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ALMOST OPEN AND ALMOST HOMEOMORPHISMS

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Abstract. This paper is devoted to the study of various notions of almost openness and almost homeomorphisms and the characterization of some of them in terms of the relative interior operator. Generally, openness (or quasi-openness) for a continuous map f is well known. We define openness (or quasi-openness) at a point x using the relative interior operator and characterize that a continuous map f is almost open, almost quasi-open, almost embedding and almost homeomorphsims.

1. Introduction and preliminaries

The main purpose of this paper is to extend the following Theorem A and Theorem B for almost continuity maps and homeomorphisms to open maps and almost homeomorphisms.

Theorem A [6]. Let E be a Baire Space, F be a second countable space and f be a mapping of E into F. Then the set of points of almost continuity of f is dense in E.

Theorem B [8]. Let $f : X \to Y$ and $g : Y \to Z$ be continuous. If $g \circ f : X \to Z$ is a homeomorphism, then g one-to-one (or f onto) implies that f and g are homeomorphisms.

We prove here the following results.

Theorem A$'$. Let $f : X_1 \to X_2$ be a continuous, closed and proper map. For every $V_1 \in \mathcal{U}_{X_1}$ the set $\text{Int}\{y \in X_2 : f \text{ is } V_1 \text{ open at each } x \in f^{-1}(y)\}$ is an open and dense subset of X_2. In particular, if X_1 is metrizable and X_2 is Baire, then $\{y \in X_2 : f \text{ is } V_1 \text{ open at each } x \in f^{-1}(y)\}$ is a residual subset of X_2.

Theorem B$'$. Let $f : X_1 \to X_2$ and $g : X_2 \to X_3$ be continuous maps. Then

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(1) Assume f is surjective. $g \circ f$ is an almost homeomorphism if and only if both f and g are almost homeomorphisms.
(2) Assume g is an almost homeomorphism. If $g \circ f$ is almost open, then f is almost quasi-open.
(3) Assume g is an almost homeomorphism. If $g \circ f$ is almost quasi-open and g is closed, then f is almost quasi-open.

We now introduce notions and definitions necessary for our works. Throu ghout the present paper, X_1 and X_2 always mean topological spaces and by $f : X_1 \to X_2$ we denote a map. Let A and B be subsets of X. The closure of A and the interior of A are denoted by $\overline{A}$ and Int$A$, respectively. A subset B of A is dense in A means that $\overline{A} \subset \overline{B}$. A subset A of X is said to be pre-open if $A \subset \text{Int}\overline{A}$ in [6]. This set is called by quasi-open in [2].

A *relation* $F : X_1 \to X_2$ is considered a map from X_1 to the power set of X_2, that is, each $x \in X_1$ corresponds to a subset $F(x)$ of X_2, or a subset of $X_1 \times X_2$ so that $y \in F(x)$ means $(x,\ y) \in F$.

For relations $F : X_1 \to X_2$ and $G : X_2 \to X_3$ we define the *inverse* $F^{-1} : X_2 \to X_1$ and the *composition* $G \circ F$(simply GF): $X_1 \to X_3$ by

$$x \in F^{-1}(y) \iff y \in F(x), \text{ i.e., } F^{-1} = \{(y,\ x) \mid (x,\ y) \in F\}.$$
$$y \in (GF)(x) \iff z \in F(x) \text{ and } y \in G(z) \text{ for some } z \in X_2.$$

In other words, GF is the projection to $X_1 \times X_3$ of the subset $\{(x,\ z,\ y) \in X_1 \times X_2 \times X_3 \mid (x,\ z) \in F \text{ and } (z,\ y) \in G\}$.

A relation $F : X_1 \to X_2$ is called a *closed relation* if it is a closed subset of $X_1 \times X_2$. It is a *pointwise closed relation* if $F(x)$ is a closed subset of X_2 for every $x \in X_1$. Clearly, a closed relation is a pointwise closed relation. F is called a *compact relation* if $F(x)$ is a compact subset of X_2 for any $x \in X_1$.

Remark that in general, a compact relation need not be a closed relation. For example, let $X = [0,\ 1]$ and $F = \{([0,\ \frac{1}{2}) \cup (\frac{1}{2},\ 1]) \times [\frac{1}{3},\ \frac{2}{3}]\} \cup \{(\frac{1}{2},\ \frac{1}{2})\}$. Then F is a compact and pointwise closed relation but F is not a closed relation. We are concerned with subsets of a cartesian product $X \times X$ of a set with itself. These subsets are relations on X. If $U = U^{-1}$, then U is called *symmetric*. The set of all pairs $(x,\ x)$ for x in X is called the *identity relation*, or the *diagonal*, and is denoted by $\triangle(X)$ or simply $\triangle$. For each subset A of X the set $U(A)$ is defined to be $\{y : (x,\ y) \in U \text{ for some } x \text{ in } A\}$, and if x is a point of X, then $U(x)$ is $U(\{x\})$. For each U and V and each A it is true that $(U \circ V)(A) = U(V(A))$. Finally, a simple definition will be needed.

Definition 1.1. [4] A *uniformity* for a set X is a non-void family $\mathcal{U}_X$ of subsets of $X \times X$ such that

(1) each member of U_X contains the diagonal $\triangle$;
(2) If $U \in \mathcal{U}_X$, then $U^{-1} \in \mathcal{U}_X$;
(3) if $U \in \mathcal{U}_X$, then there exists $V \in \mathcal{U}_X$ such that $VV = V^2 \subset U$;
(4) if U and V are members of $\mathcal{U}_X$, then $U \cap V \in \mathcal{U}_X$; and
(5) if $U \in \mathcal{U}_X$ and $U \subset V \subset X \times X$, then $V \in \mathcal{U}_X$.
The pair $(X,\ \mathcal{U}_X)$ (simply denote $\mathcal{U}_X$) is a *uniform space*.

The metric antecedents of the conditions above are not hard to discern. The first is derived from the condition that $d(x,\ x) = 0$ and the second derives from the condition that $d(x,\ y) = d(y,\ x)$. The third is a vestigial form of the triangle inequality - it says roughly that for r-spheres there are $(r/2)$- spheres. The fourth and fifth resemble axioms for the neighborhood system of a point and they will be used to derive the corresponding properties for a neighborhood system relative to a topology which will presently be defined.

There may be many different uniformities for a set X. The largest of these is the family of all those subsets of $X \times X$ which contain $\triangle$ and the smallest is the family whose only member is $X \times X$. If X is the set of real numbers the *usual uniformity* for X is the family $\mathcal{U}_X$

2. Semi-continuous relations

In this section, we develop the fundamentals of the upper and lower semi-continuous relations.

Let $f : X_1 \to X_2$ be a map. We define the *equivalence relation*:
$$E_f = (f \times f)^{-1}(1_{X_2}) = \{(x_1,\ x_2) : f(x_2) = f(x_1)\} = f^{-1} \circ f.$$

DEFINITION 2.1. Let $F : X_1 \to X_2$ and $H : X_2 \to X_2$ be relations. we define the relation F^*H on X_1 by:
$$F^*H = \{(x_1,\ x_2) : F(x_2) \subset (H \circ F)(x_1)\}.$$

PROPOSITION 2.2. *Let $F : X_1 \to X_2$ be a relation. The following properties hold:*
(1) *For relations $H_1,\ H_2$ on X_2 we have*
$$H_1 \subset H_2 \text{ implies } F^*H_1 \subset F^*H_2 \text{ and}$$
$$F^*(H_1)F^*(H_2) \subset F^*(H_1H_2).$$
(2) *For relations H on X_2 and K on X_1:*
$$FK \subset HF \text{ if and only if } K \subset F^*H.$$
For any relation H on X_2:
$$F(F^*H) \subset HF.$$
with equality when F is a surjective map.

(3) $1^*_{X_2}H = H$, *and if* $G : X_0 \to X_1$ *and* $F : X_1 \to X_2$ *are relations, then:*

$$G^*(F^*H) \subset (FG)^*H$$

with equality when G *is a map.*

(4) *For a map* $f : X_1 \to X_2$ *and relations* $F : X_0 \to X_1$ *and* $G : X_0 \to X_2$*:*

$$fF \subset G \iff F \subset f^{-1}G.$$

(5) *For a map* $f : X_1 \to X_2$ *and relation* H *on* X_2*:*

$f^*H = f^{-1}Hf = (f \times f)^{-1}(H) = \{(x_1,\ x_2) : f(x_2) \in H(f(x_1))\}$.

So when f *is a map:*

$$f^*(H^{-1}) = (f^*H)^{-1} \text{ and}$$
$$E_f(f^*H)E_f = f^*H.$$

Proof. (1) By the definition of F^*H, $F^*H_1 \subset F^*H_2$ for every $H_1 \subset H_2$. Let $(x_1,\ x_3) \in F^*(H_1)F^*(H_2)$ be given. By the definition of composition, we can find $x_2 \in X_2$ such that $(x_1,\ x_2) \in F^*(H_2)$ and $(x_2,\ x_3) \in F^*(H_1)$. In other words, $F(x_2) \subset H_2F(x_1)$ and $F(x_3) \subset H_1F(x_2)$. Hence $F(x_3) \subset H_1F(x_2) \subset H_1H_2F(x_1)$.

