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UNIQUE POINT OF COINCIDENCE FOR TWO MAPPINGS WITH φ- OR ψ-ϕ-CONTRACTIVE CONDITIONS ON 2-METRIC SPACES

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Abstract. We discuss and obtain some existence theorems of unique point of coincidence for two mappings satisfying φ-contractive conditions or ψ-ϕ-contractive conditions determined by semi-continuous functions on non-complete 2-metric spaces, in which the mappings do not satisfy commutativity and uniform boundedness. The obtained results generalize and improve many well-known and corresponding conclusions.

1. Introduction and preliminaries

There have appeared many common fixed point theorems of mappings with some contractive conditions on 2-metric spaces. But most of them held under subsidiary conditions ([9, 11]), for example; commutativity of mappings or uniform boundness of mappings at some point, and so on. The authors in ([1, 2, 3, 4, 5, 6, 7, 8, 10]) obtained generalized results of coincidence points and common fixed points for infinite or finite family of mappings satisfying generalized linear or non-linear contractive or quasi-contractive conditions and expansive conditions under removing the above subsidiary conditions. These obtained results greatly generalize and improve the corresponding conclusions.

In this paper, we will introduce three real functions with some kind of properties to establish contractive conditions of two self-mappings on 2-metric spaces, and construct convergent sequences to discuss the existence problems of unique points of coincidence of the given mappings.

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Definition 1.1. ([5, 6, 9]) A 2-metric space (X, d) consists of a nonempty set X and a function $d : X \times X \times X \to [0, +\infty)$ such that

(i) for distant elements $x, y \in X$, there exists an $u \in X$ such that $d(x, y, u) \neq 0$;
(ii) $d(x, y, z) = 0 \iff$ at least two elements in $\{x, y, z\}$ are equal;
(iii) $d(x, y, z) = d(u, v, w)$, where $\{u, v, w\}$ is any permutation of $\{x, y, z\}$;
(iv) $d(x, y, z) \leq d(x, y, u) + d(x, u, z) + d(u, y, z)$ for all $x, y, z, u \in X$.

Definition 1.2. ([5, 6, 9]) A sequence $\{x_n\}_{n\in\mathbb{N}}$ in 2-metric space (X, d) is said to be a Cauchy sequence, if for each $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that $d(x_n, x_m, a) < \varepsilon$ for all $a \in X$ and $n, m > N$. $\{x_n\}_{n\in\mathbb{N}}$ is said to be convergent to $x \in X$, if for each $a \in X$, $\lim_{n\to+\infty} d(x_n, x, a) = 0$. And we write that $x_n \to x$ and call x the limit of $\{x_n\}_{n\in\mathbb{N}}$.

Definition 1.3. ([5, 6, 9]) A 2-metric space (X, d) is said to be complete, if every cauchy sequence in X is convergent.

Lemma 1.4. ([12]) *Let $\{x_n\}$ be a sequence in 2-metric space (X, d) such that $\lim_{n\to\infty} d(x_n, x_{n+1}, a) = 0$ for all $a \in X$. If $\{x_n\}$ is not a Cauchy sequence, then there exist $a \in X$ and $\epsilon > 0$ such that for each $i \in \mathbb{N}$ there exist $m(i), n(i) \in \mathbb{N}$ with $m(i), n(i) > i$ such that*

(i) *$m(i) > n(i)$ and $n(i) \to \infty$ as $i \to \infty$;*
(ii) *$d(x_{m(i)}, x_{n(i)}, a) > \epsilon$, but $d(x_{m(i)-1}, x_{n(i)}, a) \leq \epsilon$.*

Lemma 1.5. ([5, 6, 7]) *If a sequence $\{x_n\}$ in a 2-metric space (X, d) converges to $x \in X$, then*

$$\lim_{n\to\infty} d(x_n, b, c) = d(x, b, c), \forall\, b, c \in X.$$

Definition 1.6. ([5, 6, 7]) Let $f, g : X \to X$ be two mappings. If $w = fx = gx$ for some $w, x \in X$, then x is called a coincidence point of f and g, w is called a point of coincidence of f and g.

