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A new generalization of exponentiated Frechet distribution

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Abstract: Motivated by the recent work of Cordeiro and Castro (2011), we study the Kumaraswamy exponentiated Frechet distribution (KEF). We derive some mathematical properties of the (KEF) including moment generating function, moments, quantile function and incomplete moment. We provide explicit expressions for the density function of the order statistics and their moments. In addition, the method of maximum likelihood and least squares and weighted least squares estimators are discuss for estimating the model parameters. A real data set is used to illustrate the importance and flexibility of the new distribution.

Key Words: exponentiated Frechet distribution, hazard function, Kumaraswamy distribution, maximum likelihood estimation, moments

1. INTRODUCTION AND MOTIVATION

The Frechet distribution is the most popular model for analyzing skewed data and hydrological processes. One of the important families of distributions in lifetime tests is the exponentiated Frechet (EF) distribution. The exponentiated Frechet (EF) distribution has been introduced by Nadarajah and Kotz (2003) as a generalization of the standared Frechet distribution which has cumulative distribution function (c.d.f.) and a probability density function (p.d.f.) of the form, respectively;

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$$G(x,\alpha,\lambda,\theta)=1-\left[1-e^{-(\frac{\theta}{x})^{\lambda}}\right]^{\alpha}, \alpha>0, \lambda>0, \theta>0 \text{ and } x\geq 0. \quad (1)$$

where $\theta>0$ is the scale parameter and λ,α are shape parameters respectively. The corresponding probability density function (pdf) is given by

$$g(x,\alpha,\lambda,\theta)=\alpha\lambda\theta^{\lambda}x^{-(1+\lambda)}e^{-(\frac{\theta}{x})^{\lambda}}\left[1-e^{-(\frac{\theta}{x})^{\lambda}}\right]^{\alpha-1}. \quad (2)$$

The Kumaraswamy distribution (Kumaraswamy, 1980) is not very common among statisticians and has been little explored in the literature. We refer to the Kum distribution to denote the Kumaraswamy distribution. Its cumulative distribution function (cdf) is defined by

$$F(x)=1-\left(1-x^{a}\right)^{b}, 0<x<1, \quad (3)$$

where $a>0,$ and $b>0$ are two additional parameters whose role is to introduce asymmetry and produce distributions with heavier tails. The Kum distribution does not seem to be very familiar to statisticians and has not been investigated systematically in much detail before, nor has its relative interchangeability with the beta distribution has been widely appreciated. However, in a very recent paper, Jones (2009) explored the background and genesis of the Kum distribution and, more importantly, made clear some similarities and differences between the beta and Kum distributions. He highlighted several advantages of the Kum distribution over the beta distribution: the normalizing constant is very simple; simple explicit formulae for the distribution and quantile functions which do not involve any special functions; a simple formula for random variate generation; explicit formulae for L-moments and simpler formulae for moments of order statistics. Further, according to Jones (2009), the beta distribution has the following advantages over the Kum distribution: simpler formulae for moments and moment generating function (mgf); a one-parameter sub-family of symmetric distributions; simpler moment estimation and more ways of generating the distribution via physical processes.

The probability density function (pdf) of the Kum distribution also has a simple form

$$f(x)=abx^{a-1}\left(1-x^{a}\right)^{b-1}, \quad (4)$$

and it can be unimodal, increasing, decreasing or constant, depending in the same way on the values of its parameters like the beta distribution.

If $G(x)$ is the baseline cdf of a random variable, the cdf of the Kum-generalized distribution, say $K-G$ distribution, is defined by (Cordeiro and Castro, 2010)

$$F(x)=1-\left[1-G(x)^{a}\right]^{b}. \quad (5)$$

The density function corresponding to (5) is

$$f(x)=abg(x)G(x)^{a-1}\left[1-G(x)^{a}\right]^{b-1}, \quad (6)$$

where $g(x)=\frac{d}{dx}G(x)$. The density family in (6) has many of the same properties of the class of beta-G distributions (see Eugene et al. (2002)), but has some advantages in terms of