(2) Let A be a subset of X_1 and $x \in A$. If $F(A) \subset HF(x_1)$, then $F(x) \subset HF(x_1)$. Hence $(x_1,\ x) \in F^*H$, i.e., $x \in (F^*H)(x_1)$. Conversely, we assume that $A \subset (F^*H)(x_1)$. If $y \in F(A)$, then there exists $x \in A \subset (F^*H)(x_1)$ such that $y \in F(x) \subset HF(x_1)$. Hence we obtain the following property.

$$F(A) \subset HF(x_1) \iff A \subset (F^*H)(x_1).$$

Replace A by $K(x_1)$ and we get:

$$(FK)(x_1) \subset (HF)(x_1) \iff K(x_1) \subset (F^*H)(x_1).$$

Hence $FK \subset HF \iff K \subset F^*H$.

In particular, with $K = F^*H$ we have $FK \subset HF$ if and only if $K \subset F^*H$. For the surjective map case, we first consider (5).

(5) It is clear that $(x_1, x_2) \in f^*H$ if and only if $f(x_2) \in H(f(x_1))$. This says $f^*H = (f \times f)^{-1}(H)$ and $f^*H = f^{-1}Hf$, $f^*H = f^{-1}Hf = (f \times f)^{-1}(H) = \{(x_1,\ x_2) : f(x_2) \in H(f(x_1))\}$. From this $f^*(H^{-1}) = (f^*H)^{-1}$ is obvious.

When f is a surjective map, $ff^*H = ff^{-1}Hf = Hf$ by $ff^{-1} = 1_{X_2}$.

(3) $1^*_{X_2}H = H$ is clear. By (2), $FG(G^*(F^*H)) \subset F(F^*H)G \subset HFG$. The $G^*F^*H \subset (FG)^*H$ by (2). If $G = g$ is a map, $(Fg)((Fg)^*H)g^{-1} \subset (H(Fg))g^{-1} \subset HF$ and $g((Fg)^*H)g^{-1} \subset F^*H$. By (5), $(Fg)^*H \subset g^{-1}g((Fg)^*H)g^{-1}g \subset g^{-1}(F^*H)g = g^*(F^*H)$. Alternatively, observe that $(x_1, x_2) \in (Fg)^*H$ if and only if $F(g(x_2) \subset HF(g(x_1))$ and so if and only if $(g(x_1), g(x_2)) \in F^*H$.

(4) Recall that for subsets A of X_1 and B of X_2 $f(A) \subset B \iff A \subset f^{-1}(B)$. Put $A = F(x_0)$ and $B = G(x_0)$ for given $x_0 \in X_0$. Then $fF \subset G \iff F \subset f^{-1}G$.

(5) For a map f, $E_f = (f \times f)^{-1}(1_{X_2}) = \{(x_1, x_2) : f(x_1) = f(x_2)\} = f^{-1}f = f^*1_{X_2}$. Since f is a map, $f = f1_{X_1} \subset ff^{-1}f \subset 1_{X_2}f = f$ and $f^{-1} = 1_{X_1}f^{-1} \subset f^{-1}ff^{-1} \subset f^{-1}1_{X_2} = f^{-1}$. Hence $fE_f = ff^{-1}f = f$ and $E_f f^{-1} = f^{-1}ff^{-1} = f^{-1}$.

Finally, by (1), $E_f(f^*H)E_f = f^*1_{X_2}(f^*H)f^*1_{X_2} \subset f^*(1_{X_2}H)f^*1_{X_2} = (f^*H)(f^*1_{X_2}) \subset f^*(H1_{X_2}) = f^*(H)$. Since $1_{X_1} \subset f^{-1}f$ and $E_f = f^{-1}f = f^*1_{X_2}$, $f^*H = 1_{X_1}(f^*H)1_{X_1} \subset f^{-1}f(f^*H)f^{-1}f = E_f(f^*H)E_f$ and $f^*(H^{-1}) = f^{-1}H^{-1}f = (f^{-1}Hf)^{-1} = (f^*H)^{-1}$. □

Definition 2.3. For a relation $F : X_1 \to X_2$, $V_2 \in \mathcal{U}_{X_2}$ and $x \in X_1$ we call F is V_2 *upper semicontinuous at* x, when $x \in \mathrm{Int}((F^*V_2)(x))$; V_2 *lower semicontinuous at* x, when $x \in \mathrm{Int}((F^*V_2)^{-1}(x))$; V_2 *continuous at* x, when $(x,\ x) \in \mathrm{Int}F^*V_2$ in $X_1 \times X_1$. F is *upper semicontinuous / lower semicontinuous / continuous at* x if it is V_2 upper semicontinuous / V_2 lower semicontinuous / V_2 continuous at x for all V_2 in $\mathcal{U}_{X_2}$, respectively. F is *upper semicontinuous / lower semicontinuous / continuous* if it satisfies the corresponding condition at every x in X_1. F is *uniformly continuous* provided that for every $V_2 \in \mathcal{U}_{X_2}$ there exists $V_1 \in \mathcal{U}_{X_1}$ such that $FV_1 \subset V_2F$.

The following examples show that F is upper semicontinuous but is not lower semicontinuous at x, G is lower semicontinuous but is not upper semicontinuous at x, and H is continuous at x.

Example 2.4. *Define a relation F on $\mathbb{R}$ by $F = (-\infty,\ 0) \times \{1\} \cup [0,\ \infty) \times [\frac{1}{2},\ \frac{3}{2}]$ and let $\epsilon \in (0,\ \frac{1}{2})$ be given. By definition of F^*V_ϵ, if $(x,\ y) \in F^*V_\epsilon$, then $F(y) \subset V_\epsilon(F(x)) = B(F(x),\ \epsilon)$, where $V_\epsilon = \{(x_1,\ x_2) \in \mathbb{R} \times \mathbb{R} : d(x_1,\ x_2) < \epsilon\}$. If $x \in (-\infty,\ 0)$, then $F(y) \subset B(F(x),\ \epsilon) = B(\{1\},\ \epsilon) = (1-\epsilon,\ 1+\epsilon) \subset (\frac{1}{2},\ \frac{3}{2})$. Hence $y \in (-\infty,\ 0)$. If $x \in [0,\ \infty)$, then $F(y) \subset B(F(x),\ \epsilon) = B([\frac{1}{2},\ \frac{3}{2}],\ \epsilon) = (\frac{1}{2}-\epsilon,\ \frac{3}{2}+\epsilon)$. Hence $y \in (-\infty,\ \infty)$. This means that $F^*V_\epsilon \subset (-\infty,\ 0) \times (-\infty,\ 0) \cup [0,\ \infty) \times (-\infty,\ \infty)$. Let $(x,\ y) \in (-\infty,\ 0) \times (-\infty,\ 0) \cup [0,\ \infty) \times (-\infty,\ \infty)$ be given. If $x < 0$ and $y < 0$, then $F(y) = \{1\} \subset (1-\epsilon,\ 1+\epsilon) = B(\{1\},\ \epsilon) = B(F(x),\ \epsilon) = V_\epsilon(F(x))$. Hence $(x,\ y) \in F^*V_\epsilon$. If $x \geq 0$ and $y \in \mathbb{R}$, $F(y) \subset [\frac{1}{2},\ \frac{3}{2}] \subset (\frac{1}{2}-\epsilon,\ \frac{3}{2}+\epsilon) = B([\frac{1}{2},\ \frac{3}{2}],\ \epsilon) = B(F(x),\ \epsilon) = V_\epsilon(F(x))$. Hence $(x,\ y) \in F^*V_\epsilon$. Finally, we know that $F^*V_\epsilon = (-\infty,\ 0) \times (-\infty,\ 0) \cup [0,\ \infty) \times (-\infty,\ \infty)$. Since $F^*V_\epsilon(0) = (-\infty,\ \infty)$, $0 \in (-\infty,\ \infty) = \mathrm{Int}F^*V_\epsilon(0)$. Therefore F is V_ϵ upper semicontinuous at 0. Since $(F^*V_\epsilon)^{-1} = (-\infty,\ 0) \times$*

$(-\infty,\ 0) \cup (-\infty,\ \infty) \times [0,\ \infty)$, $(F^*V_\epsilon)^{-1}(0) = [0,\ \infty)$. *It means that* $0 \notin (0,\ \infty) = \mathrm{Int}(F^*V_\epsilon)^{-1}(0)$. *Consequently,* F *is not* V_ϵ *lower semicontinuous at* 0.

Example 2.5. *Define a relation* G *on* $\mathbb{R}$ *by* $G = (-\infty,\ 0] \times \{1\} \cup (0,\ \infty) \times [\frac{1}{2},\ \frac{3}{2}]$ *and let* $\epsilon \in (0,\ \frac{1}{2})$ *be given. Since* $G^*V_\epsilon = (-\infty,\ 0] \times (-\infty,\ 0] \cup (0,\ \infty) \times (-\infty,\ \infty)$, $G^*V_\epsilon(0) = (-\infty,\ 0]$. *We know that* G *is not* V_ϵ *upper semicontinuous at* 0 *because* $0 \notin (-\infty,\ 0) = \mathrm{Int}G^*V_\epsilon(0)$. *Since* $(G^*V_\epsilon)^{-1} = (-\infty,\ 0] \times (-\infty,\ 0] \cup (-\infty,\ \infty) \times (0,\ \infty)$, $(G^*V_\epsilon)^{-1}(0) = (-\infty,\ \infty)$. *It means that* G *is* V_ϵ *lower semicontinuous at* 0 *because* $0 \in (-\infty,\ \infty) = \mathrm{Int}(G^*V_\epsilon)^{-1}(0)$.