Definition 1.7. ([5, 6, 7]) Two mappings $f, g : X \to X$ are called be weakly compatible if $fgx = gfx$ whenever $fx = gx$ for $x \in X$

Lemma 1.8. ([5, 6, 7]) *If $f, g : X \to X$ are weakly compatible and have an unique point of coincidence w, then w is the unique common fixed point of f and g.*

2. Unique point of coincidence and common fixed point

Let $\varphi : [0, \infty) \to [0, \infty)$ be a non-decreasing function satisfying the following conditions:

(φ_1): $\varphi(0)=0$; (φ_2): $0<\varphi(t)<t$ for all $t>0$; (φ_3): $\sum_{n=1}^{\infty}\varphi^n(t)<\infty$ for each $t\in(0,\infty)$.

THEOREM 2.1. *Let (X,d) be a 2-metric space, $f,g:X\to X$ two mappings such that $fX\subset gX$. Suppose that*

$$d(fx,fy,a)\le\varphi\big(M(x,y,a)\big),\ \forall\ x,y,a\in X, \tag{2.1}$$

where $M(x,y,a)=\max\big\{d(gx,gy,a),d(gx,fx,a),d(gy,fy,a),\frac{1}{2}[d(gx,fy,a)+d(gy,fx,a)]\big\}$ and φ is upper semi-continuous. If fX or gX is complete, and X is bounded (i.e., $\sup_{x,y,z\in X}d(x,y,z)<+\infty$), then f and g have a unique point of coincidence. Furthermore, if f and g are weakly compatible, then f and g have a unique common fixed point.

Proof. Take $x_0\in X$ and construct sequences $\{x_n\}$ and $\{y_n\}$ satisfying

$$y_n=fx_n=gx_{n+1},\ n=0,1,2,\cdots.$$

For any fixed $n\ge1$, in view of (2.1),

$$\begin{aligned}
&d(y_n,y_{n+1},y_{n+2})\\
=\ &d(fx_{n+2},fx_{n+1},y_n)\\
\le\ &\varphi(M(x_{n+2},x_{n+1},y_n)),\\
=\ &\varphi\big(\max\big\{d(gx_{n+2},gx_{n+1},y_n),d(gx_{n+2},fx_{n+2},y_n),d(gx_{n+1},fx_{n+1},y_n),\\
&\qquad\frac{d(gx_{n+2},fx_{n+1},y_n)+d(gx_{n+1},fx_{n+2},y_n)}{2}\big\}\big)\\
=\ &\varphi\big(d(y_{n+2},y_{n+1},y_n)\big).
\end{aligned}$$

Hence by (φ_2),

$$d(y_n,y_{n+1},y_{n+2})=0,\ \forall\ n=1,2,,\cdots.$$

Suppose that $d(y_k,y_n,y_{n+1})=0$ for $n-k\ge1$, then by (2.1),

$$d(y_{n+1},y_{n+2},y_k)=d(fx_{n+1},fx_{n+2},y_k)\le\varphi(M(x_{n+1},x_{n+2},y_k)),$$

where

$$\begin{aligned}
&M(x_{n+1},x_{n+2},y_k)\\
=\ &\max\big\{d(gx_{n+1},gx_{n+2},y_k),d(gx_{n+1},fx_{n+1},y_k),d(gx_{n+2},fx_{n+2},y_k),\\
&\qquad\frac{d(gx_{n+1},fx_{n+2},y_k)+d(gx_{n+2},fx_{n+1},y_k)}{2}\big\}\\
=\ &\max\big\{d(y_{n+1},y_{n+2},y_k),\frac{d(y_n,y_{n+2},y_k)}{2}\big\}
\end{aligned}$$

But $d(y_n, y_{n+2}, y_k) \le d(y_n, y_{n+1}, y_k) + d(y_{n+1}, y_{n+2}, y_k) + d(y_n, y_{n+1}, y_{n+2}) = d(y_{n+1}, y_{n+2}, y_k)$, hence

$$M(x_{n+1}, x_{n+2}, y_k) = d(y_{n+1}, y_{n+2}, y_k),$$

therefore

$$d(y_k, y_{n+1}, y_{n+2}) \le \varphi(d(y_{n+1}, y_{n+2}, y_k)),$$

which implies that $d(y_k, y_{n+1}, y_{n+2}) = 0$ by (φ_2). Hence

$$d(y_k, y_n, y_{n+1}) = 0, \ \forall\ k, n \in \mathbb{N}, k \le n. \tag{2.2}$$

For every $m, n, l \in \mathbb{N}$ with $m \le n \le l$, using (2.2), we obtain

$$\begin{aligned} & d(y_m, y_n, y_l) \\ \le\ & d(y_m, y_n, y_{l-1}) + d(y_m, y_{l-1}, y_l) + d(y_n, y_{l-1}, y_l) = d(y_m, y_n, y_{l-1}) \\ \le\ & \cdots \\ \le\ & d(y_m, y_n, y_{n+1}) \\ =\ & 0. \end{aligned}$$