tractability, since it does not involve any special function such as the beta function. So, the new $K-G$ distribution is obtained by adding two parameters a and b to the quantile function of the G distribution. This generalization contains distributions with unimodal and bathtub shaped hazard rate functions. It also contemplates a broad class of models with monotone risk functions. Some mathematical properties of the $K-G$ distribution derived by Cordeiro and Castro (2010) are usually much simpler than those properties of the beta G distribution (Eugene et al., 2002).
In this note, we combine the works of Kumaraswamy (1980) and Cordeiro and de Castro (2011) to derive some mathematical properties of a new model, called the Kumaraswamy exponentiated Frechet (KEF) distribution. Equivalently, as occurs with the beta-G family of distributions. The special $K-G$ distributions can be generated as follows: the K-normal distribution is obtained by taking $G(x)$ in (4) to be the normal cumulative function. Analogously, the K-Weibull (Cordeiro et al.(2010)), General results for the Kumaraswamy-G distribution (Nadarajah et al.(2011)). K-generalized gamma (Pascoa et al.(2011)), K-Birnbaum-Saunders (Saulo et al. (2011)) Beta-Linear Failure Rate Distribution and its Applications (see Jafari et al.(2012)) and K- Gumbel (Cordeiro et al. (2011)) distributions are obtained by taking $G(x)$ to be the cdf of the Weibull, generalized gamma, Birnbaum-Saunders and Gumbel distributions, respectively, among several others. Hence, each new $K-G$ distribution can be generated from a specified G distribution.
A physical interpretation of the $K-G$ distribution given by (5) and (6) (for a and b positive integers) is as follows. Suppose a system is made of b independent components and that each component is made up of a independent subcomponents. Suppose the system fails if any of the b components fails and that each component fails if all of the a subcomponents fail. Let $X_{j1}, X_{j2}, ..., X_{ja}$ denote the life times of the subcomponents with in the j_{th} component, $j=1,...,b$ with common (cdf) G. Let X_j denote the lifetime of the j_{th} component, $j=1,...,b$ and let X denote the lifetime of the entire system. Then the (cdf) of X is given by

$$\begin{aligned}P(X\le x) &= 1-P(X_1>x, X_2>x,...,X_b>x)\\ &= 1-\left[P(X_1>x)\right]^b = 1-\{1-P(X_1\le x)\}^b\\ &= 1-\{1-P(X_{11}\le x, X_{12}\le x,...,X_{1a}\le x)\}^b\\ &= 1-\left\{1-P\left[X_{11}\le x\right]^a\right\}^b = 1-\left\{1-G^a(x)\right\}^b.\end{aligned}$$

So, it follows that the $K-G$ distribution given by (3) and (4) is precisely the time to failure distribution of the entire system. The rest of the article is organized as follows. In Section 2, we define the cumulative function, probability density function and hazard functions of the

KEF distribution and some special cases. Quantile function, median, moment generating function and moments are discussed in Section 3. Section 4 included the order statistics. The least squares and weighted least squares estimators are introduced in Section 5. Maximum likelihood estimation is performed and the observed information matrix is determined in Section 6. Section 7 gives applications involving a real data set. The probability density function in Equation (6) does not involve any complicated function. If X is a random variable with pdf in (6), we write $X \sim KEF(a,b,\alpha,\theta,\lambda)$.

2. KUMARASWAMY EXPONENTIATED FRECHET DISTRIBUTION

Let $G(x,a,b,\alpha,\theta,\lambda)$ is the exponentiated Frechet cumulative distribution with parameters a,b,α,θ and λ, then the Equation (5) yields the Kumaraswamy exponentiated Frechet (KEF) cumulative distribution

$$F_{X|\{a,b,\alpha,\theta,\lambda\}}(x) = 1-\left[1-\left(1-\left(1-e^{-(\frac{\theta}{x})^{\lambda}}\right)^{\alpha}\right)^{a}\right]^{b}, \tag{7}$$

