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UNIFORMLY LIPSCHITZ STABILITY AND ASYMPTOTIC BEHAVIOR OF PERTURBED DIFFERENTIAL SYSTEMS

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Abstract. In this paper we show that the solutions to the perturbed differential system

$$y' = f(t,y) + \int_{t_0}^{t} g(s, y(s), Ty(s))ds$$

have uniformly Lipschitz stability and asymptotic behavior by imposing conditions on the perturbed part $\int_{t_0}^{t} g(s, y(s), Ty(s))ds$ and the fundamental matrix of the unperturbed system $y' = f(t,y)$.

1. Introduction

The notion of uniformly Lipschitz stability (ULS) was introduced by Dannan and Elaydi [8]. This notion of ULS lies somewhere between uniformly stability on one side and the notions of asymptotic stability in variation of Brauer[4] and uniformly stability in variation of Brauer and Strauss[3] on the other side. An important feature of ULS is that for linear systems, the notion of uniformly Lipschitz stability and that of uniformly stability are equivalent. However, for nonlinear systems, the two notions are quite distinct. Also, Elaydi and Farran[9] introduced the notion of exponential asymptotic stability(EAS) which is a stronger notion than that of ULS. They investigated some analytic criteria for an autonomous differential system and its perturbed systems to be EAS. Pachpatte[16,17] investigated the stability and asymptotic behavior of solutions of the functional differential equation. Gonzalez and Pinto[10]

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proved theorems which relate the asymptotic behavior and boundedness of the solutions of nonlinear differential systems. Choi et al.[7] studied Lipschitz and exponential asymptotic stability for nonlinear functional systems. Also, Goo[11,12,13] and Choi and Goo[5,6] and Goo et al.[14] investigated Lipschitz and asymptotic stability for perturbed differential systems.

In this paper, we investigate ULS and asymptotic behavior for solutions of the perturbed differential systems using integral inequalities.

2. Preliminaries

We consider the nonautonomous differential system

$$x' = f(t,x), \; x(t_0) = x_0, \tag{2.1}$$

where $f \in C(\mathbb{R}^+ \times \mathbb{R}^n, \mathbb{R}^n)$, $\mathbb{R}^+ = [0,\infty)$ and $\mathbb{R}^n$ is the Euclidean n-space. We assume that the Jacobian matrix $f_x = \partial f/\partial x$ exists and is continuous on $\mathbb{R}^+ \times \mathbb{R}^n$ and $f(t,0) = 0$. Also, we consider the perturbed functional differential system of (2.1)

$$y' = f(t,y) + \int_{t_0}^{t} g(s, y(s), Ty(s))ds, \; y(t_0) = y_0, \tag{2.2}$$

where $g \in C(\mathbb{R}^+ \times \mathbb{R}^n \times \mathbb{R}^n, \mathbb{R}^n)$, $g(t,0,0) = 0$, and $T : C(\mathbb{R}^+, \mathbb{R}^n) \to C(\mathbb{R}^+, \mathbb{R}^n)$ is a continuous operator .

The symbol $|\cdot|$ will be used to denote any convenient vector norm in $\mathbb{R}^n$. For an $n \times n$ matrix A, define the norm $|A|$ of A by $|A| = \sup_{|x|\le 1} |Ax|$.

Let $x(t, t_0, x_0)$ denote the unique solution of (2.1) with $x(t_0, t_0, x_0) = x_0$, existing on $[t_0, \infty)$. Then we can consider the associated variational systems around the zero solution of (2.1) and around $x(t)$, respectively,

$$v'(t) = f_x(t,0)v(t), \; v(t_0) = v_0 \tag{2.3}$$

and

$$z'(t) = f_x(t, x(t,t_0,x_0))z(t), \; z(t_0) = z_0. \tag{2.4}$$

The fundamental matrix $\Phi(t, t_0, x_0)$ of (2.4) is given by

$$\Phi(t, t_0, x_0) = \frac{\partial}{\partial x_0} x(t, t_0, x_0),$$

and $\Phi(t, t_0, 0)$ is the fundamental matrix of (2.3).

We now give the following fundamental concept[8].