Hence we have

$$d(y_m, y_n, y_l) = 0, \forall\ m, n, l \in \mathbb{N}. \tag{2.3}$$

For any fixed $n \ge 1$ and $a \in X$,

$$d(y_n, y_{n+1}, a) = d(fx_n, fx_{n+1}, a) \le \varphi(M(x_n, x_{n+1}, a)),$$

where

$$\begin{aligned} & M(x_n, x_{n+1}, a) \\ =\ & \max\Big\{d(gx_n, gx_{n+1}, a), d(gx_n, fx_n, a), d(gx_{n+1}, fx_{n+1}, a), \\ & \qquad \frac{d(gx_n, fx_{n+1}, a) + d(gx_{n+1}, fx_n, a)}{2}\Big\} \\ =\ & \max\Big\{d(y_{n-1}, y_n, a), d(y_n, y_{n+1}, a), \frac{d(y_{n-1}, y_{n+1}, a)}{2}\Big\}. \end{aligned}$$

But using Definition 1.1(iv) and (2.3), we obtain

$$d(y_{n-1}, y_{n+1}, a) \le d(y_{n-1}, y_n, a) + d(y_n, y_{n+1}, a),$$

hence

$$M(x_n, x_{n+1}, a) = \max\big\{d(y_{n-1}, y_n, a), d(y_n, y_{n+1}, a)\big\}.$$

If there exists $a \in X$ such that $d(y_{n-1}, y_n, a) < d(y_n, y_{n+1}, a)$, then $M(x_n, x_{n+1}, a) = d(y_n, y_{n+1}, a) > 0$, hence by (φ_2), we have

$$d(y_n, y_{n+1}, a) \le \varphi(M(x_n, x_{n+1}, a)) = \varphi(d(y_n, y_{n+1}, a)) < d(y_n, y_{n+1}, a),$$

which is a contradiction. So $M(x_n, x_{n+1}, a) = d(y_{n-1}, y_n, a)$ for all $n \in \mathbb{N}$ and $a \in X$, therefore, for all $n \in \mathbb{N}$ and $a \in X$,

$$d(y_n, y_{n+1}, a) \leq \varphi(M(x_n, x_{n+1}, a)) = \varphi(d(y_{n-1}, y_n, a)).$$

Continuing this process, we obtain that for all $n \in \mathbb{N}$ and $a \in X$,

$$d(y_n, y_{n+1}, a) \leq \varphi^n(d(y_0, y_1, a)).$$

Since X is bounded, there exists $M > 0$ such that $d(y_0, y_1, a) < M$ for all $a \in X$. So we have

$$d(y_n, y_{n+1}, a) \leq \varphi^n(M), \ \forall n \in \mathbb{N}, \, a \in X.$$

For any $m, n \in \mathbb{N}$ with $n > m$ and $a \in X$,

$$\begin{aligned}
& d(y_m, y_n, a) \\
\leq\ & d(y_m, y_{m+1}, a) + d(y_{m+1}, y_n, a) + d(y_m, y_n, y_{m+1}) \\
=\ & d(y_m, y_{m+1}, a) + d(y_{m+1}, y_n, a) \\
\leq\ & d(y_m, y_{m+1}, a) + d(y_{m+1}, y_{m+2}, a) + d(y_{m+2}, y_n, a) + d(y_{m+1}, y_n, y_{m+2}) \\
=\ & d(y_m, y_{m+1}, a) + d(y_{m+1}, y_{m+2}, a) + d(y_{m+2}, y_n, a) \\
\leq\ & \cdots \\
\leq\ & d(y_m, y_{m+1}, a) + d(y_{m+1}, y_{m+2}, a) + \cdots + d(y_{n-1}, y_n, a) \\
\leq\ & \sum_{k=m}^{n-1} \varphi^k(M).
\end{aligned}$$

Hence by (φ_3), we know that $\{y_n\}$ is a Cauchy sequence in X.

Suppose that gX is complete, then there exist $u, v \in X$ such that $y_n = fx_n = gx_{n+1} \to u = gv$ as $n \to \infty$.

