-m)}{3} t), \\
&\qquad\qquad N'(L^n \phi(2x, -x, -x), \frac{(1-L)(1-m)}{3} t)\Big\}
\end{aligned}$$

for all $x \in X$, $t > 0$, and all $n \in \mathbb{N}$. Hence, letting $n \to \infty$ in the above inequality, we have $A(x) = A_0(x)$ for all $x \in X$. □

Related with Theorem 2.2, we can also have the following theorem. And the proof is similar to that of Theorem 2.2.

Theorem 2.3. *Assume that $\phi : X^3 \longrightarrow [0, \infty)$ is a function such that*

$$N'\Big(\phi\Big(\frac{x}{2}, \frac{y}{2}, \frac{z}{2}\Big), t\Big) \geq N'\Big(\frac{L}{2}\phi(x, y, z), t\Big) \tag{2.14}$$

for all $x, y, z \in X$, $t > 0$ and some L with $0 < L < 1$. Let $f : X \longrightarrow Y$ be a mapping with $f(0) = 0$ and (2.7). Then there exists a unique additive mapping $A : X \longrightarrow Y$ such that

$$N\Big(f(x) - A(x), \frac{1}{1-L} t\Big) \geq \Psi(x, t) \tag{2.15}$$

for all $x \in X$ and all $t > 0$, where

$$\Psi(x, t) = \min\Big\{N'(\phi(x, -x, 0), \frac{2(1-m)}{3} t), N'(\phi(2x, -x, -x), \frac{2(1-m)}{3} t)\Big\}.$$

Proof. Letting $y = -x$ and $z = 0$ in (2.7), by (N2), we have

$$\begin{aligned}
&N(f(x) + f(-x), t) \\
&\geq \min\{N(0, mt), N'(\phi(x, -x, 0), (1-m)t)\} \\
&= N'(\phi(x, -x, 0), (1-m)t)
\end{aligned} \tag{2.16}$$

for all $x \in X$, $t > 0$ and letting $y = z = -\frac{x}{2}$ in (2.7), by (N5), we have

$$\begin{aligned} N\Big(f(x) + 2f\Big(-\frac{x}{2}\Big), t\Big) &\geq N'\Big(\phi\Big(x, -\frac{x}{2}, -\frac{x}{2}\Big), (1-m)t\Big) \\ &\geq N'\Big(\phi(2x, -x, -x), \frac{2(1-m)}{L}t\Big) \\ &\geq N'(\phi(2x, -x, -x), 2(1-m)t), \end{aligned} \tag{2.17}$$

because $L < 1$. By (N5), (2.16), and (2.17), we have

$$\begin{aligned} &N\Big(f(x) - 2f\Big(\frac{x}{2}\Big), t\Big) \\ &\geq \min\Big\{N(f(-x) + 2f(\frac{x}{2}), \frac{1}{3}t), N(f(x) + f(-x), \frac{2}{3}t)\Big\} \geq \Psi(x, t) \end{aligned} \tag{2.18}$$

for all $x \in X$ and all $t > 0$.

Consider the set $S = \{g \mid g : X \longrightarrow Y\}$ and the generalized metric d on S defined by

$$d(g, h) = \inf\{c \in [0, \infty) \mid N(g(x) - h(x), ct) \geq \Psi(x, t), \forall x \in X, \forall t > 0\}.$$

Then (S, d) is a complete metric space(see [19]). Define a mapping $J : S \longrightarrow S$ by $Jg(x) = 2g(\frac{1}{2}x)$ for all $x \in X$ and all $g \in S$. Let $g, h \in S$ and $d(g, h) \leq c$ for some $c \in [0, \infty)$. Then by (2.14), we have

$$\begin{aligned} &N(Jg(x) - Jh(x), cLt) \\ &= N\Big(2g\Big(\frac{1}{2}x\Big) - 2h\Big(\frac{1}{2}x\Big), cLt\Big) \geq \Psi\Big(\frac{1}{2}x, \frac{L}{2}t\Big) \geq \Psi(x, t) \end{aligned}$$

for all $x \in X$ and all $t > 0$. Hence $N(Jg(x) - Jh(x), cLt) \geq \Psi(x, t)$ for all $x \in X$, $t > 0$ and thus $d(Jg, Jh) \leq Ld(g, h)$ for any $g, h \in S$. Moreover, by (2.18), we have $d(f, Jf) \leq 1 < \infty$. The rest of the proof is similar to Theorem 2.2. $\square$

As examples of $\phi(x, y, z)$ and $N'(x, t)$ in Theorem 2.2 and Theorem 2.3, we can take $\phi(x, y, z) = \epsilon(\|x\|^p + \|y\|^p + \|z\|^p)$ and

$$N'(x, t) = \begin{cases} \dfrac{t}{t + k|x|}, & \text{if } t > 0 \\ 0, & \text{if } t \leq 0 \end{cases}$$

for all $x \in \mathbb{R}$ and all $t > 0$, where $k = 1, 2$. Then we can formulate the following corollary

Corollary 2.4. *Let X be a normed space and (Y, N) a fuzzy Banach space. Let $f : X \longrightarrow Y$ be a mapping such that*

(2.19)
$$N(f(x) + f(y) + f(z), t) \geq \min\left\{N(f(x + y + z), mt),\ \frac{(1-m)t}{(1-m)t + k\epsilon(\|x\|^p + \|y\|^p + \|z\|^p)}\right\}$$

for all $x, y, z \in X$, $t > 0$, a fixed real number p with $p \neq 1$, and some fixed real number m with $\frac{1}{3} < m \leq 1$. Then there is a unique additive mapping $A : X \longrightarrow Y$ such that

$$N(f(x) - A(x), t) \geq \begin{cases} \dfrac{(2-2^p)(1-m)t}{(2-2^p)(1-m)t + 3(2+2^p)\epsilon k\|x\|^p}, & \text{if } 0 < p < 1, \\[2ex] \dfrac{(2^p-2)(1-m)t}{(2^p-2)(1-m)t + 3 \times 2^{p-1}(2+2^p)\epsilon k\|x\|^p}, & \text{if } 1 < p \end{cases}$$

for all $x \in X$ and all $t > 0$.

