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Dynamic Characteristics of a System Composed of a Supporting Structure and Subsystems

지지구조물과 부시스템으로 이루어진 계의 동적 특성

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Key Words : Approximate Natural Frequency(근사적 고유 진동수), Mode Shape(모드형), Supporting Structure(지지구조물), Subsystem(부시스템), Dunkerley's Formula(Dunkerley 공식), Harmonic Excitation(조화가진)

ABSTRACT

The natural frequencies and mode shapes of a system composed of subsystems on a supporting structure have been derived approximately. The system is modeled as a discrete system, and it is assumed that the masses of subsystems are much smaller than the mass of a supporting structure. It has been found that the fundamental frequency corresponds to the supporting structure, and each higher frequency corresponds to each subsystem. A relation between the vibration amplitudes of the supporting structure and subsystems is also derived for the case when the supporting structure is excited by a harmonic force.

요 약

지지구조물 위의 부시스템으로 이루어진 계의 고유 진동수와 모드형을 근사적으로 구하는 식을 유도하였다. 이 계는 이산계로 모형화되었고, 부시스템의 질량은 지지구조물의 질량에 비하여 매우 작다고 가정하였다. 기본진동수는 지지구조물에 대응하고, 고차 진동수는 각각의 부시스템에 대응한다는 것이 밝혀졌다. 또한 지지구조물이 조화력에 의해 가진될 때 지지구조물과 부시스템의 진동 진폭 사이의 관계를 유도하였다.

1. Introduction

We can observe many systems around us which are composed of subsystems on a supporting structure. Those systems can be modeled as discrete systems shown in Fig. 1. One example of such a system is a cockpit seat of a helicopter. A seat is a subsystem and a fuselage of a helicopter is a supporting structure. In many cases the mass of a subsystem is much smaller than that of a supporting structure. For dynamic analysis of such a system, the natural frequencies and the mode shapes of the system can be calculated by solving

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a corresponding eigenvalue problem. Although this is an exact method, the expansion of the characteristic determinant and the solution of the resulting polynomial equation can become quite tedious for large values of degree-of-freedom. Several analytical and numerical methods have been developed to compute approximately the natural frequencies and mode shapes of multi degree-of-freedom systems. They include Dunkerley's formula(1), Rayleigh's method(2), Holzer's method(3), and the matrix iteration method(4). Approximate expressions for the fundamental frequency of several dynamic systems have been derived using these methods and are used frequently(5).

In this research approximate expressions for the natural frequencies and the mode shapes of the above mentioned systems are derived. If the supporting structure of the system is excited by a harmonic force, both the supporting structure and subsystems vibrate harmonically. A relation between the vibration amplitudes is also derived.

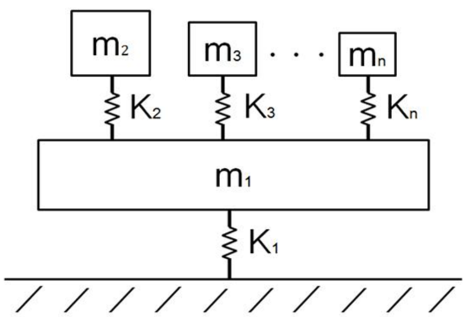

Fig. 1 A system composed of a supporting structure and subsystems

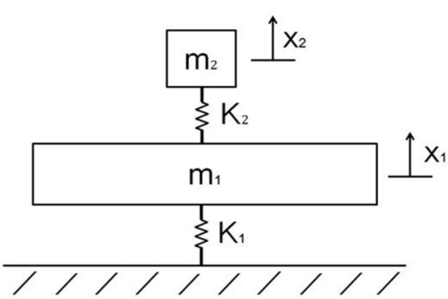

Fig. 2 A system with a supporting structure and a subsystem

2. Characteristics of the Composed Systems

2.1 Natural Frequencies

Consider a system composed of a supporting structure and a subsystem shown in Fig. 2. The mass and stiffness matrices of the system are given by

$$[\mathrm{m}] = \begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix} \tag{1}$$

and

$$[\mathrm{k}] = \begin{bmatrix} k_1 + k_2 & -k_2 \\ -k_2 & k_2 \end{bmatrix} \tag{2}$$

The characteristic equation is given by

$$\left|[\mathrm{k}] - \omega^2 [m]\right| = m_1 m_2 \omega^4 - \left\{\left(k_1 + k_2\right) m_2 + k_2 m_1\right\} \omega^2 + k_1 k_2 = 0 \tag{3}$$