For any n and $a \in X$,

$$\begin{aligned}
& d(y_n, fv, a) = d(fx_n, fv, a) \\
\leq\ & \varphi(M(x_n, v, a) \\
=\ & \varphi(\max\{d(gx_n, gv, a), d(gx_n, fx_n, a), d(gv, fv, a), \frac{d(gx_n, fv, a) + d(gv, fx_n, a)}{2}\}) \\
=\ & \varphi(\max\{d(y_{n-1}, gv, a), d(y_{n-1}, y_n, a), d(gv, fv, a), \frac{d(y_{n-1}, fv, a) + d(gv, y_n, a)}{2}\}).
\end{aligned}$$

Letting $n \to \infty$, then we obtain from the above and Lemma 1.5 that

$$
\begin{aligned}
d(gv, fv, a) &= \lim_{n\to\infty} d(y_n, fv, a) \\
&\le \limsup_{n\to\infty} \varphi(\max\{d(y_{n-1}, gv, a), d(y_{n-1}, y_n, a), d(gv, fv, a), \\
&\qquad \frac{d(y_{n-1}, fv, a) + d(gv, y_n, a)}{2}\}) \\
&\le \varphi(\lim_{n\to\infty} \max\{d(y_{n-1}, gv, a), d(y_{n-1}, y_n, a), d(gv, fv, a), \\
&\qquad \frac{d(y_{n-1}, fv, a) + d(gv, y_n, a)}{2}\}) \\
&= \varphi(d(gv, fv, a)).
\end{aligned}
$$

This implies that

$$d(gv, fv, a) = 0, \ \forall\ a \in X,$$

hence

$$fv = gv = u.$$

So u is a point of coincidence of f and g.

Suppose that u_1 is another point of coincidence of f and g, then there exists v_1 satisfying $u_1 = fv_1 = gv_1$, and there exists $a \in X$ satisfying $d(u, u_1, a) > 0$. By (2.1) and (φ_2), we obtain the following contradiction

$$
\begin{aligned}
d(u, u_1, a) &= d(fv, fv_1, a) \\
&\le \varphi(M(v, v_1, a)) = \varphi(\max\{d(gv, gv_1, a), d(gv, fv, a), d(gv_1, fv_1, a), \\
&\qquad \frac{d(gv, fv_1, a) + d(gv_1, fv, a)}{2}\}) \\
&= \varphi(d(u, u_1, a)) < d(u, u_1, a).
\end{aligned}
$$

Hence u is the unique point of coincidence of f and g.

Suppose that fX is complete. Then there exist $u, v, w \in X$ such that $y_n = fx_n \to u = fw = gv$ since $fX \subset gX$, hence the corresponding conclusion follows from the similar discussion, and the rest proof follows from Lemma 1.8. □

A mapping $\psi : [0, \infty) \to [0, \infty)$ is called an altering distance function if ψ is continuous and non-decreasing and $\psi(t) = 0 \Leftrightarrow t = 0$.

Theorem 2.2. *Let (X, d) be a 2-metric space, $f, g : X \to X$ two mappings such that $fX \subset gX$. Suppose that*

(2.4) $\quad \psi(d(fx, fy, a)) \le \psi(M(x, y, a)) - \phi(M(x, y, a)), \forall\ x, y, a \in X,$

where, $M(x, y, a)$ is that in Theorem 2.1, ψ is an altering distance function, $\phi : [0, \infty) \to [0, \infty)$ is a lower semi-continuous function such that

$\phi(t) = 0 \Leftrightarrow t = 0$. *If* fX *or* gX *is complete, then* f *and* g *have a unique point of coincidence. Furthermore, if* f *and* g *are weakly compatible, then* f *and* g *have a unique common fixed point.*

Proof. Take $x_0 \in X$ and construct sequences $\{x_n\}$ and $\{y_n\}$ satisfying

$$y_n = fx_n = gx_{n+1}, \ \forall n = 0, 1, 2, 3, \cdots .$$

For any $n \in \mathbb{N}$,

$$\psi(d(fx_{n+1}, fx_{n+2}, y_n)) \le \psi(M(x_{n+1}, x_{n+2}, y_n)) - \phi(M(x_{n+1}, x_{n+2}, y_n)),$$

where $M(x_{n+1}, x_{n+2}, y_n) = d(y_{n+1}, y_{n+2}, y_n)$ (see Theorem 2.1). Hence