The corresponding probability density function is given by

$$f_{X|\{a,b,\alpha,\theta,\lambda\}}(x) = a\, b\alpha\lambda\ \theta^{\lambda} x^{-(1+\lambda)}\ e^{-(\frac{\theta}{x})^{\lambda}}\left[1-e^{-(\frac{\theta}{x})^{\lambda}}\right]^{\alpha-1}\left[1-\left[1-e^{-(\frac{\theta}{x})^{\lambda}}\right]^{\alpha}\right]^{a-1} \times\left[1-\left[1-\left[1-e^{-(\frac{\theta}{x})^{\lambda}}\right]^{\alpha}\right]^{a}\right]^{b-1}, \tag{8}$$

In Figures 1 and 2, we plot the KEF pdf and cdf for selected parameter values respectively.

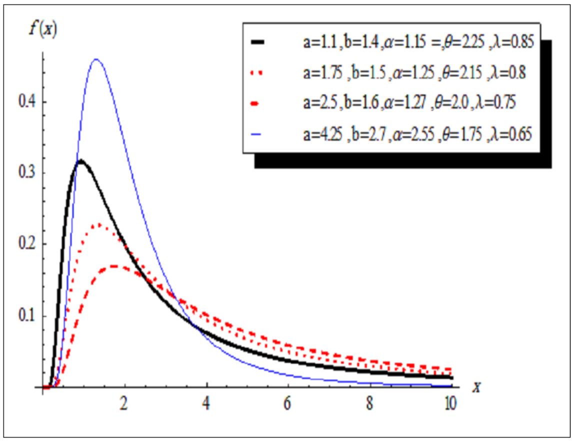

Figure 1. Plots of the KEF density for selected parameter values

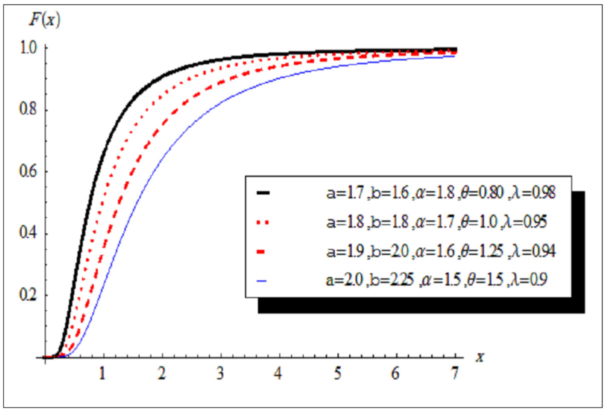

Figure 2. Plots of the KEF cdf for selected parameter values

The associated hazard (failure) rate (HR) and reversed hazard rate (RHR) function are given respectively by,

$$h_{X|\{a,b,\alpha,\theta,\lambda\}}(x)=\frac{f(x,a,b,\alpha,\theta,\lambda)}{1-F(x,a,b,\alpha,\theta,\lambda)}$$

$$=\frac{a\, b\alpha\lambda\ \theta^{\lambda}x^{-(1+\lambda)}\ e^{-(\frac{\theta}{x})^{\lambda}}\left[1-e^{-(\frac{\theta}{x})^{\lambda}}\right]^{\alpha-1}\left[1-\left[1-e^{-(\frac{\theta}{x})^{\lambda}}\right]^{\alpha}\right]^{a-1}}{\left[1-\left(1-\left(1-e^{-(\frac{\theta}{x})^{\lambda}}\right)^{\alpha}\right)^{a}\right]}, \tag{9}$$

and

$$\tau_{X|\{a,b,\alpha,\theta,\lambda\}}(x)=\frac{f(x,a,b,\alpha,\theta,\lambda)}{F(x,a,b,\alpha,\theta,\lambda)}$$

$$=\frac{a\, b\alpha\lambda\ \theta^{\lambda}x^{-(1+\lambda)}\ e^{-(\frac{\theta}{x})^{\lambda}}\left[1-e^{-(\frac{\theta}{x})^{\lambda}}\right]^{\alpha-1}\times\left[1-\left[1-e^{-(\frac{\theta}{x})^{\lambda}}\right]^{\alpha}\right]^{a-1}\times\left[1-\left[1-\left[1-e^{-(\frac{\theta}{x})^{\lambda}}\right]^{\alpha}\right]^{a}\right]^{b-1}}{1-\left[1-(1-\left(1-e^{-(\frac{\theta}{x})^{\lambda}})^{\alpha}\right)^{a}\right]^{b}}. \tag{10}$$

In Figures 3, we plot the KEF hazared rate for selected parameter values.