Letting $\lambda = \omega^2$, the solution of the above quadratic equation is obtained as

$$\lambda = \frac{k_1 + k_2}{2m_1} + \frac{k_2}{2m_2} \pm \sqrt{\frac{(k_1 + k_2)^2}{4m_1^2} + \frac{k_2{}^2}{4m_2^2} + \frac{k_2^2 - k_1 k_2}{2m_1 m_2}} \tag{4}$$

Letting $\mu = m_2/m_1$ and $\upsilon = k_2/k_1$, the above equation can be expressed as

$$\lambda = \frac{k_1}{m_1}\left[\frac{1+\upsilon}{2} + \frac{\upsilon}{2\mu} \pm \frac{\sqrt{\mu^2(1+\upsilon)^2 + \upsilon^2 + 2\mu\upsilon(\upsilon - 1)}}{2\mu}\right] \tag{5}$$

If we let $x = \mu^2(1+\upsilon)^2 + 2\mu\upsilon(\upsilon - 1)$, the numerator of the third term in the bracket of Eq. (5) is of the form, $\sqrt{x+\upsilon^2}$. As μ approaches zero, x approaches zero and $\sqrt{x+\upsilon^2}$ can be expressed around $x=0$ as follows using the Taylor series.

$$\sqrt{x+\upsilon^2} = \upsilon + \frac{x}{2\upsilon} \tag{6}$$

Under this condition Eq. (5) can be expressed as

$$\lambda = \frac{k_1}{m_1}\left[\frac{1+\upsilon}{2} + \frac{\upsilon}{2\mu} \pm \left(\frac{\upsilon}{2\mu} + \frac{\mu(1+\upsilon)^2}{4\upsilon} + \frac{\upsilon-1}{2}\right)\right] \tag{7}$$

The above solutions become

$$\lambda_1 = \frac{k_1}{m_1}\left[1 - \frac{\mu(1+\upsilon)^2}{4\upsilon}\right] \quad \text{and} \quad \lambda_2 = \frac{k_1}{m_1}\left[\upsilon + \frac{\upsilon}{\mu} + \frac{\mu(1+\upsilon)^2}{4\upsilon}\right] \tag{8}$$

If the term $\mu(1+\upsilon)^2/4\upsilon$ is very small compared to the other terms, the above solutions approach

$$\lambda_1 = \frac{k_1}{m_1} \quad \text{and} \quad \lambda_2 = \frac{k_1}{m_1}\left[\upsilon + \frac{\upsilon}{\mu}\right] = \frac{k_2}{m_1} + \frac{k_2}{m_2} \tag{9}$$

The above mentioned condition is satisfied when μ is very small while υ is neither very small nor very large, in other words, when m_2 is very small compared to m_1, while k_1 and k_2 have magnitudes of similar orders. Finally the two natural frequencies can be expressed under this condition as

$$\omega_1 = \sqrt{\frac{k_1}{m_1}} \quad \text{and} \quad \omega_2 = \sqrt{\frac{k_2}{m_1} + \frac{k_2}{m_2}} \tag{10}$$

Examining the above equation, we can find that the second natural frequency is expressed in terms of the mass and stiffness of the subsystem and the mass of the supporting structure.

To obtain a more accurate fundamental frequency, Dunkerley's formula is applied. The formula gives an approximate fundamental frequency of a n-dof system with n masses m_1, m_2, $\cdots$, m_n as follows.

$$\frac{1}{\omega_1^2} \approx a_{11}m_1 + a_{22}m_2 + \cdots + a_{nn}m_n \tag{11}$$

where a_{ij} represents the flexibility influence coefficient which is defined as the deflection at point i (point where mass m_i is located) due to a unit load at point j. For the system shown in Fig. 2, a_{11} and a_{22} become[6]

$$a_{11} = \frac{1}{k_1} \quad \text{and} \quad a_{22} = \frac{1}{k_1} + \frac{1}{k_2} \tag{12}$$

Inserting these values into Eq. (11), we obtain an approximate fundamental frequency

$$\omega_1 = \sqrt{\frac{k_1}{m_1 + m_2}} \tag{13}$$

Next, consider a system composed of a supporting structure and two subsystems as shown in Fig. 3. On the analogy of the results in Eq. (10), we can expect the second and third natural frequencies as follows.

$$\omega_2 = \sqrt{\frac{k_2}{m_1} + \frac{k_2}{m_2}} \quad \text{and} \quad \omega_3 = \sqrt{\frac{k_3}{m_1} + \frac{k_3}{m_3}} \tag{14}$$