$$\psi(d(y_{n+1}, y_{n+2}, y_n)) \le \psi(d(y_{n+1}, y_{n+2}, y_n)) - \phi(d(y_{n+1}, y_{n+2}, y_n)),$$

so

$$\phi(d(y_{n+1}, y_{n+2}, y_n)) = 0, \ \forall \ n \in \mathbb{N}.$$

By the property of ϕ,

$$d(y_n, y_{n+1}, y_{n+2}) = 0, \ \forall \ n \in \mathbb{N}. \tag{2.5}$$

Suppose that $d(y_k, y_n, y_{n+1}) = 0$, where $n \ge k+1$. Using (2.4), we have

$$\psi(d(fx_{n+1}, fx_{n+2}, y_k)) \le \psi(M(x_{n+1}, x_{n+2}, y_k)) - \phi(M(x_{n+1}, x_{n+2}, y_k)),$$

where $M(x_{n+1}, x_{n+2}, y_k) = \max\{d(y_{n+1}, y_{n+2}, y_k), \frac{d(y_n, y_{n+2}, y_k)}{2}\}$(see Theorem 2.1). By (2.5) and the assumption,

$$\begin{aligned} d(y_n, y_{n+2}, y_k) &\le d(y_n, y_{n+1}, y_{n+2}) + d(y_{n+1}, y_{n+2}, y_k) + d(y_n, y_{n+1}, y_k) \\ &= d(y_{n+1}, y_{n+2}, y_k), \end{aligned}$$

so

$$M(x_{n+1}, x_{n+2}, y_k) = d(y_{n+1}, y_{n+2}, y_k).$$

Hence

$$\psi(d(y_k, y_{n+1}, y_{n+2})) \le \psi(d(y_k, y_{n+1}, y_{n+2})) - \phi(d(y_k, y_{n+1}, y_{n+2})),$$

which implies that

$$\phi(d(y_k, y_{n+1}, y_{n+2})) = 0 \Longrightarrow d(y_k, y_{n+1}, y_{n+2}) = 0. \tag{2.6}$$

Therefore, in view of (2.5) and (2.6), we have the next fact:

$$d(y_k, y_n, y_{n+1}) = 0, \ \forall \ n, k \in \mathbb{N}, \ n \ge k \ge 1. \tag{2.7}$$

For all $m, n, k \in \mathbb{N}$ with $k \ge n \ge m$, using (2.7), we have

$$\begin{aligned} d(y_m, y_n, y_k) &\le d(y_m, y_n, y_{k-1}) + d(y_m, y_{k-1}, y_k) + d(y_n, y_{k-1}, y_k) \\ &= d(y_m, y_n, y_{k-1}). \end{aligned}$$

Continuing this process, we obtain the following fact: for all $k \ge n \ge m$,

$$d(y_m, y_n, y_k) \le d(y_m, y_n, y_{k-1}) \le \cdots \le d(y_m, y_n, y_{n+1}) = 0. \tag{2.8}$$

For any fixed $n \in \mathbb{N}$ and any $a \in X$,

$$\begin{aligned}\psi(d(y_{n+1}, y_{n+2}, a)) &= \psi(d(fx_{n+1}, fx_{n+2}, a)) \\ &\leq \psi(M(x_{n+1}, x_{n+2}, a)) - \phi(M(x_{n+1}, x_{n+2}, a)),\end{aligned}$$

where $M(x_{n+1}, x_{n+2}, a) = \max\{d(y_n, y_{n+1}, a), (y_{n+1}, y_{n+2}, a)\}$(see Theorem 2.1).

If $d(y_n, y_{n+1}, a) < (y_{n+1}, y_{n+2}, a)$ for some $a \in X$, then $M(x_{n+1}, x_{n+2}, a) = d(y_{n+1}, y_{n+2}, a) > 0$. Hence using the property of ϕ, we have

$$\begin{aligned}\psi(d(y_{n+1}, y_{n+2}, a)) &\leq \psi(d(y_{n+1}, y_{n+2}, a)) - \phi(d(y_{n+1}, y_{n+2}, a)) \\ &< \psi(d(y_{n+1}, y_{n+2}, a)),\end{aligned}$$

which is a contradiction. Hence

$$M(x_{n+1}, x_{n+2}, a) = d(y_n, y_{n+1}, a), \ \forall\, a, n$$

and we have

$$\begin{aligned}\psi(d(y_{n+1}, y_{n+2}, a)) &\leq \psi(d(y_n, y_{n+1}, a)) - \phi(d(y_n, y_{n+1}, a)) \\ &\leq \psi(d(y_n, y_{n+1}, a)), \ \forall\, a, n.\end{aligned} \tag{2.9}$$