Example 2.6. *Define a relation* H *on* $\mathbb{R}$ *by* $H = (-\infty,\ \infty) \times [\frac{1}{2},\ \frac{3}{2}]$ *and let* $\epsilon > 0$ *be given. Since* $H^*V_\epsilon = (-\infty,\ \infty) \times (-\infty,\ \infty)$, H *is* V_ϵ *continuous at* 0 *because* $(0,\ 0) \in (-\infty,\ \infty) \times (-\infty,\ \infty) = \mathrm{Int}H^*V_\epsilon$.

Lemma 2.7. *Let* $F : X_1 \to X_2$ *be a relation and* $V_2 \in \mathcal{U}_{X_2}$.

(1) *Let* $\tilde{V}_2, \subset V_2$. *If* F *is* $\tilde{V}_2$ *upper semicontinuous at* x / $\tilde{V}_2$ *lower semicontinuous at* x / $\tilde{V}_2$ *continuous at* x, *then* F *satisfies the corresponding property for* V_2.

(2) *If* F *is* V_2 *continuous at* x, *then* F *is* V_2 *upper semicontinuous at* x *and* V_2 *lower semicontinuous at* x. *If* $\tilde{V}_2 \in \mathcal{U}_{X_2}$ *with* $\tilde{V}_2^2 \subset V_2$ *and* F *is* $\tilde{V}_2$ *upper semicontinuous at* x *and* $\tilde{V}_2$ *lower semicontinuous at* x, *then* F *is* V_2 *continuous at* x.

(3) F *is continuous at* x *if and only if* F *is upper semicontinuous at* x *and lower semicontinuous at* x. *If* F *is uniformly continuous, then* F *is continuous. If* F *is continuous and* X_1 *is compact, then* F *is uniformly continuous.*

Proof. (1) This follows from Proposition 2.2 (1).

(2) If F be V_2 continuous at x, $(x,x) \in \mathrm{Int}F^*V_2$. This means that $x \in \mathrm{Int}(F^*V_2)(x)$ and $x \in \mathrm{Int}(F^*V_2)^{-1}(x)$. Therefore, F is V_2 upper semicontinuous at x and V_2 lower semicontinuous at x. Let $\tilde{V}_2 \in \mathcal{U}_{X_2}$ with $\tilde{V}_2^2 \subset V_2$, F be $\tilde{V}_2$ upper semicontinuous at x and $\tilde{V}_2$ lower semicontinuous at x. Since $\tilde{V}_2 \subset V_2$, $x \in \mathrm{Int}((F^*V_2)(x))$ and $x \in \mathrm{Int}((F^*V_2)^{-1}(x))$. It means that F is V_2 continous at x.

(3) By Definition 2.3, the proof of (3) is obvious. □

Theorem 2.8. *Let* $F : X_1 \to X_2$ *be a pointwise closed relation. Assume that* F *is upper semicontinuous and* $F(x)$ *is compact for every* $x \in X_1$. *For every* $V_2 \in U_{X_2}$ *the set of* V_2 *continuity points,* $\{x\ :\ (x,\ x) \in \mathrm{Int}F^*V_2\}$, *is open and dense in* X_1.

Proof. The set of V_2 continuity points, $\{x : (x,\ x) \in \mathrm{Int}F^*V_2\}$, is always open. By Lemma 2.7 (2), it suffices to prove that the set of V_2 upper semicontinuous points and the set of V_2 lower semicontinuous points are each dense.

Fix $V_2 \in \mathcal{U}_{X_2}$, $x_1 \in X_1$, and O_1 an open set containing x_1. We claim that $O_1 \cap \{x\ :\ (x,\ x) \in \mathrm{Int}F^*V_2\} \neq \emptyset$.

We produce $x_0 \in O_1$ at which F is V_2 lower semicontinuous. Choose $\tilde{V},\ W \in \mathcal{U}_{X_2}$, symmetric, with $\tilde{V}$ closed and such that $\tilde{V}^2 \subset V_2$, $W^2 \subset \tilde{V}$ and $W^2 \subset \tilde{V}$. Because F is upper semicontinuous, there is an open set $\tilde{O} \subset O_1$ such that $x \in \tilde{O}$ implies $F(x) \subset W(F(x_1))$, i.e. $(x_1,\ x) \in F^*W$. Because $F(x_1)$ is compact, it has a finite subset R such that $F(x_1) \subset W(R)$. Define for $x \in \tilde{O}$:

$$R(x) = R \cap \tilde{V}(F(x)).$$

Clearly, for $x \in \tilde{O}$:

$$R(x) \subset \tilde{V}(F(x)) \quad \text{and} \quad (\mathrm{R} - \mathrm{R(x)}) \cap \tilde{\mathrm{V}}(\mathrm{R(x)}) = \emptyset. \tag{2.1}$$

If there exists $y \in F(x) \cap \tilde{V}(R - R(x))$, then $y \in F(x)$ and $y \in \tilde{V}(F(x))$. This is a contradiction for $(R - R(x)) \cap \tilde{V}(R(x)) = \emptyset$. Hence we get:

$$\begin{aligned} &F(x) \cap \tilde{V}(R - R(x)) = \emptyset \quad \text{and} \quad \mathrm{F(x)} \subset \tilde{\mathrm{V}}(\mathrm{R(x)}); \\ &F(x) \cap \tilde{V}(R - R(x)) = \emptyset \quad \text{and} \quad \mathrm{F(x)} \subset \tilde{\mathrm{V}}(\mathrm{R(x)}). \end{aligned} \tag{2.2}$$

Since F is upper semicontinuous and $F(x_1) \subset W(R)$, we obtain the following property.

$$F(x) \subset W(F(x_1)) \subset W^2(R) \subset \tilde{V}(R(x)) \cup \tilde{V}(R - R(x)).$$

Now choose $x_0 \in \tilde{O}$ so that $R(x_0)$ is minimal in the family $\{R(x) : x \in \tilde{O}\}$ of subsets of the finite set R.

Since $F(x_0)$ is compact and $\tilde{V}(R - R(x_0))$ is the finite union of closed sets, and thus is closed, there exists $W_0 \in \mathcal{U}_{X_2}$ such that

$$W_0(F(x_0)) \cap \tilde{V}(R - R(x_0)) = \emptyset.$$

Since F is upper semicontinuous:

$$O_0 \equiv \{x \in \tilde{O} : (x_0,\ x) \in F^*W_0\}$$

is a neighborhood of x_0. For $x \in O_0$, $W_0(F(x_0)) \cap \tilde{V}(R - R(x_0)) = \emptyset$ implies $R(x) \cap (R - R(x_0)) = \emptyset$, so $R(x) \subset R(x_0)$. By the minimality of $R(x_0)$, we have $R(x) = R(x_0)$ for all $x \in O_0$. Therefore we obtain that

$$F(x_0) \subset \tilde{V}(R(x_0)) = \tilde{V}(R(x)) \subset \tilde{V}^2(F(x)) \subset V_2(F(x))$$

for all $x \in O_0$.

This means $O_0 \times \{x_0\} \subset F^*V_2$, so F is V_2 lower semicontinuous at x_0, i.e. $x_0 \in O_1 \cap \{x : (x,\ x) \in \mathrm{Int}F^*V_2\}$. Hence the set of V_2 upper semicontinuous points is dense. Similarly, we can derive the fact that the set of V_2 lower semicontinuous points is dense. □

THEOREM 2.9. *Let $F : X_1 \to X_2$ be a pointwise closed relation. Assume that F is lower semicontinuous and X_2 is compact. For every $V_2 \in \mathcal{U}_{X_2}$ the set of V_2 continuity points, $\{x : (x,\ x) \in \mathrm{Int}F^*V_2\}$, is open and dense in X_1.*

Proof. If $x_1 \in X_1$ and O_1 is an open set containing x_1, then we can choose $\tilde{V},\ W \in \mathcal{U}_{X_2}$ symmetric with $\tilde{V}$ open, $\tilde{V}^2 \subset V_2$, and $W^2 \subset \tilde{V}$. Choose R a finite subset of X_2 such that $W(R) = X_2$, and define $R(x)$ for all $x \in X_2$ according to $R(x) = R \cap \tilde{V}(F(x))$. Then for all $x \in X_2$, (2-1) and (2-2) hold as before.