The fundamental frequency is obtained using Dunkerley's formula. For the system in Fig. 3, a_{11} and a_{22} are the same as those in Eq. (12), and a_{33} is obtained as

$$a_{33} = \frac{1}{k_1} + \frac{1}{k_3} \tag{15}$$

Inserting the a_{ii} values into Eq. (11), we obtain an

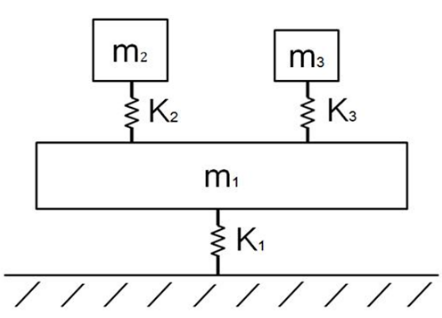

Fig. 3 A system with a supporting structure and two subsystems

approximate fundamental frequency for the system with two subsystems.

$$\omega_1 = \sqrt{\frac{k_1}{m_1 + m_2 + m_3}} \tag{16}$$

From the above results we can generalize that the natural frequencies of a system with a supporting structure and n-1 subsystems are expressed as

$$\omega_1 = \sqrt{\frac{k_1}{m_1 + m_2 + \cdots + m_n}} \tag{17}$$

and

$$\omega_i = \sqrt{\frac{k_i}{m_1} + \frac{k_i}{m_i}}\ ,\ i \geq 2 \tag{18}$$

The above equations imply that the fundamental frequency is expressed in terms of the stiffness of the supporting structure and the total mass of the system, and the higher natural frequency corresponds to each subsystem and is expressed in terms of the parameters (mass and stiffness) of the subsystem and the mass of the supporting structure.

2.2 Mode Shapes

The mode shape vector $\vec{X}$ of a discrete system can be obtained by solving the following equation with the previously obtained natural frequencies.

$$\left([\mathrm{k}] - \omega^2 [m]\right)\vec{X} = 0 \tag{19}$$

For the system in Fig. 3 the above equation becomes

$$\begin{Bmatrix} \left(k_1 + k_2 + k_3 - \omega^2 m_1\right)X_1 - k_2 X_2 - k_3 X_3 = 0 \\ -k_2 X_1 + \left(k_2 - \omega^2 m_2\right)X_2 = 0 \\ -k_3 X_1 + \left(k_3 - \omega^2 m_3\right)X_3 = 0 \end{Bmatrix} \tag{20}$$

where X_i represents the amplitude of mass m_i. From the second and third equations of Eq. (20) we obtain

$$X_2 = \frac{k_2}{k_2 - \omega^2 m_2} X_1 \tag{21}$$

and

$$X_3 = \frac{k_3}{k_3 - \omega^2 m_3} X_1 \tag{22}$$

respectively. Then the mode shape vector becomes

$$\vec{X} = \begin{Bmatrix} X_1 \\ X_2 \\ X_3 \end{Bmatrix} = \begin{Bmatrix} 1 \\ \dfrac{k_2}{k_2 - \omega^2 m_2} \\ \dfrac{k_3}{k_3 - \omega^2 m_3} \end{Bmatrix} \tag{23}$$

If we insert the previously obtained natural frequencies ω_1^2, ω_2^2, and ω_3^2 into Eq. (23), we obtain the corresponding mode shapes.

The above result can be generalized for n-dof systems with n-1 subsystems as follows.

$$\vec{X} = \begin{Bmatrix} 1 \\ \dfrac{k_2}{k_2 - \omega^2 m_2} \\ \vdots \\ \dfrac{k_n}{k_n - \omega^2 m_n} \end{Bmatrix} \tag{24}$$

Inserting the natural frequencies ω_i^2 for ω^2 in the above equation, we obtain the ith mode shape vector.

To evaluate the accuracy of the above approximate expressions, a system with three subsystems has been considered. The given parameters of the system are: m_1 = 1 kg, m_2 = 0.01 kg, m_3 = 0.03 kg, m_4 = 0.05 kg; k_1 = 10 000 N/m, k_2 = 6000 N/m, k_3 = 1000 N/m, k_4 = 10 000 N/m. These parameters satisfy the condition that the masses of subsystems are very small compared to the mass of supporting structure, and the stiffness of springs are of similar orders. The approximate natural frequencies and mode shapes are compared with the exact ones obtained by solving the corresponding eigenvalue problem in Tables 1 and 2, respectively. The approximate mode shape vectors are multiplied by some constants so that the largest components become the same to those of the exact vectors. The tables show that the errors between the exact and approximate frequencies are less than 1 %, and the approximate mode shapes are very close to the exact ones. Examining the obtained mode shapes, we can find that all masses of the system vibrate with similar amplitudes for the fundamental mode, while the mass corresponding to a higher mode vibrates with a much larger amplitude than the other masses for higher modes.