DEFINITION 2.1. The system (2.1) (the zero solution $x = 0$ of (2.1)) is called
(S)*stable* if for any $\epsilon > 0$ and $t_0 \geq 0$, there exists $\delta = \delta(t_0, \epsilon) > 0$ such that if $|x_0| < \delta$, then $|x(t)| < \epsilon$ for all $t \geq t_0 \geq 0$,
(US)*uniformly stable* if the δ in (S) is independent of the time t_0,
(ULS) *uniformly Lipschitz stable* if there exist $M > 0$ and $\delta > 0$ such that $|x(t)| \leq M|x_0|$ whenever $|x_0| \leq \delta$ and $t \geq t_0 \geq 0$
(ULSV) *uniformly Lipschitz stable in variation* if there exist $M > 0$ and $\delta > 0$ such that $|\Phi(t, t_0, x_0)| \leq M$ for $|x_0| \leq \delta$ and $t \geq t_0 \geq 0$,
(EAS) *exponentially asymptotically stable* if there exist constants $K > 0$, $c > 0$, and $\delta > 0$ such that

$$|x(t)| \leq K\,|x_0|e^{-c(t-t_0)}, 0 \leq t_0 \leq t$$

provided that $|x_0| < \delta$,
(EASV) *exponentially asymptotically stable in variation* if there exist constants $K > 0$ and $c > 0$ such that

$$|\Phi(t, t_0, x_0)| \leq K\,e^{-c(t-t_0)}, 0 \leq t_0 \leq t$$

provided that $|x_0| < \infty$.

REMARK 2.2. [10] The last definition implies that for $|x_0| \leq \delta$

$$|x(t)| \leq K\,|x_0|e^{-c(t-t_0)}, 0 \leq t_0 \leq t.$$

For the proof we prepare some related properties.

We need Alekseev formula to compare between the solutions of (2.1) and the solutions of perturbed nonlinear system

$$y' = f(t, y) + g(t, y),\; y(t_0) = y_0, \tag{2.5}$$

where $g \in C(\mathbb{R}^+ \times \mathbb{R}^n, \mathbb{R}^n)$ and $g(t, 0) = 0$. Let $y(t) = y(t, t_0, y_0)$ denote the solution of (2.5) passing through the point (t_0, y_0) in $\mathbb{R}^+ \times \mathbb{R}^n$.

The following is a generalization to nonlinear system of the variation of constants formula due to Alekseev [1].

LEMMA 2.3. [2] *Let x and y be a solution of (2.1) and (2.5), respectively. If $y_0 \in \mathbb{R}^n$, then for all $t \geq t_0$ such that $x(t, t_0, y_0) \in \mathbb{R}^n$, $y(t, t_0, y_0) \in \mathbb{R}^n$,*

$$y(t, t_0, y_0) = x(t, t_0, y_0) + \int_{t_0}^{t} \Phi(t, s, y(s))\, g(s, y(s))\, ds.$$

Lemma 2.4. *(Bihari-type inequality) Let $u, \lambda \in C(\mathbb{R}^+)$, $w \in C((0,\infty))$ and $w(u)$ be nondecreasing in u. Suppose that, for some $c > 0$,*

$$u(t) \le c + \int_{t_0}^{t} \lambda(s) w(u(s)) ds, \; t \ge t_0 \ge 0.$$

Then

$$u(t) \le W^{-1}\Big[W(c) + \int_{t_0}^{t} \lambda(s) ds\Big],$$

where $t_0 \le t < b_1$, $W(u) = \int_{u_0}^{u} \frac{ds}{w(s)}$, $W^{-1}(u)$ is the inverse of $W(u)$, and

$$b_1 = \sup\Big\{t \ge t_0 : W(c) + \int_{t_0}^{t} \lambda(s) ds \in \mathrm{domW^{-1}}\Big\}.$$

Lemma 2.5. [5] *Let $u, \lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5, \lambda_6 \in C(\mathbb{R}^+)$, $w \in C((0,\infty))$ and $w(u)$ be nondecreasing in u, $u \le w(u)$. Suppose that for some $c > 0$,*

$$\begin{aligned} u(t) \le c &+ \int_{t_0}^{t} \lambda_1(s) u(s) ds + \int_{t_0}^{t} \lambda_2(s) w(u(s)) ds \\ &+ \int_{t_0}^{t} \lambda_3(s) \int_{t_0}^{s} \lambda_4(\tau) u(\tau) d\tau ds \\ &+ \int_{t_0}^{t} \lambda_5(s) \int_{t_0}^{s} \lambda_6(\tau) w(u(\tau)) d\tau ds, \;\; 0 \le t_0 \le t. \end{aligned}$$