Choose $x_0 \in O_1$ so that $R(x_0)$ is maximal in the family $\{R(x) : x \in O_1\}$. Since $\tilde{V}$ is open and $F(x_0)$ is compact, there exists $W_0 \in \mathcal{U}_{X_2}$ symmetric such that for each $y \in R(x_0)$ there exists $z_y \in F(x_0)$ such that $W_0(z_y) \subset \tilde{V}(y)$. Because F is lower semicontinuous:

$$O_0 \equiv \{x \in O_1 : F(x_0) \subset W_0(F(x))\}$$

is a neighborhood of x_0. For each $x \in O_0$ and $y \in R(x_0)$, $\tilde{V}(y)$ contains $W_0(z_y)$, which meets $F(x)$. Thus $y \in \tilde{V}(F(x))$. Consequently, $R(x_0) \subset R(x)$, so by the maximality of $R(x_0)$, we have $R(x_0) = R(x)$ for $x \in O_0$. Hence for all $x \in O_0$:

$$F(x) \subset \tilde{V}(R(x)) = \tilde{V}(R(x_0)) \subset \tilde{V}^2(F(x_0)) \subset V_2(F(x_0)).$$

This says $\{x_0\} \times O_0 \subset F^*V_2$, so F is upper semicontinuous at x_0. □

LEMMA 2.10. *X is a Baire space if and only if given any countable collection $\{U_n\}$ of open sets in X, each of which is dense in X, their intersection $\cap U_n$ is also dense in X.*

Proof. See [9] Lemma 7.1. □

LEMMA 2.11. *Let $f : X_1 \to X_2$ be a continuous map.*

(1) *Let f be a closed map. Assume A is an E_f invariant set, i.e., $A = E_f(A)$. If B is closed and $B \subset A$, then $B_1 = E_f(B)$ is an E_f invariant closed subset of X_1 satisfying $B \subset B_1 \subset A$. If O is open and $A \subset O$, then $O_1 = \{x : E_f(x) \subset O\}$ is an E_f invariant open subset of X_1 satisfying $A \subset O_1 \subset O$. The relations E_f on X_1 and $f^{-1} : X_2 \to X_1$ are upper semicontinuous.*

(2) *If f is a proper map(i.e., point inverses are compact) and f^{-1} is upper semicontinuous, then f is a closed map.*

Proof. (1) If B is closed, then $B_1 = E_f(B) = f^{-1}f(B)$ is closed when f is a continuous, closed map. $B \subset A = E_f(A)$ implies $B \subset$

$E_f(B) \subset E_f(A) = A$. Since O is open, $O_1 = \{x : E_f(x) \subset O\} = X_1 - E_f(X - O)$ is open. If A is invariant, then $A_1 = \{x : E_f(x) \subset A\} = A$. Therefore $A \subset O$ implies $A = A_1 \subset O_1 \subset O$. Similarly, $\{y \in X_2 : f^{-1}(y) \subset O\} = X_2 - f(X_1 - O)$ is open in X_2. In particular, for any $V_1 \in \mathcal{U}_{X_1}$ and $y \in X_2$, $y \in \{y_1 \in X_2 : f^{-1}(y_1) \subset \mathrm{Int}V_1(f^{-1}(y))\} \subset ((f^{-1})^*V_1)(y)$. Then $y \in \mathrm{Int}((f^{-1})^*V_1)(y)$ for any $V_1 \in \mathcal{U}_{X_1}$. Thus f^{-1} is upper semicontinuous at y. By Proposition 2.2 (3), since f is a map, $E_f^*V_1 = f^*(f^{-1})^*V_1 = f^{-1}((f^{-1})^*V_1)f$. Hence for any $x \in X_1$, $(E_f^*V_1)(x) = f^{-1}((f^{-1})^*V_1(f(x)))$. Then $f(x) \in \mathrm{Int}(f^{-1})^*V_1(f(x))$ implies $x \in \mathrm{Int}(E_f)^*V_1(x)$, so E_f is upper semicontinuous at x.

(2) If A is closed in X_1, $y \notin f(A)$ and $f^{-1}(y)$ is compact, then there exists $V_1 \in \mathcal{U}_{X_1}$ such that $V_1(f^{-1}(y)) \cap A = \emptyset$. If f^{-1} is upper semicontinuous at y, then there exists O_2 open with $y \in O_2$ and $f^{-1}(O_2) \subset V_1(f^{-1}(y))$. Then $O_2 \cap f(A) = \emptyset$ and $y \notin \overline{f(A)}$. Hence f is a closed map. □

3. Genericity and almost homeomorphisms

In this section, we define openness (or quasi-openness) at a point x using the relative interior operator and characterize that a continuous map f is almost open, almost quasi-open, almost embedding and almost homeomorphsim.

For a continuous map $f : X_1 \to X_2$ and let $A \subset X_1$, we define the *relative interior operator*:

$$\mathrm{Int}_f A = (\mathrm{Int}A) \cap f^{-1}(\mathrm{Int}f(A)).$$

Clearly, $\mathrm{Int}_f A$ is an open subset of $\mathrm{Int}A$. For $x \in X_1$, $x \in \mathrm{Int}_f A$ if and only if $x \in \mathrm{Int}A$ and $f(x) \in \mathrm{Int}f(A)$.

Definition 3.1. Let $f : X_1 \to X_2$ be a continuous map. For $V_1 \in \mathcal{U}_{X_1}$ and $x \in X_1$, we call f is V_1 *open at* x if $x \in \mathrm{Int}_f V_1(x)$. f is *open at* x provided it is V_1 open at x for all $V_1 \in \mathcal{U}_{X_1}$ and f is *open* if f is open at x for every $x \in X_1$.

The equivalent conditions of V_1 openness are the following.

Lemma 3.2. *Let $f : X_1 \to X_2$ be a continuous map. Then the following statements are equivalent:*

(1) *f is V_1 open at x.*
(2) *If U is a neighborhood of x in X_1, then $f(U)$ is a neighborhood of $f(x)$ in X_2.*
(3) *For any subset A of X_1, $x \in \mathrm{Int}A$ implies $x \in \mathrm{Int}_f A$.*

Proof. (1)⇒(2). If f is V_1 open at x for all $V_1 \in \mathcal{U}_{X_1}$, then $x \in \mathrm{Int}_f V_1(x)$. Let U be a neighborhood of x in X_1. By the definition of the relative interior operator, $x \in f^{-1}(\mathrm{Int} f(V_1(x)))$. It means that $f(x) \in f(f^{-1}(\mathrm{Int} f(V_1))) \subset \mathrm{Int} f(V_1(x)) \subset f(U)$.

(2)⇒(3). Let A be a subset of X_1 and $x \in \mathrm{Int} A$. By the hypothesis, $f(\mathrm{Int} A)$ is a neighborhood of $f(x)$. It means that $x \in f^{-1}(\mathrm{Int} f(A))$. Thus, $x \in \mathrm{Int}_f A$.

(3)⇒(1). Since $x \in V_1(x)$, $x \in \mathrm{Int} V_1(x)$. By (3), $x \in \mathrm{Int}_f V_1(x)$. It means that f is V_1 open at x. □

LEMMA 3.3. *Let $f : X_1 \to X_2$ be a continuous map and let $y \in X_2$ and V_1 be a symmetric element of $\mathcal{U}_{X_1}$. If f^{-1} is V_1 lower semicontinuous at y, then f is V_1 open at x and E_f is V_1 lower semicontinuous at x for all $x \in f^{-1}(y)$. If $f^{-1}(y)$ is compact and f is V_1 open at every $x \in f^{-1}(y)$, then f^{-1} is V_1^2 lower semicontinuous at y. In particular, if $f^{-1}(y) = \emptyset$, then f^{-1} is lower semicontinuous at y.*

Proof. If f^{-1} is V_1 lower semicontinuous at y, then there exists $V_2 \in \mathcal{U}_{X_2}$ such that $V_2(y) \subset ((f^{-1})^* V_1)^{-1}(y)$. This is, $y_1 \in V_2(y)$ implies $f^{-1}(y) \subset V_1(f^{-1}(y_1))$, so $V_2(y) \subset f(V_1(x))$ for all $x \in f^{-1}(y)$. For $x_1 \in f^{-1}(V_2(y))$ setting $y_1 = f(x_1)$ shows $E_f(x) = f^{-1}(y) \subset V_1(E_f(x_1))$. Thus E_f is V_1 lower semicontinuous at x_1.

If $f^{-1}(y)$ is compact, then there exists $\{x_1, \cdots, x_n\}$ in $f^{-1}(y) \subset \cup_{i=1}^n V_1(x_i)$. If f is V_1 open at each x_i, then there exists $V_2 \in \mathcal{U}_{X_2}$ such that $V_2(y) \subset \cap_{i=1}^n f(V_1(x_i))$. If $y_1 \in V_2(y)$, $f^{-1}(y_1) \cap V_1(x_i) \neq \emptyset$, so $f^{-1}(y_1) \cap V_1^2(x) \neq \emptyset$ for every $x \in f^{-1}(y)$. That is $V_2(y) \subset ((f^{-1})^*(V_1^2))^{-1}(y)$ and f^{-1} is V_1^2 lower semicontinuous at y. □

PROPOSITION 3.4. *Let $f : X_1 \to X_2$ be a continuous map. Then the following statements are equivalent:*

(1) *f is open.*

(2) *If U is open in X_1, then $f(U)$ is open in X_2.*

(3) *For all $A \subset X_1$, $\mathrm{Int} A = \mathrm{Int}_f A$.*

Proof. This is obvious from Lemma 3.2. □

DEFINITION 3.5. Let $f : X_1 \to X_2$ be a continuous map. We call f is *almost open* provided that for all $A \subset X_1$, $\mathrm{Int} A \neq \emptyset$ implies $\mathrm{Int} f(A) \neq \emptyset$.

The equivalent conditions of almost openness are the following.

THEOREM 3.6. *Let $f : X_1 \to X_2$ be a continuous map. Then the following statements are equivalent:*

(1) *f is almost open.*

(2) *D dense in X_2 implies $f^{-1}(D)$ is dense in X_1.*

(3) *If U is open in X_1, then $\mathrm{Int}_f U = U \cap f^{-1}(\mathrm{Int} f(U))$ is dense in U.*

(4) *For every $V_1 \in \mathcal{U}_{X_1}$, $\{x : f$ is V_1 open at $x\}$ is dense in X_1.*

(5) *For every $V_1 \in \mathcal{U}_{X_1}$, $\{x : (x, f(x)) \in \mathrm{Int}(f \circ V_1)\}$ is open and dense in X_1.*

Proof. (1)⇒(2). Suppose that $f^{-1}(D)$ is not dense in X_1. Then $U = X_1 - \overline{f^{-1}(D)}$ is a nonempty open set. By (1), $\mathrm{Int} f(U) \neq \emptyset$. Since $f(U) \cap D = \emptyset$, $\mathrm{Int} f(U) \cap D = \emptyset$. This is a contradiction.