We remark that the functional inequality (1.5) is not stable for $p = 1$ in Corollary 2.4. The following example shows that the (2.19) is not stable for $p = 1$, especially in the case of $k = 1$, $\epsilon = 48$, and $m = \frac{1}{2}$.

Example 2.5. Let $s : \mathbb{R} \longrightarrow \mathbb{R}$ be a mapping defined by

$$s(x) = \begin{cases} x, & \text{if } |x| < 1 \\ 1, & \text{if } x > 1 \\ -1, & \text{if } x < -1 \end{cases}$$

and define a mapping $f : \mathbb{R} \longrightarrow \mathbb{R}$ by $f(x) = \sum_{n=0}^{\infty} \frac{s(2^n x)}{2^n}$. Let

$$N(x, t) = N'(x, t) = \begin{cases} \frac{t}{t+|x|}, & \text{if } t > 0 \\ 0, & \text{if } t \leq 0. \end{cases}$$

for all $x \in X$. Let $\phi(x, y, z) = 48(|x| + |y| + |z|)$.

We will show that (2.19) holds, but there do not exist an additive mapping $A : \mathbb{R} \longrightarrow \mathbb{R}$ and a positive constant K such that

$$|A(x) - f(x)| \leq K|x| \tag{2.20}$$

for all $x \in \mathbb{R}$.

Proof. Now, we claim that for any $x, y, z \in X$

$$|f(x) + f(y) + f(z)| \le 2|f(x + y + z)| \tag{2.21}$$

or

$$|f(x) + f(y) + f(z)| \le \phi(x, y, z). \tag{2.22}$$

Take any $x, y, z \in X$. Suppose that (2.21) does not hold. Then we have

$$|f(x) + f(y) + f(z)| > 2|f(x + y + z)|. \tag{2.23}$$

First, suppose that $|x| + |y| + |z| \ge \frac{1}{4}$. Since $|f(x)| \le 2$, we have

$$|f(x) + f(y) + f(z)| \le 24(|x| + |y| + |z|) \le \phi(x, y, z).$$

Suppose that $|x| + |y| + |z| < \frac{1}{4}$. Then there is a non-negative integer k such that

$$\frac{1}{2^{k+3}} \le |x| + |y| + |z| < \frac{1}{2^{k+2}}.$$

Then we have

$$2^k|x| < \frac{1}{4},\ \ 2^k|y| < \frac{1}{4},\ \ 2^k|z| < \frac{1}{4}$$

and

$$2^k|x + y + z| \le 2^k(|x| + |y| + |z|) < \frac{1}{4}.$$

For any $n = 0, 1, 2, \cdots, k$, we have

$$s(2^n x) + s(2^n y) + s(2^n z) = s(2^n(x + y + z))$$

and so by (2.23), we get

$$\begin{aligned}
&|f(x) + f(y) + f(z)| \\
&\le \Big|\sum_{n=0}^{k} \frac{s(2^n x) + s(2^n y) + s(2^n z)}{2^n}\Big| + \Big|\sum_{n=k+1}^{\infty} \frac{s(2^n x) + s(2^n y) + s(2^n z)}{2^n}\Big| \\
&\le \Big|\sum_{n=0}^{k} \frac{s(2^n(x + y + z))}{2^n}\Big| + \frac{3}{2^k} \\
&\le |f(x + y + z)| + 24(|x| + |y| + |z|) \\
&\le \frac{1}{2}|f(x) + f(y) + f(z)| + 24(|x| + |y| + |z|).
\end{aligned}$$

Thus we have (2.22). By (2.21) and (2.22), we have

$$N(f(x) + f(y) + f(z), t) \ge N\Big(f(x + y + z), \frac{1}{2}t\Big)$$

or

$$N(f(x) + f(y) + f(z), t) \ge N'\Big(\phi(x, y, z), t\Big)$$

for all $x, y, z \in X$ and all $t > 0$. Hence we have (2.19).

Suppose that there exists an additive mapping $A : \mathbb{R} \longrightarrow \mathbb{R}$ and a positive constant K with (2.20). Since $|f(x)| \leq 2$,

$$-K|x| - 2 \leq A(x) \leq K|x| + 2$$

for all $x \in X$ and since A is additive,

$$-K|x| - \frac{2}{n} \leq A(x) \leq K|x| + \frac{2}{n}$$

for all $x \in X$ and all $n \in \mathbb{N}$. Hence we have $|A(x)| \leq K|x|$ for all $x \in X$ and so, by (2.20), we have $|f(x)| \leq 2K|x|$ for all $x \in X$. Take a positive integer l such that $l > 2K$, and pick $x \in \mathbb{R}$ with $0 < 2^l x < 1$. Then

$$f(x) = \sum_{n=0}^{\infty} \frac{s(2^n x)}{2^n} > \sum_{n=0}^{l-1} \frac{s(2^n x)}{2^n} = \sum_{n=0}^{l-1} x = lx > 2Kx$$

which is a contradiction. $\square$

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