Then

$$\begin{aligned} u(t) \le W^{-1}\Big[W(c) + \int_{t_0}^{t} \Big(\lambda_1(s) + \lambda_2(s) + \lambda_3(s) \int_{t_0}^{s} \lambda_4(\tau) d\tau \\ + \lambda_5(s) \int_{t_0}^{s} \lambda_6(\tau) d\tau\Big) ds\Big], \end{aligned}$$

where $t_0 \le t < b_1$, W, W^{-1} are the same functions as in Lemma 2.4, and

$$\begin{aligned} b_1 = \sup\Big\{t \ge t_0 : W(c) + \int_{t_0}^{t} \Big(\lambda_1(s) + \lambda_2(s) + \lambda_3(s) \int_{t_0}^{s} \lambda_4(\tau) d\tau \\ + \lambda_5(s) \int_{t_0}^{s} \lambda_6(\tau) d\tau\Big) ds \in \mathrm{domW^{-1}}\Big\}. \end{aligned}$$

For the proof we need the following corollary.

Corollary 2.6. *Let $u, \lambda_1, \lambda_2, \lambda_3, \lambda_4 \in C(\mathbb{R}^+)$, $w \in C((0,\infty))$ and $w(u)$ be nondecreasing in u, $u \le w(u)$. Suppose that for some $c > 0$ and*

$0 \le t_0 \le t$,

$$u(t) \le c + \int_{t_0}^{t} \lambda_1(s)u(s)ds + \int_{t_0}^{t} \lambda_2(s)w(u(s))ds + \int_{t_0}^{t} \lambda_3(s)\int_{t_0}^{s} \lambda_4(\tau)w(u(\tau))d\tau ds.$$

Then

$$u(t) \le W^{-1}\Big[W(c) + \int_{t_0}^{t}\Big(\lambda_1(s) + \lambda_2(s) + \lambda_3(s)\int_{t_0}^{s}\lambda_4(\tau)d\tau\Big)ds\Big],$$

where $t_0 \le t < b_1$, W, W^{-1} *are the same functions as in Lemma 2.4, and*

$$b_1 = \sup\Big\{t \ge t_0 : W(c) + \int_{t_0}^{t}\Big(\lambda_1(s) + \lambda_2(s) + \lambda_3(s)\int_{t_0}^{s}\lambda_4(\tau)d\tau\Big)ds \in \mathrm{domW^{-1}}\Big\}.$$

THEOREM 2.7. [14] *Suppose that* $x = 0$ *of (2.1) is ULS. Let the following condition hold for (2.2):*

$$\int_{t_0}^{t} |g(s, y(s), Ty(s))|ds \le W(t, |y|, T|y|),$$

where $0 \le t_0 \le t$, $W(t, u, v) \in C(\mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R}^+, \mathbb{R}^+)$ *is monotone nondecreasing in* u *and* v *for each fixed* $t \ge 0$ *with* $W(t, 0, 0) = 0$. *Assume that* $u(t)$ *is any the solution of the scalar differential equation*

$$u'(t) = KW(t, u, Tu), u(t_0) = u_0 > 0, K \ge 1, \tag{2.6}$$

existing on $\mathbb{R}^+$ *such that* $|y(t_0)| < u(t_0)$. *If* $u = 0$ *of (2.6) is ULS, then* $y = 0$ *of (2.2) is also ULS whenever* $K|y_0| < u_0$.