2.3 Vibration Amplitudes in Case of Harmonic Excitation

Consider the case when the supporting structure in Fig. 3 is excited by a harmonic force $F_0 \cos\omega t$. The equation of motion of the system is similar to Eq. (20) and is expressed as follows.

$$\begin{Bmatrix} \left(k_1+k_2+k_3-\omega^2 m_1\right)X_1-k_2X_2-k_3X_3=F_0 \\ -k_2X_1+\left(k_2-\omega^2 m_2\right)X_2=0 \\ -k_3X_1+\left(k_3-\omega^2 m_3\right)X_3=0 \end{Bmatrix} \tag{25}$$

where X_i represents the amplitude of the mass m_i. Since the second and third equations of Eq. (25) are identical to those of Eq. (20), the ratios of amplitudes are identical to Eq. (21) and (22). Consequently the amplitude ratio between masses can be obtained from Eq. (23) by inserting the excitation frequency ω into the equation. Similarly the same relation can be obtained from Eq. (24) for a general n-dof system. The result implies that the amplitude ratio of a subsystem to a supporting structure is not affected by addition of another subsystem.

When the supporting structure in Fig. 2 is excited by a harmonic force, the vibration amplitude of the subsystem may be excessive. In that case it can be reduced by attaching to the subsystem a vibration absorber which is composed of a mass, m_a and a spring with stiffness, k_a. It can be shown that the vibration amplitude of the subsystem can be reduced to zero if the natural frequency of the absorber is equal to the excitation frequency. That is,

$$\omega=\sqrt{\frac{k_a}{m_a}} \tag{26}$$

Next, consider a system with two subsystems on a supporting structure which is harmonically excited. Following a similar procedure as above it can be shown that the vibration amplitude of each subsystem can be reduced to zero by attaching an

Table 1 Comparison of the exact and approximate natural frequencies for a 4-dof system

Natural frequency	Exact value(rad/s)	Approximate value(rad/s)	% Error
	95.19	95.78	0.62
	186.04	185.29	-0.40
	458.69	458.26	-0.09
	778.63	778.46	-0.02

Table 2 Comparison of the exact and approximate mode shapes for a 4-dof system

Mode shape	Exact value	Approximate value
$\vec{X}^1$	$\begin{Bmatrix} 0.9442 \\ 0.9587 \\ 1.2966 \\ 0.9890 \end{Bmatrix}$	$\begin{Bmatrix} 0.9398 \\ 0.9544 \\ 1.2966 \\ 0.9850 \end{Bmatrix}$
$\vec{X}^2$	$\begin{Bmatrix} -0.2158 \\ -0.2290 \\ 5.6259 \\ -0.2609 \end{Bmatrix}$	$\begin{Bmatrix} -0.1688 \\ -0.1790 \\ 5.6259 \\ -0.2037 \end{Bmatrix}$
$\vec{X}^3$	$\begin{Bmatrix} -0.2262 \\ -0.3484 \\ 0.0426 \\ 4.3533 \end{Bmatrix}$	$\begin{Bmatrix} -0.2177 \\ -0.3349 \\ 0.0411 \\ 4.3533 \end{Bmatrix}$
$\vec{X}^4$	$\begin{Bmatrix} -0.2262 \\ -0.3484 \\ 0.0426 \\ 4.3533 \end{Bmatrix}$	$\begin{Bmatrix} -0.0995 \\ 9.9452 \\ 0.0058 \\ 0.0490 \end{Bmatrix}$

absorber which meets the above requirement to each subsystem. The result implies that the vibration amplitude of each subsystem on a supporting structure can be reduced by using an absorber, and the absorber can be designed independently without considering the other subsystems.

3. Conclusions

A n-dof system composed of n-1 subsystems on a supporting structure has been considered. The system is modeled as a discrete system, and it is assumed that the masses of subsystems are much smaller than the mass of the supporting structure. The natural frequencies and mode shapes of the system have been derived approximately. The fundamental frequency is expressed in terms of the stiffness of the supporting structure and the total mass of the system. The higher natural frequency corresponds to each subsystem and is expressed in terms of the parameters (mass and stiffness) of the subsystem and the mass of the supporting structure. The mode shapes are obtained from the same expression by inserting the approximate natural frequencies. Numerical simulations have proved the accuracy of the approximate natural frequencies and mode shapes.

When the supporting structure is excited by a harmonic force, the amplitude ratio between masses can be obtained from the above mode shape vector. The amplitude ratio of a subsystem to the supporting structure does not change by addition of another subsystem. A vibration absorber for each subsystem can be designed independently without considering other subsystems.

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