By the property of ψ, we obtain that

$$d(y_{n+1}, y_{n+2}, a) \leq d(y_n, y_{n+1}, a), \ \forall\, a, n.$$

So for any fixed $a \in X$, $\{d(y_n, y_{n-1}, a)\}$ is a non-increasing and non-negative real sequence, hence there exists $r(a) \geq 0$ such that

$$\lim_{n\to\infty} d(y_n, y_{n-1}, a) = r(a).$$

Letting $n \to \infty$ in the both sides of the first inequality in (2.9), we obtain

$$\begin{aligned}\psi(r(a)) \leq&\ \psi(r(a)) - \liminf_{n\to\infty} \phi(d(y_n, y_{n+1}, a)) \\ \leq&\ \psi(r(a)) - \phi(\lim_{n\to\infty} d(y_n, y_{n+1}, a)) \\ =&\ \psi(r(a)) - \phi(r(a)),\end{aligned}$$

hence $\phi(r(a)) = 0$, which implies that $r(a) = 0$. Therefore, we have

$$\lim_{n\to\infty} d(y_n, y_{n+1}, a) = 0, \forall\, a \in X. \tag{2.10}$$

If $\{y_n\}$ is not Cauchy, then by Lemma 1.4, there exist $a \in X$ and $\epsilon > 0$ such that for any $i \in \mathbb{N}$ there exist $m(i) > n(i) \in \mathbb{N}$ satisfying

(i) $m(i), n(i) > i$, $m(i) > n(i) + 1$ and $n(i) \to \infty$ as $i \to \infty$;

(ii) $d(y_{m(i)}, y_{n(i)}, a) > \epsilon$, but $d(y_{m(i)-1}, y_{n(i)}, a) \leq \epsilon, i = 1, 2, \cdots$.

Using (2.8) and (2.10) and the following result

$$d(y_{m(i)}, y_{n(i)}, a) \\ \leq d(y_{m(i)}, y_{m(i)-1}, a) + d(y_{m(i)-1}, y_{n(i)}, a) + d(y_{m(i)}, y_{n(i)}, y_{m(i)-1}),$$

we obtain

$$\lim_{i\to\infty} d(y_{m(i)}, y_{n(i)}, a) = \lim_{i\to\infty} d(y_{m(i)-1}, y_{n(i)}, a) = \epsilon. \tag{2.11}$$

The following two inequalities hold

$$|d(y_{m(i)}, y_{n(i)}, a) - d(y_{m(i)}, y_{n(i)-1}, a)| \\ \leq d(y_{n(i)-1}, y_{n(i)}, a) + d(y_{m(i)}, y_{n(i)}, y_{n(i)-1})$$

and

$$|d(y_{m(i)-1}, y_{n(i)-1}, a) - d(y_{m(i)}, y_{n(i)-1}, a)| \\ \leq d(y_{m(i)-1}, y_{m(i)}, a) + d(y_{m(i)}, y_{m(i)-1}, y_{n(i)-1}),$$

hence using (2.8), (2.10) and (2.11), we have

$$\begin{aligned} &\lim_{n\to\infty} d(y_{m(i)}, y_{n(i)}, a) = \lim_{n\to\infty} d(y_{m(i)-1}, y_{n(i)}, a) \\ &= \lim_{i\to\infty} d(y_{m(i)}, y_{n(i)-1}, a) = \lim_{i\to\infty} d(y_{m(i)-1}, y_{n(i)-1}, a) = \epsilon. \end{aligned} \tag{2.12}$$

Since

(2.13)

$$\begin{aligned} \psi(d(y_{m(i)}, y_{n(i)}, a)) &= \psi(d(fx_{m(i)}, fx_{n(i)}, a)) \\ &\leq \psi(M(x_{m(i)}, x_{n(i)}, a)) - \phi(M(x_{m(i)}, x_{n(i)}, a)), \end{aligned}$$

where

$$\begin{aligned} &M(x_{m(i)}, x_{n(i)}, a) \\ = \ &\max\Big\{d(gx_{m(i)}, gx_{n(i)}, a), d(gx_{m(i)}, fx_{m(i)}, a), d(gx_{n(i)}, fx_{n(i)}, a), \\ &\qquad \frac{d(gx_{m(i)}, fx_{n(i)}, a) + d(gx_{n(i)}, fx_{m(i)}, a)}{2}\Big\} \\ = \ &\max\Big\{d(y_{m(i)-1}, y_{n(i)-1}, a), d(y_{m(i)-1}, y_{m(i)}, a), d(y_{n(i-1)}, y_{n(i)}, a), \\ &\qquad \frac{d(y_{m(i)-1}, y_{n(i)}, a) + d(y_{n(i)-1}, y_{m(i)}, a)}{2}\Big\}. \end{aligned}$$