LEMMA 2.8. [12] *Let* $u, \lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5, \lambda_6, \lambda_7, \lambda_8 \in C(\mathbb{R}^+)$, $w \in C((0, \infty))$, *and* $w(u)$ *be nondecreasing in* u, $u \le w(u)$. *Suppose that for some* $c > 0$ *and* $0 \le t_0 \le t$,

$$u(t) \le c + \int_{t_0}^{t} \lambda_1(s)u(s)ds + \int_{t_0}^{t} \lambda_2(s)w(u(s))ds + \int_{t_0}^{t} \lambda_3(s)\int_{t_0}^{s} (\lambda_4(\tau)u(\tau) + \lambda_5(\tau)\int_{t_0}^{\tau} \lambda_6(r)w(u(r))dr)d\tau ds + \int_{t_0}^{t} \lambda_7(s)\int_{t_0}^{s} \lambda_8(\tau)w(u(\tau))d\tau ds.$$

Then

$$u(t) \leq W^{-1}\Big[W(c) + \int_{t_0}^{t} \Big(\lambda_1(s) + \lambda_2(s) + \lambda_3(s) \int_{t_0}^{s} (\lambda_4(\tau) + \lambda_5(\tau) \int_{t_0}^{\tau} \lambda_6(r)dr)d\tau + \lambda_7(s) \int_{t_0}^{s} \lambda_8(\tau)d\tau\Big)ds\Big],$$

where $t_0 \leq t < b_1$, W, W^{-1} *are the same functions as in Lemma 2.4, and*

$$b_1 = \sup\Big\{t \geq t_0 : W(c) + \int_{t_0}^{t} \Big(\lambda_1(s) + \lambda_2(s) + \lambda_3(s) \int_{t_0}^{s} (\lambda_4(\tau) + \lambda_5(\tau) \int_{t_0}^{\tau} \lambda_6(r)dr)d\tau + \lambda_7(s) \int_{t_0}^{s} \lambda_8(\tau)d\tau\Big)ds \in \mathrm{domW}^{-1}\Big\}.$$

For the proof we prepare the following corollary.

Corollary 2.9. *Let* $u, \lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5, \lambda_6, \in C(\mathbb{R}^+)$, $w \in C((0,\infty))$, *and* $w(u)$ *be nondecreasing in* u, $u \leq w(u)$. *Suppose that for some* $c > 0$ *and* $0 \leq t_0 \leq t$,

$$u(t) \leq c + \int_{t_0}^{t} \lambda_1(s) \int_{t_0}^{s} (\lambda_2(\tau)u(\tau) + \lambda_3(\tau) \int_{t_0}^{\tau} \lambda_4(r)w(u(r))dr)d\tau ds + \int_{t_0}^{t} \lambda_5(s) \int_{t_0}^{s} \lambda_6(\tau)w(u(\tau))d\tau ds.$$

Then

$$u(t) \leq W^{-1}\Big[W(c) + \int_{t_0}^{t} \Big(\lambda_1(s) \int_{t_0}^{s} (\lambda_2(\tau) + \lambda_3(\tau) \int_{t_0}^{\tau} \lambda_4(r)dr)d\tau + \lambda_5(s) \int_{t_0}^{s} \lambda_6(\tau)d\tau\Big)ds\Big],$$

where $t_0 \leq t < b_1$, W, W^{-1} *are the same functions as in Lemma 2.4, and*

$$b_1 = \sup\Big\{t \geq t_0 : W(c) + \int_{t_0}^{t} \Big(\lambda_1(s) \int_{t_0}^{s} (\lambda_2(\tau) + \lambda_3(\tau) \int_{t_0}^{\tau} \lambda_4(r)dr)d\tau + \lambda_5(s) \int_{t_0}^{s} \lambda_6(\tau)d\tau\Big)ds \in \mathrm{domW}^{-1}\Big\}.$$

Lemma 2.10. [13] *Let* $u, \lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5, \lambda_6, \lambda_7 \in C(\mathbb{R}^+)$, $w \in C((0,\infty))$, *and* $w(u)$ *be nondecreasing in* u, $u \leq w(u)$. *Suppose that for some* $c > 0$ *and* $0 \leq t_0 \leq t$,

$$u(t) \le c + \int_{t_0}^{t} \lambda_1(s) \int_{t_0}^{s} \Big(\lambda_2(\tau)u(\tau) + \lambda_3(\tau)w(u(\tau)) + \lambda_4(\tau) \int_{t_0}^{\tau} \lambda_5(r)u(r)dr + \lambda_6(\tau) \int_{t_0}^{\tau} \lambda_7(r)w(u(r))dr \Big) d\tau ds.$$