(2)⇒(3). For any $A \subset X_1$, $D = (\mathrm{Int} f(A)) \cup (X_2 - f(A))$ is dense in X_2. By (2), $f^{-1}(A)$ is dense in X_1. If U is an open subset of X_1, then $\mathrm{Int}_f U = U \cap f^{-1}(D)$ is dense in U.

(3)⇒(4). Let $V_1 \in \mathcal{U}_{X_1}$ and $x \in X_1$. Also, let V_2 be an open, symmetric element of $\mathcal{U}_{X_1}$ such that $V_2^2 \subset V_1$. By (3), $V_2(x) \cap f^{-1}(\mathrm{Int} f(V_2(x)))$ is open and dense in $V_2(x)$. Choose $x_1 \in V_2(x) \cap f^{-1}(\mathrm{Int} f(V_2(x)))$. Then $f(x_1) \in f(f^{-1}(\mathrm{Int} f(V_2(x)))) \subset \mathrm{Int} f(V_2(x))$. Since V_2 can be chosen arbitrarily small, f is V_1 open at points of a dense set.

(4)⇒(1). For all $A \subset X_1$, let $\mathrm{Int} A \neq \emptyset$. If $V_1^2(x) \subset \mathrm{Int} A$ let $x_1 \in V_1(x)$ at which f is V_1 open. Since $f(V_1(x_1)) \subset f(V_1^2(x)) \subset f(A)$, $f(V_1(x_1))$ is a neighborhood of $f(x_1)$.

(4) $\Longleftrightarrow$ (5). For V_1 symmetric in $\mathcal{U}_{X_1}$, $\{x : (x, f(x)) \in \mathrm{Int}(f \circ V_1)\}$ $\subset \{x : f$ is V_1 open at $x\} \subset \{x : (x, f(x)) \in \mathrm{Int}(f \circ V_1^2)\}$.

Observe that $U_1 \times U_2 \subset f \circ V_1$ if and only if $U_2 \subset f(V_1(x_1))$ for all $x_1 \in U_1$. The first inclusion is clear, and the second follows from $f(V_1(x)) \subset f(V_1^2(x_1))$ for all $x \in V_1(x)$. Together these inclusions yield the equivalence. □

THEOREM 3.7. *Let $f : X_1 \to X_2$ and $g : X_2 \to X_3$ be continuous maps. If both f and g are almost open, then $g \circ f$ is almost open. If $g \circ f$ is almost open and f is surjective, then g is almost open.*

Proof. Let f and g be almost open and let $\mathrm{Int} A$ be a nonempty subset of X_1. Since f and g are almost open, $\mathrm{Int} f(A)$ and $\mathrm{Int} g(f(A))$ are nonempty subsets of X_2 and X_3, respectively. Let $g \circ f$ be almost open, f be surjective and $\mathrm{Int} A_2$ be a nonempty subset of X_2. Since f is continuous and surjective, $f^{-1}(\mathrm{Int} A_2)$ is a nonempty open subset of X_1. $\mathrm{Int}((g \circ f)(f^{-1}(\mathrm{Int} A_2))) = \mathrm{Int} g(\mathrm{Int} A_2)$ is a nonempty subset of X_3 because $g \circ f$ is almost open. Therefore g is almost open. □

THEOREM 3.8. *Let $f : X_1 \to X_2$ be a continuous, closed and proper map. For every $V_1 \in \mathcal{U}_{X_1}$, the set:*

$$\mathrm{Int}\{y \in X_2 : f \text{ is } V_1 \text{ open at each } x \in f^{-1}(y)\}$$

is an open and dense subset of X_2. *In particular, if* X_1 *is metrizable and* X_2 *is Baire, then*

$$\{y \in X_2 : f \text{ is } V_1 \text{ open at each } x \in f^{-1}(y)\}$$

is a residual subset of X_2.

Proof. By Lemma 2.11 (1), f^{-1} is an upper semicontinuous relation. By Theorem 2.8, $\{y : (y,y) \in \mathrm{Int}(f^{-1})^*V_1\}$ is open and dense in X_2 and it is contained in the set of points at which f^{-1} is a lower semicontinuous relation. By Lemma 3.3, f is V_1 open at every point $x \in X_1$ such that $y = f(x)$ lies in this set. As usual when X_1 is metrizable, we can intersect over a countable base for $\mathcal{U}_{X_1}$ to obtain a residual subset of X_2. □

Generally $\{x : f \text{ is almost open at } x\}$ is not dense when f is a continuous, closed and proper map.

EXAMPLE 3.9. *Define* $f : [0,1] \to [0,2]$ *by* $f(x) = 1$. *Then* f *is a continuous, closed and proper map. But* $\{x : f \text{ is almost open at } x\} = \emptyset$.

A subset A of X is called *quasi-open* if $A \subset \mathrm{Int}\overline{A}$. For any subset A of X define the *quasi-interior*:

$$\mathcal{Q}\mathrm{Int}A = A \cap \mathrm{Int}\overline{A}.$$

For a continuous map $f : X_1 \to X_2$ and a subset A of X_1, define

$$\mathcal{Q}\mathrm{Int}_f A = (\mathcal{Q}\mathrm{Int}A) \cap f^{-1}(\mathcal{Q}\mathrm{Int}f(A)) = (\mathcal{Q}\mathrm{Int}A) \cap f^{-1}(\mathrm{Int}\overline{f(A)}).$$

The two definitions agree because $f^{-1}(f(A)) \supset A$. In particular:

$$x \in \mathcal{Q}\mathrm{Int}_f A \text{ if and only if } x \in \mathcal{Q}\mathrm{Int}A \text{ and } f(x) \in \mathrm{Int}\overline{f(A)}.$$

PROPOSITION 3.10. *The following properties are hold*

(1) A *is quasi-open in* X *if and only if there exist open subset* U *and dense set* D *such that* $A = U \cap D$. *In particular, any open set or any dense set is quasi-open.*

(2) The arbitrary union of quasi-open sets is quasi-open. If A *is quasi-open and* U *is open, then* $A \cap U$ *is quasi-open.*

(3) The $\mathcal{Q}\mathrm{Int}A$ *is dense in* $\mathrm{Int}\overline{A}$. $\mathcal{Q}\mathrm{Int}A$ *is the largest quasi-open set contained in* A. *In particular,* $\mathcal{Q}\mathrm{Int}(\mathcal{Q}\mathrm{Int}A) = \mathcal{Q}\mathrm{Int}A$. *The set* A *is quasi-open if and only if* $A = \mathcal{Q}\mathrm{Int}A$. $\mathcal{Q}\mathrm{Int}A = \emptyset$ *if and only if* A *is nowhere dense.*

(4) If $f : X_1 \to X_2$ *is continuous and* $A \subset X_1$, *then* $\mathcal{Q}\mathrm{Int}_f A$ *is a quasi-open subset of* $\mathcal{Q}\mathrm{Int}A$. *If* A *is quasi-open, then* $f(\mathcal{Q}\mathrm{Int}_f A) = \mathcal{Q}\mathrm{Int}f(A)$ *which is quasi-open in* X_2. *If* U *is open, then* $\mathcal{Q}\mathrm{Int}_f U$ *is open.*

Proof. (1) Let A be quasi-open in X, i.e., $A \subset \mathrm{Int}\overline{A}$. Put $D = A \cup (X - \overline{A})$. Then D is dense in X. Since A is quasi-open, $A =$

$D \cap \mathrm{Int}\overline{A} = (A \cup (X - \overline{A})) \cap \mathrm{Int}\overline{A}$. Conversely, let $A = U \cap D$ where U is open and D is dense in X. Then A is dense in U, i.e., $\overline{A} = \overline{U}$. Thus $A \subset U \subset \mathrm{Int}\overline{U} = \mathrm{Int}\overline{A}$. Since $A \cap X = A$, any open set or any dense set is quasi-open.

(2) Let $\{A_\alpha : \alpha \in \Lambda\}$ be quasi-open, i.e., $A_\alpha \subset \mathrm{Int}\overline{A_\alpha}$ for all $\alpha \in \Lambda$. Since A_α is quasi-open for all $\alpha \in \Lambda$, $\cup_{\alpha\in\Lambda} A_\alpha \subset \cup_{\alpha\in\Lambda}\mathrm{Int}\overline{A_\alpha} \subset \mathrm{Int}(\cup_{\alpha\in\Lambda}\overline{A_\alpha}) \subset \mathrm{Int}(\overline{\cup_{\alpha\in\Lambda} A_\alpha})$.

Let A be quasi-open and let U be open. By (1), $A = V \cap D$ for some open V and dense D. Thus $A \cap U = (U \cap V) \cap D$ is quasi-open.