By (2.10) and (2.12), we know

$$\lim_{i\to\infty} M(x_{m(i)}, x_{n(i)}, a) = \epsilon. \tag{2.14}$$

Letting $i \to \infty$ in (2.13) and using (2.12) and (2.14), we obtain

$$\begin{aligned}\psi(\epsilon) &\leq \psi(\epsilon) - \liminf_{n\to\infty} \phi(M(x_{m(i)}, x_{n(i)}, a)) \\ &\leq \psi(\epsilon) - \phi(\lim_{n\to\infty} M(x_{m(i)}, x_{n(i)}, a)) = \psi(\epsilon) - \phi(\epsilon),\end{aligned}$$

which implies that $\phi(\epsilon) = 0$, i.e., $\epsilon = 0$. This is a contradiction, hence $\{y_n\}$ is a Cauchy sequence.

Suppose that gX is complete. Then there exist $u, v \in X$ such that $y_n = fx_n = gx_{n+1} \to u = gv$. For any n and $a \in X$, we have that

$$\psi(y_n, fv, a) = d(fx_n, fv, a) \leq \psi(M(x_n, v, a) - \phi(M(x_n, v, a)), \quad (2.15)$$

where

$$\begin{aligned}&M(x_n, v, a)\\ =\ &\max\big\{d(gx_n, gv, a), d(gx_n, fx_n, a), d(gv, fv, a), \frac{d(gx_n, fv, a) + d(gv, fx_n, a)}{2}\big\}\\ =\ &\max\big\{d(y_{n-1}, gv, a), d(y_{n-1}, y_n, a), d(gv, fv, a), \frac{d(y_{n-1}, fv, a) + d(gv, y_n, a)}{2}\big\}.\end{aligned}$$

Let $n \to \infty$, then

$$\lim_{n\to\infty} M(x_n, v, a) = d(gv, gv, a),\ \forall\ a \in X.$$

Hence Letting $n \to \infty$ in (2.15), we obtain that

$$\begin{aligned}&\psi(fv, gv, a)\\ \leq\ &\psi(d(fv, gv, a)) - \liminf_{n\to\infty} \phi(M(x_n, v, a))\\ \leq\ &\psi(d(fv, gv, a)) - \phi(\lim_{n\to\infty} M(x_n, v, a))\\ =\ &\psi(d(fv, gv, a)) - \phi(d(fv, gv, a)),\ \forall\ a \in X.\end{aligned}$$

This implies that $\phi(d(fv, gv, a)) = 0,\ \forall\ a \in X \implies d(fv, gv, a) = 0,\ \forall\ a \in X$, hence

$$fv = gv = u.$$

Suppose that u_1 is another point of coincidence of f and g, then there exists v_1 satisfying $u_1 = fv_1 = gv_1$, and there exists $a \in X$ satisfying $d(u, u_1, a) > 0$. By (2.4)

$$\psi(d(u, u_1, a)) = \psi(d(fv, fv_1, a)) \leq \psi(M(v, v_1, a)) - \phi(M(v, v_1, a)),$$

where $M(v, v_1, a) = d(u, u_1, a)$, hence

$$\psi(d(u, u_1, a)) \leq \psi(d(u, u_1, a)) - \phi(d(u, u_1, a)),$$

which implies $d(u, u_1, a) = 0$, a contradiction. So u is the unique pint of coincidence of f and g.

Suppose that fX is complete. Then there exist $u, v, w \in X$ such that $y_n = fx_n \to u = fw = gv$ since $fX \subset gX$. The rest proof follows from the similar discussion and Lemma 1.8. □

Remark 2.3. (1) φ in Theorem 2.1 need not be a strictly increasing function.

(2) We find that the condition $\psi(t) = 0 \Longleftrightarrow t = 0$ in Theorem 2.2 is superfluous. Hence we only need that ψ is a continuous and non-decreasing function.

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