Then

$$u(t) \le W^{-1}\Big[W(c) + \int_{t_0}^{t} \lambda_1(s) \int_{t_0}^{s} \Big(\lambda_2(\tau) + \lambda_3(\tau) + \lambda_4(\tau) \int_{t_0}^{\tau} \lambda_5(r)dr + \lambda_6(\tau) \int_{t_0}^{\tau} \lambda_7(r)dr \Big) d\tau ds \Big],$$

where $t_0 \le t < b_1$, W, W^{-1} *are the same functions as in Lemma 2.4, and*

$$b_1 = \sup \Big\{ t \ge t_0 : W(c) + \int_{t_0}^{t} \lambda_1(s) \int_{t_0}^{s} \Big(\lambda_2(\tau) + \lambda_3(\tau) + \lambda_4(\tau) \int_{t_0}^{\tau} \lambda_5(r)dr + \lambda_6(\tau) \int_{t_0}^{\tau} \lambda_7(r)dr \Big) d\tau ds \in \mathrm{domW}^{-1} \Big\}.$$

3. Main results

In this section, we investigate uniformly Lipschitz stability and asymptotic property for solutions of the perturbed differential systems.

THEOREM 3.1. *For the perturbed (2.2), we assume that*

$$|g(t, y, Ty)| \le a(t)|y(t)| + b(t)w(|y(t)|) + |Ty(t)| \tag{3.1}$$

and

$$|Ty(t)| \le c(t) \int_{t_0}^{t} k(s)w(|y(s)|)ds \tag{3.2}$$

where $a, b, c, k \in C(\mathbb{R}^+)$, $a, b, c, k, w \in L^1(\mathbb{R}^+)$, $w \in C((0, \infty))$, *and* $w(u)$ *is nondecreasing in* u, $u \le w(u)$, *and* $\frac{1}{v}w(u) \le w(\frac{u}{v})$ *for some* $v > 0$,

$$M(t_0) = W^{-1}\Big[W(M) + M \int_{t_0}^{\infty} \int_{t_0}^{s} \Big(a(\tau) + b(\tau) + c(\tau) \int_{t_0}^{\tau} k(r)dr \Big) d\tau ds \Big], \tag{3.3}$$

where $M(t_0) < \infty$ *and* $b_1 = \infty$. *If the zero solution of (2.1) is ULSV, then the zero solution of (2.2) is ULS.*

Proof. Using the nonlinear variation of constants formula of Alekseev[1], the solutions of (2.1) and (2.2) with the same initial value are related by

$$y(t,t_0,y_0) = x(t,t_0,y_0) + \int_{t_0}^{t} \Phi(t,s,y(s)) \int_{t_0}^{s} g(\tau,y(\tau),Ty(\tau))d\tau ds.$$

Since $x = 0$ of (2.1) is ULSV, it is ULS([8],Theorem 3.3). Using the ULSV condition of $x = 0$ of (2.1), together with (3.1) and (3.2), we have

$$\begin{aligned} |y(t)| &\le |x(t)| + \int_{t_0}^{t} |\Phi(t,s,y(s))| \int_{t_0}^{s} |g(\tau,y(\tau),Ty(\tau))| d\tau ds \\ &\le M|y_0| + \int_{t_0}^{t} M \int_{t_0}^{s} \Big(a(\tau)|y(\tau)| + b(\tau)w(|y(\tau)|) \\ &\quad + c(\tau) \int_{t_0}^{\tau} k(r)w(|y(r)|)dr \Big) d\tau ds. \end{aligned}$$

It follows that

$$\begin{aligned} |y(t)| \le M|y_0| + \int_{t_0}^{t} M|y_0| \int_{t_0}^{s} \Big(a(\tau)\frac{|y(\tau)|}{|y_0|} + b(\tau)w\Big(\frac{|y(\tau)|}{|y_0|}\Big) \\ + c(\tau) \int_{t_0}^{\tau} k(r)w\Big(\frac{|y(r)|}{|y_0|}\Big)dr \Big) d\tau ds \end{aligned}$$

since $\frac{1}{v}w(u) \le w(\frac{u}{v})$ for some $v > 0$. Define $u(t) = |y(t)||y_0|^{-1}$. Then, an application of Corollary 2.9 yields

$$|y(t)| \le |y_0| W^{-1}\Big[W(M) + M \int_{t_0}^{t} \int_{t_0}^{s} \Big(a(\tau) + b(\tau) + c(\tau) \int_{t_0}^{\tau} k(r)dr \Big) d\tau ds \Big],$$

Thus, by (3.3), we have $|y(t)| \le M(t_0)|y_0|$ for some $M(t_0) > 0$ whenever $|y_0| < \delta$. Hence, the proof is complete. □

Remark 3.2. Letting $a(t) = 0$ in Theorem 3.1, we obtain the same result as that of Theorem 3.2 in [6].