(3) Since A is dense in $\overline{A}$ and $\mathrm{Int}\overline{A}$ is open in $\overline{A}$, $\mathcal{Q}\mathrm{Int}A = A \cap \mathrm{Int}\overline{A}$ is dense in $\mathrm{Int}\overline{A}$. We have that $\mathcal{Q}\mathrm{Int}(\mathcal{Q}\mathrm{Int}A) = \mathcal{Q}\mathrm{Int}A \cap \mathrm{Int}(\overline{\mathcal{Q}\mathrm{Int}A}) = \mathcal{Q}\mathrm{Int}A \cap \mathrm{Int}(\mathrm{Int}\overline{A}) = A \cap \mathrm{Int}\overline{A} \cap \mathrm{Int}\overline{A} = \mathcal{Q}\mathrm{Int}A$. Clearly, A is quasi-open if and only if $A = \mathcal{Q}\mathrm{Int}A$. Thus $\mathcal{Q}\mathrm{Int}A$ is qusi-open.

(4) Since $\mathcal{Q}\mathrm{Int}A$ is quasi-open and $f^{-1}(\mathrm{Int}\overline{f(A)})$ is open, $\mathcal{Q}\mathrm{Int}_f A = \mathcal{Q}\mathrm{Int}A \cap f^{-1}(\mathcal{Q}\mathrm{Int}f(A)) = \mathcal{Q}\mathrm{Int}A \cap f^{-1}(\mathrm{Int}\overline{f(A)})$ is quasi-open which also shows that $\mathcal{Q}\mathrm{Int}_f A$ is open when A is open. By (3), $f(\mathcal{Q}\mathrm{Int}_f A) = f(\mathcal{Q}\mathrm{Int}A \cap f^{-1}(\mathrm{Int}\overline{f(A)})) = f(\mathcal{Q}\mathrm{Int}A) \cap \mathrm{Int}\overline{f(A)} = f(A) \cap \mathrm{Int}\overline{f(A)} = \mathcal{Q}\mathrm{Int}f(A)$. □

Generally, $\mathrm{Int}_f A \subset \mathrm{Int}A \subset \mathcal{Q}\mathrm{Int}A \subset A$ and $\mathrm{Int}_f A \subset \mathcal{Q}\mathrm{Int}_f A \subset \mathcal{Q}\mathrm{Int}A \subset A$ for any subset A of X_1. The followings are examples in which equalities are not hold by the above inclusion relations.

Example 3.11. *If* $A = ([0,1] \cap \mathbb{Q}) \cup [1,2]$, *then* $\mathrm{Int}A \subsetneq \mathcal{Q}\mathrm{Int}A \subsetneq A$.

Define $f : \mathbb{R} \to \mathbb{R}$ *by* $f(x) = |x|$. *Let* $B = [-1,1] - \{-\frac{1}{2}, \frac{1}{2}\}$. *Then* $\mathrm{Int}_f B \subsetneq \mathrm{Int}B$.

Define $f : \mathbb{R} \to \mathbb{R}$ *by* $f(x) = |x-1|$. *Let* $C = ([0,1] \cap \mathbb{Q}) \cup [1,2]$. *Then* $\mathrm{Int}_f C \subsetneq \mathcal{Q}\mathrm{Int}_f C$.

Definition 3.12. Let $f : X_1 \to X_2$ be a continuous map. For $x \in X_1$ and $V_1 \in \mathcal{U}_{X_1}$, we call f is V_1 *quasi-open at* x if $x \in \mathcal{Q}\mathrm{Int}_f V_1(x)$. If f is V_1 quasi-open at x for every $V_1 \in \mathcal{U}_{X_1}$, then we call f is *quasi-open at* x. If f is quasi-open at every $x \in X_1$, then we call f is *quasi-open.*

Theorem 3.13. *Let* $f : X_1 \to X_2$ *be a continuous map. For* $x \in X_1$ *the following conditions are equivalent.*

(1) f *is quasi-open at* x.

(2) For all $A \subset X_1$, $x \in \mathcal{Q}\mathrm{Int}A$ *implies* $x \in \mathcal{Q}\mathrm{Int}_f A$.

(3) If U *is an open neighborhood of* x *in* X_1, *then* $\mathcal{Q}\mathrm{Int}_f U$ *is an open neighborhood of* x *in* X_1 *with* $f(x)$ *in* $f(\mathcal{Q}\mathrm{Int}_f U) = \mathcal{Q}\mathrm{Int}f(U)$ *quasi-open in* X_2.

(4) If U is a neighborhood of x in X_1, then $\overline{f(U)}$ is a neighborhood of $f(x)$ in X_2.

Proof. (1)⇒(2). Let $A \subset X_1$ and $x \in \mathcal{Q}\text{Int}A$. Assume V_1 is open in $\mathcal{U}_{X_1}$ with $V_1(x) \subset \overline{A}$. Since $f(x) \in \text{Int}\overline{f(V_1(x))}$. there exists open U_2 in X_2 such that $f(x) \in U_2 \subset \text{Int}\overline{f(V_1(x))} \subset \overline{f(V_1(x))} \subset \overline{f(\overline{A})} \subset \overline{f(A)}$. This means that $x \in f^{-1}(U_2) \subset f^{-1}(\text{Int}\overline{f(A)})$, i.e., $x \in \mathcal{Q}\text{Int}_f A$.

(2)⇒(3). Since U is open, U is quasi-open. By Proposition 3.10, $f(\mathcal{Q}\text{Int}_f U) = \mathcal{Q}\text{Int}f(U)$. If $x \in U$, then $x \in \mathcal{Q}\text{Int}_f U$. This means that $f(x) \in f(\mathcal{Q}\text{Int}_f U) = \mathcal{Q}\text{Int}f(U)$.

(3)⇒(4). Let U be a neighborhood of x. By (3), $f(x) \in f(\mathcal{Q}\text{Int}_f U) = \mathcal{Q}\text{Int}f(U)$. This means that $f(x) \in \mathcal{Q}\text{Int}f(U) \subset \text{Int}\overline{\mathcal{Q}\text{Int}f(U)} \subset \overline{f(U)}$.

(4)⇒(1). Let $V_1 \in \mathcal{U}_{X_1}$ be given. Since $V_1(x)$ is neighborhood of x, $\overline{f(V_1(x))}$ is a neighborhood of $f(x)$. □

THEOREM 3.14. *Let $f : X_1 \to X_2$ be a continuous map. The following conditions are equivalent.*

(1) f is quasi-open.

(2) For all $A \subset X_1$, $\mathcal{Q}\text{Int}A = \mathcal{Q}\text{Int}_f A$.

(3) If A is quasi-open in X_1, then $f(A)$ is quasi-open in X_2.

(4) If U is open in X_1, then $f(U)$ is quasi-open in X_2.

(5) For all U open in X_1, $U = \mathcal{Q}\text{Int}_f U$.

Proof. (1)⇒(2). This is obvious from Theorem 3.13 (1) $\iff$ (2).

(2)⇒(3). Let A be quasi-open in X_1. Since $A = \mathcal{Q}\text{Int}A = \mathcal{Q}\text{Int}_f A$, $A \subset f^{-1}(\text{Int}\overline{f(A)})$. This means that $f(A) \subset \text{Int}\overline{f(A)}$.

(3)⇒(4). Since U is open, U is quasi-open. By (3), $f(U)$ is quasi-open.

(4)⇒(1). Let $V_1 \in \mathcal{U}_{X_1}$ and $x \in X_1$. Since $V_1(x)$ is an open neighborhood of x, $f(x) \in \text{Int}\overline{f(V_1(x))}$.

(2)⇒(5). Since U open in X_1, $\mathcal{Q}\text{Int}_f U = \mathcal{Q}\text{Int}U = U$.

(5)⇒(4). Let U be open in X_1. By (5), $U = \mathcal{Q}\text{Int}_f(U) \subset f^{-1}(\text{Int}\overline{f(U)})$. This means that $f(U) \subset \text{Int}\overline{f(U)}$, i.e., $f(U)$ is quasi-open in X_2. □

THEOREM 3.15. *Let $f : X_1 \to X_2$ and $g : X_2 \to X_3$ be continuous maps. If both f and g are quasi-open, then $g \circ f$ is quasi-open. If $g \circ f$ is quasi-open and f is surjective, then g is quasi-open.*

Proof. Let f and g be quasi-open and let A_1 be quasi-open in X_1. By Proposition 3.14 (3), $f(A_1)$ and $g(f(A_1))$ are quasi-open in X_2 and X_3, respectively. Let $g \circ f$ be quasi-open, f be surjective and U_2 be an open subset of X_2. Since f is continuous, $f^{-1}(U_2)$ is open in X_1.

$(g \circ f)(f^{-1}(U_2)) = g(U_2)$ is quasi-open because $g \circ f$ is quasi-open and f is surjective. □

Definition 3.16. Let $f : X_1 \to X_2$ be a continuous map. We call f is *almost quasi-open* when for all $A \subset X_1$, $\mathrm{Int}A \neq \emptyset$ implies $\mathrm{Int}\overline{f(A)} \neq \emptyset$.