Theorem 3.3. *For the perturbed (2.2), we assume that*

$$\int_{t_0}^{t} |g(s,y(s),Ty(s))| ds \le a(t)|y(t)| + b(t)w(|y(t)|) + |Ty(t)| \tag{3.4}$$

and

$$|Ty(t)| \le c(t) \int_{t_0}^{t} k(s)w(|y(s)|)ds \tag{3.5}$$

where $a, b, c, k \in C(\mathbb{R}^+)$, $a, b, c, k, w \in L^1(\mathbb{R}^+)$, $w \in C((0,\infty))$, T *is a continuous operator, and* $w(u)$ *is nondecreasing in* u, $u \le w(u)$, *and* $\frac{1}{v}w(u) \le w(\frac{u}{v})$ *for some* $v > 0$,

$$(3.6)\quad M(t_0) = W^{-1}\Big[W(M)+M\int_{t_0}^{\infty}\Big(a(s)+b(s)+c(s)\int_{t_0}^{s}k(\tau)d\tau\Big)ds\Big],$$

where $M(t_0) < \infty$ *and* $b_1 = \infty$. *If the zero solution of (2.1) is ULSV, then the zero solution of (2.2) is ULS.*

Proof. Let $x(t) = x(t, t_0, y_0)$ and $y(t) = y(t, t_0, y_0)$ be solutions of (2.1) and (2.2), respectively. Since $x = 0$ of (2.1) is ULSV, it is ULS. Applying Lemma 2.3, together with (3.4) and (3.5), we have

$$|y(t)| \le M|y_0| + \int_{t_0}^{t} M\Big(a(s)|y(s)| + b(s)w(|y(s)|) + c(s)\int_{t_0}^{s}k(\tau)w(|y(\tau)|)d\tau\Big)ds.$$

which leads to

$$|y(t)| \le M|y_0| + \int_{t_0}^{t} M|y_0|\Big(a(s)\frac{|y(s)|}{|y_0|} + b(s)w\Big(\frac{|y(s)|}{|y_0|}\Big) + c(s)\int_{t_0}^{s}k(\tau)w\Big(\frac{|y(\tau)|}{|y_0|}\Big)d\tau\Big)ds$$

since $\frac{1}{v}w(u) \le w(\frac{u}{v})$ for some $v > 0$. Defining $u(t) = |y(t)||y_0|^{-1}$, then it follows from Corollary 2.6 that

$$|y(t)| \le |y_0|W^{-1}\Big[W(M) + M\int_{t_0}^{t}\Big(a(s) + b(s) + c(s)\int_{t_0}^{s}k(\tau)d\tau\Big)ds\Big].$$

Hence, by (3.6), we have $|y(t)| \le M(t_0)|y_0|$ for some $M(t_0) > 0$ whenever $|y_0| < \delta$. This completes the proof. □

Remark 3.4. Letting $a(t) = 0$ in Theorem 3.3, we obtain the same result as that of Theorem 3.3 in [6].

To obtain the asymptotic property, the following assumptions are needed:

(H1) The solution $x = 0$ of (2.1) is EASV.

(H2) $w(u)$ is nondecreasing in u, $u \le w(u)$.

Theorem 3.5. *Suppose that (H1), (H2), and the perturbing term* $g(t, y, Ty)$ *satisfies*

$$(3.7)\qquad |g(t, y(t), Ty(t))| \le e^{-\alpha t}\Big(a(t)|y(t)| + b(t)w(|y(t)|) + |Ty(t)|\Big)$$

and

$$|Ty(t)| \le c(t)\int_{t_0}^{t} k(s)|y(s)|ds + d(t)\int_{t_0}^{t} q(s)w(|y(s)|)ds \tag{3.8}$$

where $\alpha > 0$, $a,b,c,k,d,q \in C(\mathbb{R}^+)$, $a,b,c,k,d,q,w \in L^1(\mathbb{R}^+)$, $w \in C((0,\infty))$, T *is a continuous operator. If*

$$\begin{aligned} M(t_0) = W^{-1}\Big[W(c) + M\int_{t_0}^{\infty} e^{\alpha s}\int_{t_0}^{s}\Big(a(\tau)+b(\tau) \\ + c(\tau)\int_{t_0}^{\tau} k(r)dr + d(\tau)\int_{t_0}^{\tau} q(r)dr\Big)d\tau ds\Big] < \infty, \end{aligned} \tag{3.9}$$

where $t \ge t_0$ *and* $c = |y_0|Me^{\alpha t_0}$, *then all solutions of (2.2) approach zero as* $t \to \infty$

Proof. Let $x(t) = x(t,t_0,y_0)$ and $y(t) = y(t,t_0,y_0)$ be solutions of (2.1) and (2.2), respectively. Since the solution $x = 0$ of (2.1) is EASV, it is EAS by Remark 2.2. Using Lemma 2.3, together with (3.7) and (3.8), we have

$$\begin{aligned} &|y(t)| \\ &\le |x(t)| + \int_{t_0}^{t} |\Phi(t,s,y(s))|\int_{t_0}^{s} |g(\tau,y(\tau),Ty(\tau))|d\tau ds \\ &\le M|y_0|e^{-\alpha(t-t_0)} + \int_{t_0}^{t} Me^{-\alpha(t-s)}\int_{t_0}^{s} e^{-\alpha\tau}\Big(a(\tau)|y(\tau)| \\ &\quad + b(\tau)w(|y(\tau)|) + c(\tau)\int_{t_0}^{\tau} k(r)|y(r)|dr + d(\tau)\int_{t_0}^{\tau} q(r)w(|y(r)|)dr\Big)d\tau ds. \end{aligned}$$

Applying the assumption (H2), we obtain

$$\begin{aligned} |y(t)| \le M|y_0|e^{-\alpha(t-t_0)} + \int_{t_0}^{t} Me^{-\alpha(t-s)}\int_{t_0}^{s}\Big(a(\tau)|y(\tau)|e^{\alpha\tau} \\ +b(\tau)w(|y(\tau)|e^{\alpha\tau}) + c(\tau)\int_{t_0}^{\tau} k(r)|y(r)|e^{\alpha r}dr \\ +d(\tau)\int_{t_0}^{\tau} q(r)w(|y(r)|e^{\alpha r})dr\Big)d\tau ds. \end{aligned}$$

Let $u(t) = |y(t)|e^{\alpha t}$. An application of Lemma 2.10 and (3.9) yields

$$|y(t)| \le ce^{-\alpha t}W^{-1}\Big[W(c) + M\int_{t_0}^{t} e^{\alpha s}\int_{t_0}^{s}\Big(a(\tau)+b(\tau)$$
$$+c(\tau)\int_{t_0}^{\tau}k(r)dr + d(\tau)\int_{t_0}^{\tau}q(r)dr\Big)d\tau ds\Big] \le ce^{-\alpha t}M(t_0),$$

where $t \ge t_0$ and $c = M|y_0|e^{\alpha t_0}$. Hence, all solutions of (2.2) approach zero as $t \to \infty$. □

REMARK 3.6. Letting $b(t) = 0$ in Theorem 3.5, we obtain the same result as that of Theorem 3.1 in [13].