Theorem 3.17. *Let $f : X_1 \to X_2$ be a continuous map. The following conditions are equivalent.*

(1) *f is almost quasi-open.*
(2) *For every $V_1 \in \mathcal{U}_{X_1}$, $\{x : f$ is V_1 quasi-open at $x\}$ is dense in X_1.*
(3) *For every $V_1 \in \mathcal{U}_{X_1}$, $\mathrm{Int}\{x : f$ is V_1 quasi-open at $x\}$ is open and dense in X_1.*
(4) *If A is quasi-open in X_1, then $\mathcal{Q}\mathrm{Int}_f A$ is dense in A.*
(5) *For all $A \subset X_1$, $\mathcal{Q}\mathrm{Int}A \neq \emptyset$ implies $\mathcal{Q}\mathrm{Int}f(A) \neq \emptyset$.*
(6) *B is nowhere dense in X_2 implies $f^{-1}(B)$ is nowhere dense in X_1.*
(7) *D is open and dense in X_2 implies $f^{-1}(D)$ is dense in X_1.*

Proof. (1)⇒(2). Let $V_1 \in \mathcal{U}_{X_1}$ and $x \in X_1$. Let $V = V^{-1}$ be open in $\mathcal{U}_{X_1}$, with $V^2 \subset V_1$. By (1), $\mathrm{Int}\overline{f(V(x))} \neq \emptyset$. Since $\mathcal{Q}\mathrm{Int}_f V(x) = \mathcal{Q}\mathrm{Int}V(x) \cap f^{-1}(\mathrm{Int}\overline{f(V(x))})$ is open and nonemptyset, we can find $x_1 \in \mathcal{Q}\mathrm{Int}_f V(x)$ and notice that $f(x_1) \in \mathrm{Int}\overline{f(V(x))} \subset \mathrm{Int}\overline{f(V_1(x_1))}$. Thus, f is V_1 quasi-open at x_1 . Since V can be chosen arbitrarily small, $\{x : f$ is V_1 quasi-open at $x\}$ is dense.

(2)⇒(3). Let $V_1 \in \mathcal{U}_{X_1}$ be given and let $V = V^{-1}$ be open in $\mathcal{U}_{X_1}$ with $V^2 \subset V_1$. f is V quasi-open at x if and only if there is an open neighborhood U of x such that $f(U) \subset \mathrm{Int}\overline{f(V(x))}$. By (2), $V(x) \cap \{x : f$ is V_1 quasi-open at $x\} \neq \emptyset$ for all $x \in X$. If $x_1 \in U \cap V(x)$, then $f(U) \subset \mathrm{Int}\overline{f(V_1(x_1))}$. So f is V_1 quasi-open at every point of a neighborhood of x.

(3)⇒(4). Let A be quasi-open in X_1. Since $\mathcal{Q}\mathrm{Int}_f A \subset A$, $\overline{\mathcal{Q}\mathrm{Int}_f A} \subset \overline{A}$. For a neighborhood U of $x \in A$, choose V_1 open in $\mathcal{U}_{X_1}$ such that $V_1(x) \subset U$. Since A is quasi-open in X_1, there exists $x_1 \in V_1(x) \cap A$ such that f is V_1 quasi-open at x_1. This means that $f(x_1) \in \mathrm{Int}\overline{f(V_1(x))} \subset \mathrm{Int}\overline{f(U)} = \mathrm{Int}\overline{f(A)}$. Since $x_1 \in A = \mathcal{Q}\mathrm{Int}A$, it follows that $x_1 \in \mathcal{Q}\mathrm{Int}_f A$.

(4)⇒(5). Let A be a subset of X_1 with $\mathcal{Q}\mathrm{Int}A \neq \emptyset$. Put $A_1 = \mathcal{Q}\mathrm{Int}A$. By (4), $\mathcal{Q}\mathrm{Int}_f A_1$ is dense in A_1. Since A_1 is quasi-open, $\mathcal{Q}\mathrm{Int}f(A_1) = f(\mathcal{Q}\mathrm{Int}_f A_1) \neq \emptyset$.

(5)⇒(1). Let A be a subset of X_1 with $\mathrm{Int}A \neq \emptyset$. Put $U = \mathrm{Int}A$. Since U is nonempty open, U is quasi-open. By (5), $\emptyset \neq \mathcal{Q}\mathrm{Int}f(U) \subset \mathrm{Int}\overline{f(A)}$.

(1)⇒(6). Let $\text{Int}\overline{f^{-1}(B)} \neq \emptyset$ and let $A = f^{-1}(\overline{B})$. Since f is continuous, A is closed in X_1 and $\overline{f^{-1}(B)} \subset A$. Since $\text{Int}\overline{f^{-1}(B)} \neq \emptyset$, $\text{Int}A \neq \emptyset$. By (1), $\emptyset \neq \text{Int}\overline{f(A)} \subset \text{Int}\overline{B}$.

(6)⇒(7). Let D be open and dense in X_2 and put $B = X_2 - D$. Then B is nowhere dense in X_2. By (6), $f^{-1}(B)$ is nowhere dense in X_1. This means that $\emptyset = \text{Int}\overline{f^{-1}(B)} = \text{Int}f^{-1}(B) = \text{Int}f^{-1}(X_2 - D) = \text{Int}(X_1 - f^{-1}(D))$.

(7)⇒(1). Let A be a subset of X_1 with $\text{Int}A \neq \emptyset$ and let $\text{Int}\overline{f(A)} = \emptyset$. Put $D = X_2 - \overline{f(A)}$ and D is open and dense. By (7), f^{-1} is open and dense. Since $A \subset f^{-1}(f(A)) \subset f^{-1}(\overline{f(A)})$ and $f^{-1}(D) = X_1 - f^{-1}(\overline{f(A)})$, $f^{-1}(D) \cap A = \emptyset$. This is a contradiction. Thus $\text{Int}\overline{f(A)} \neq \emptyset$. □

THEOREM 3.18. *Let $f : X_1 \to X_2$ and $g : X_2 \to X_3$ be continuous maps. If both f and g are almost quasi-open, then $g \circ f$ is almost quasi-open. If $g \circ f$ is almost quasi-open and f is surjective, then g is almost quasi-open.*

Proof. Let f and g be almost quasi-open and let $Q\text{Int}A$ be a nonempty subset of X_1. By Theorem 3.17, $Q\text{Int}f(A)$ and $Q\text{Int}g(f(A))$ are nonempty subsets of X_2 and X_3, respectively. Let $g \circ f$ be almost quasi-open, f be surjective and let $U_2 \equiv \text{Int}A_2$ be a nonempty set, where $A_2 \subset X_2$. Since f is continuous and surjective, $\overline{f^{-1}(U_2)} = \text{Int}\overline{f^{-1}(U_2)}$ is nonempty. By Theorem 3.17, $\emptyset \neq \text{Int}\overline{(g \circ f)(f^{-1}(U_2))} = \text{Int}\overline{g(U_2)} \subset \text{Int}\overline{g(A_2)}$. □

REMARK 3.19. If for every dense G_δ set B in X_2 $f^{-1}(B)$ is dense in X_1, then f satisfies (7) of Theorem 3.17, so f is almost quasi-open. The converse is true as well if X_1 is Baire.

DEFINITION 3.20. Let $f : X_1 \to X_2$ be a continuous map. For $x \in X_1$ and $V_1 \in \mathcal{U}_{X_1}$, we call f is a V_1 *embedding at* x if there exists $V_2 \in \mathcal{U}_{X_2}$ such that

$$(f^*V_2)(x) = f^{-1}(V_2(f(x))) \subset V_1(x).$$

This just says that the preimage of some neighborhood of $f(x)$ is contained in $V_1(x)$. Clearly f is a V_1 embedding at x if and only if the associated surjective map $f : X_1 \to f(X_1)$ is a V_1 embedding at x. If f is surjective and is a V_1 embedding at x, then f is V_1 open at x because $V_2(f(x)) \subset f(V_1(x))$. If f is a V_1 embedding at x for all $V_1 \in \mathcal{U}_{X_1}$, then we call f is an *embedding at* x.

THEOREM 3.21. *Let $f : X_1 \to X_2$ be a continuous map. For $x \in X_1$ the following conditions are equivalent.*

(1) *f is an embedding at x.*
(2) *$f^{-1}u[f(x)] = u[x]$, where $u[f(x)]$ is an uniform neighborhood of $f(x)$.*
(3) *For every $V_1 \in \mathcal{U}_{X_1}$, there exists U_2 a neighborhood of $f(x)$ such that $f^{-1}(U_2) \times f^{-1}(U_2) \subset V_1$.*

If f is an embedding at x, then $f^{-1}(f(x)) = E_f(x)$ is $\{x\}$. Conversely if f is a closed map and $E_f(x) = \{x\}$, then f is an embedding at x.

Proof. If $V = V^{-1}$ and $V^2 \subset V_1$, then $f^{-1}(U) \subset V(x)$ implies $f^{-1}(U) \times f^{-1}(U) \subset V_1$. So (1)⇒(3). (3) obviously implies (1) which is equivalent to (2), i.e.,$\{(f^*V_2)\ (x) = f^{-1}(V_2(f(x))) : V_2 \in \mathcal{U}_{X_2}\}$ is a base for the neighborhood of x. Clearly if f is a V_1 embedding at x, then $E_f(x) \subset V_1(x)$. If f is an embedding at x, $E_f(x) = \{x\}$. If V_1 is open and f is closed, then $f(X_1 - V_1(x))$ is closed in X_2 and $f(X_1 - V_1(x)) \cap f(x) = \emptyset$ if $E_f(x) = \{x\}$. In that case the complement U satisfies $f^{-1}(U) \subset V_1(x)$. The last assertion is already true at the V_1 level as previously discussed. □

Notice that f is an embedding at x, for all $x \in X_1$ if and only if f is an embedding, i.e., if and only if the surjective map $f : X_1 \to f(X_1)$ is a homeomorphism.