THEOREM 3.7. *Suppose that (H1), (H2), and the perturbed term* $g(t, y, Ty)$ *satisfies*

(3.10)
$$\int_{t_0}^{t}|g(s,y(s),Ty(s))|ds \le e^{-\alpha t}\Big(a(t)|y(t)| + b(t)w(|y(t)|) + |Ty(t)|\Big),$$

and

(3.11)
$$|Ty(t)| \le c(t)\int_{t_0}^{t} q(s)w(|y(s)|)ds$$

where $\alpha > 0$, $a, b, c, q, w \in C(\mathbb{R}^+)$, $a, b, c, q, w \in L^1(\mathbb{R}^+)$, T *is a continuous operator. If*

(3.12)
$$M(t_0) = W^{-1}\Big[W(c) + M\int_{t_0}^{\infty}\Big(a(s)+b(s)+c(s)\int_{t_0}^{s}q(\tau)d\tau\Big)ds\Big] < \infty,$$

where $b_1 = \infty$ *and* $c = M|y_0|e^{\alpha t_0}$, *then all solutions of (2.2) approach zero as* $t \to \infty$

Proof. Let $x(t) = x(t, t_0, y_0)$ and $y(t) = y(t, t_0, y_0)$ be solutions of (2.1) and (2.2), respectively. Since the solution $x = 0$ of (2.1) is EASV, it is EAS. Using Lemma 2.3, together with (3.10) and (3.11), we have

$$|y(t)| \le M|y_0|e^{-\alpha(t-t_0)} + \int_{t_0}^{t} Me^{-\alpha(t-s)}\Big(e^{-\alpha s}(a(s)|y(s)|$$
$$+ b(s)w(|y(s)|)) + c(s)\int_{t_0}^{s}q(\tau)w(|y(\tau)|)d\tau\Big)ds.$$

By the assumption (H2), we obtain

$$|y(t)| \le M|y_0|e^{-\alpha(t-t_0)} + \int_{t_0}^{t} Me^{-\alpha t}\Big(a(s)|y(s)|e^{\alpha s} + b(s)w(|y(s)|e^{\alpha s})$$
$$+c(s)\int_{t_0}^{s} q(\tau)w(|y(\tau)|e^{\alpha\tau})d\tau\Big)ds.$$

Set $u(t) = |y(t)|e^{\alpha t}$. Then, an application of Corollary 2.6 and (3.12) obtains

$$|y(t)| \le e^{-\alpha t}W^{-1}\Big[W(c) + M\int_{t_0}^{t}\Big(a(s) + b(s) + c(s)\int_{t_0}^{s} q(\tau)d\tau\Big)ds\Big]$$
$$\le e^{-\alpha t}M(t_0),$$

where $c = M|y_0|e^{\alpha t_0}$. Therefore, all solutions of (2.2) approach zero as $t \to \infty$. □

Remark 3.8. Letting $a(t) = 0$ in Theorem 3.7, we obtain the same result as that of Theorem 3.7 in [14].

Theorem 3.9. *Suppose that $x = 0$ of (2.1) is ULS and (H2). Consider the scalar differential equation*

(3.13)
$$u'(t) = KW(t,u,Tu) = K\Big(a(t)u(t)+b(t)w(u(t))+c(t)\int_{t_0}^{t} k(s)w(u(s))ds\Big),$$

where $w \in C((0,\infty)$, $u(t_0) = u_0 \ge 1, K \ge 1$ and $a,b,c,k \in C(\mathbb{R}^+)$ satisfy the conditions:

(a) $\int_{t_0}^{t} |g(s,y(s),Ty(s))|ds \le W(t,|y|,T|y|)$,where $\int_{t_0}^{t} g(s,y(s),Ty(s))ds$ is in (2.2),

(b) $M(t_0) = W^{-1}[W(u_0) + K\int_{t_0}^{\infty}(a(s) + b(s) + c(s)\int_{t_0}^{s} k(\tau)d\tau)ds] < \infty$, $b_1 = \infty$, and $a,b,c,k,w \in L^1(\mathbb{R}^+)$. Then, $y = 0$ of (2.2) is ULS.

Proof. Let $u(t) = u(t,t_0,u_0)$ be any solution of (3.13). Then, by Corollary 2.6, we obtain

$$|u(t)| \le W^{-1}\Big[W(u_0) + K\int_{t_0}^{t}\Big(a(s) + b(s) + c(s)\int_{t_0}^{s} k(\tau)d\tau\Big)ds)\Big]$$
$$\le M(t_0) \le M(t_0)|u_0|,$$

Hence $u = 0$ of (3.13) is ULS. This implies that the solution $y = 0$ of (2.2) is ULS by Theorem 2.7. □

Remark 3.10. Letting $b(t) = 0$ in Theorem 3.9, we obtain the same result as that of Corollary 3.3 in [14].

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