Definition 3.22. A continuous map $f : X_1 \to X_2$ is called an *almost embedding* if U_1 is open and nonempty in X_1, then there exists U_2 open in X_2 such that $f^{-1}(U_2)$ is a nonempty subset of U_1.

Theorem 3.23. *Let $f : X_1 \to X_2$ be a continuous map. The following conditions are equivalent.*

(1) *f is an almost embedding.*
(2) *For every $V_1 \in \mathcal{U}_{X_1}$, $\{x \in X_1 : f^{-1}(U) \times f^{-1}(U) \subset V_1$ for some neighborhood U of $f(x)\}$ is open and dense in X_1.*
(3) *For every $V_1 \in \mathcal{U}_{X_1}$, $\{x \in X_1 : f$ is a V_1 embedding at $x\}$ is dense in X_1.*
(4) *For $D \subset X_1$, $f(D)$ dense in $f(X_1)$ implies D dense in X_1.*
(5) *For all U open in X_1 the open set:*
$$U^f \equiv X_1 - f^{-1}(\overline{f(X_1 - U)}) = f^{-1}(\mathrm{Int}(X_2 - f(X_1 - U)))$$
is dense in U.

Proof. (1)⇒(2). If U_1 is open and nonempty, then we shrink to get $U_1 \times U_1 \subset V_1$. By (1), there exist $x \in U_1$ and U_2 a neighborhood of $f(x)$ such that $f^{-1}(U_2) \subset U_1$.

(2)⇒(3). This is obvious.

(3)⇒(4). Assume $f(D)$ is dense in $f(X_1)$. Given $x_0 \in X_1$ and $W \in \mathcal{U}_{X_1}$, we show that $W(x_0) \cap D \neq \emptyset$. Choose $V_1 \in \mathcal{U}_{X_1}$ symmetric such that $V_1^2 \subset W$. By (3), there exists $x_1 \in V(x_0)$ and f is a V_1 embedding at x_1. This means that $f^{-1}(V_2(f(x_1))) = (f^*V_2)(x_1) \subset V_1(x_1)$ for some $V_2 \in \mathcal{U}_{X_2}$. Since $f(D)$ is dense in $f(X_1)$, we can find $y_1 \in f(D) \cap V_2(f(x_1))$. Thus there exists $x_2 \in D$ such that $f(x_2) = y_1$ and $f(x_2) \in V_2(f(x_1))$. Therefore $x \in f^{-1}(V_2(f(x_1))) = (f^*V_2)(x_1) \subset V_1(x_1) \subset V_1(V_1(x_0)) = V_1^2(x_0) \subset W(x_0)$. It means that $x_2 \in D \cap W(x_0)$. Thus D is dense in X_1.

(4)⇒(5). Let $D = U^f \cap (X_1 - U) = f^{-1}(\mathrm{Int}(X_2 - f(X_1 - U))) \cup (X_1 - U)$. $f(D) = [f(X_1) \cap \mathrm{Int}(X_2 - f(X_1 - U))] \cup f(X_1 - U)$ which is dense in $f(X_1)$. By (4), D is dense in X_1, so $D \cap U = (U^f \cup (X_1 - U)) \cap U = U^f \cap U$ is dense in U.

(5)⇒(1). Let U_1 be open and nonempty in X_1. Put $U_2 = \mathrm{Int}(X_2 - f(X_1 - U_1))$. By (5), $f^{-1}(U_2)$ is dense in U_1. Thus $f^{-1}(U_2)$ is nonempty. □

Definition 3.24. A continuous map f is called *an almost homeomorphism* if it is a surjective almost embedding.

Example 3.25. *Let $X = \{a, b, c\}$ and $\tau = \{\emptyset, \{a, b, c\}\}$. Let $Y = \{1, 2\}$ and $\sigma = \{\emptyset, \{1, 2\}\}$. Define a map $f : (X, \tau) \to (Y, \sigma)$ as follows: $f(a) = 1$, $f(b) = 1$ and $f(c) = 2$. Then f is an almost homeomorphism. However, f is not a homeomorphism.*

Theorem 3.26. *If f is an almost homeomorphism, then for any V_1 closed in $\mathcal{U}_{X_1}$*

$$\{x : f \text{ is } V_1 \text{ open at } x\} = \{x : f \text{ is a } V_1 \text{ embedding at } x\},$$
$$\{x : f \text{ is open at } x\} = \{x : f \text{ is an embedding at } x\}.$$

Proof. In general, for a surjective continuous map f and $V_1 \in \mathcal{U}_{X_1}$, if f is V_1 embedding at x, then there exists $V_2 \in \mathcal{U}_{X_2}$ such that $(f^*V_2)(x) = f^{-1}(V_2(f(x))) \subset V_1(x)$. It means that $f(x) \in V_2(f(x)) \subset f(V_1(x))$. Thus f is V_1 open at x.

If f is an almost embedding, A is closed in X_1, and U is open in X_2, then the following property holds ;

$$\text{if } U \subset f(A), \text{ then } f^{-1}(U) \subset A.$$

If not, then $U_1 = f^{-1}(U) - A$ is a nonempty open subset of X_1. Since f is almost embedding, there exists nonempty open U_2 in X_2 such that $f^{-1}(U_2) \subset U_1$. But $U_2 = f(f^{-1}(U_2)) \subset f(U_1) \subset U \subset f(A)$. This is a contradiction.

If V_1 is closed and f is V_1 open at x, then there exists open U of X_2 such that $f(x) \in U \subset f(V_1(x))$. Hence there exists V_2 open in $\mathcal{U}_{X_2}$ such that $V_2(f(x)) \subset U \subset f(V_1(x))$. By the above property, $f^{-1}(V_2(f(x))) \subset V_1(x)$. Hence f is V_1 embedding at x.

$\{x : f \text{ is open at } x\} = \{x : f \text{ is an embedding at } x\}$ is clear. □

THEOREM 3.27. *Let $f : X_1 \to X_2$ and $g : X_2 \to X_3$ be continuous maps. Then*

(1) *Assume f is surjective. $g \circ f$ is an almost homeomorphism if and only if both f and g are almost homeomorphisms.*
(2) *Assume g is an almost homeomorphism. If $g \circ f$ is almost open, then f is almost quasi-open.*
(3) *Assume g is an almost homeomorphism. If $g \circ f$ is almost quasi-open and g is closed, then f is almost quasi-open.*

Proof. (1) Let U_1 be a nonempty subset of X_1. Since $g \circ f$ is an almost homeomorphism, there exists nonempty open $U_3 \subset X_3$ such that $(g \circ f)^{-1}(U_3)$ is a nonempty subset of U_1, i.e., $f^{-1}(g^{-1}(U_3)) \subset U_1$. Put $U_2 \equiv g^{-1}(U_3)$. Since g is continuous, U_2 is a nonempty open subset of X_2. It follows that $f^{-1}(U_2) \subset U_1$. Hence f is an almost homeomorphism. Let U_2 be a nonempty open subset of X_2. Since f is surjective continuous, $f^{-1}(U_2)$ is nonempty open in X_1. By the definition of the almost homeomorphism of $g \circ f$, there exists nonempty open $U_3 \subset X_3$ such that $(g \circ f)^{-1}(U_3) \subset f^{-1}(U_2)$. It follows that $g^{-1}(U_3) = f(f^{-1}(g^{-1}(U_3))) \subset f(f^{-1}(U_2)) = U_2$. Hence g is an almost homeomorphism.

Conversely, let f and g be almost homeomorphisms and let U_1 be a nonempty subset of X_1. By the definition of the almost homeomorphism of f and g, we can find nonempty open $U_2 \subset X_2$ and $U_3 \subset X_3$ such that $f^{-1}(U_2) \subset U_1$ and $g^{-1}(U_3) \subset U_2$. This means that $(g \circ f)^{-1}(U_3) = f^{-1}(g^{-1}(U_3)) \subset f^{-1}(U_2) \subset U_1$.

(2) Let A be a subset of X_1 with $\mathrm{Int}A \neq \emptyset$. Since $g \circ f$ is almost open, $\mathrm{Int}(g \circ f)(\mathrm{Int}A) \neq \emptyset$. Since g is surjective and continuous, $g^{-1}(\mathrm{Int}g(f(\mathrm{Int}A)))$ is nonempty open in X_2. Since g is an almost homeomorhpism, there exists nonempty open U_3 in X_3 such that $g^{-1}(U_3) \subset g^{-1}(\mathrm{Int}g(f(\mathrm{Int}A)))$.

(3) Let A be a subset of X_1 with $\mathrm{Int}A \neq \emptyset$. Put $U_1 = \mathrm{Int}A$. Since $g \circ f$ is almost quasi-open, $\mathrm{Int}\overline{(g \circ f)(U_1)} \neq \emptyset$. Since g is continuous and closed, $\mathrm{Int}\overline{g(f(U_1))} = \mathrm{Int}g(\overline{f(U_1)})$. Put $U_3 = \mathrm{Int}\overline{g(f(U_1))}$. Then U_3 is nonempty open and satisfies $U_3 \subset g(\overline{f(U_1)})$ and $g^{-1}(U_3) \subset \overline{f(U_1)}$. This means that $\emptyset \neq g^{-1}(U_3) \subset \mathrm{Int}\overline{f(U_1)} \subset \mathrm{Int}\overline{f(A)}$. Thus, f is amlost quasi-open. □

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