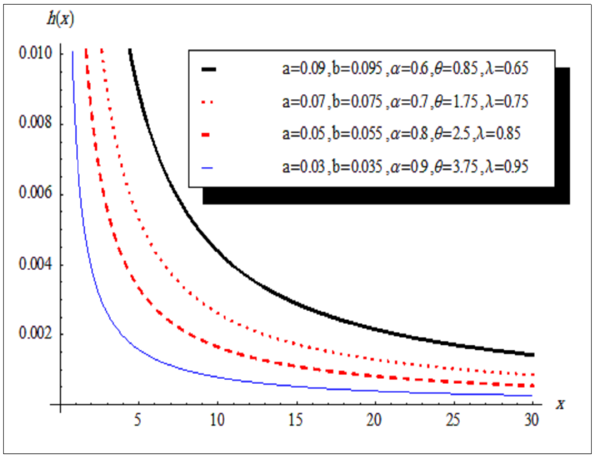

Figure 3. Plots of the KEF hazard rate for selected parameter values

Note that the Kumaraswamy exponentiated Frechet distribution is very flexible model that approaches to different distributions when its parameters are changed. In addition to some standard distribution the KEF distribution includes the following well-known distributions as special models.

1) If $a=b=1$, the exponentiated Frechet distribution is obtained.

2) When $a=b=\alpha=1,$ we get Frechet distribution.

3) When $\alpha=1,$ we get Kumaraswamy Frechet distribution.

4) If $\lambda=\alpha=1,$ we get kumaraswamy inverse exponential distribution.

5) If $\lambda=1,$ we get kumaraswamy exponentiated inverse exponential distribution.

2.1 Expansion for the pdf and cdf of distribution

In this subsection, we present two formulae for the cdf of the KEF distribution depending if the parameter $b>0$ is real non-integer or integer. First, if $|z|<1$ and $b>0$ is real non-integer, we have

$$(1-z)^{b-1}=\sum_{i=0}^{\infty}(-1)^{i}\binom{b-1}{i}z^{i}=\frac{(-1)^{i}\Gamma(b)}{i!\Gamma(b-i)}z^{i}. \tag{11}$$

Using the expansion (2.5) in (2.1), the cdf of the KEF distribution when $b>0$ is real non-integer follows

$$F_{X\mid\{a,b,\alpha,\theta,\lambda\}}(x)=1-\sum_{i=0}^{\infty}(-1)^{i}\binom{b-1}{i}\left(1-\left(1-e^{-(\frac{\theta}{x})^{\lambda}}\right)^{\alpha}\right)^{ai}.$$

when $b \circledast 0$ is integer, using the expansion (11) in (7) , we get

$$F_{X\mid\{a,b,\alpha,\theta,\lambda\}}(x)=1-\sum_{i=0}^{b}(-1)^{i}\binom{b-1}{i}\left(1-\left(1-e^{-(\frac{\theta}{x})^{\lambda}}\right)^{\alpha}\right)^{ai}, \tag{12}$$

also using the power series of Equation (11) the pdf Equation (8) becomes

$$f_{X\mid\{a,b,\alpha,\theta,\lambda\}}(x)=\sum_{i=0}^{\infty}(-1)^{i}\binom{b-1}{i}a\ b\ \alpha\lambda\theta^{\lambda}x^{-(1+\lambda)}e^{-(\frac{\theta}{x})^{\lambda}}\left(1-e^{-(\frac{\theta}{x})^{\lambda}}\right)^{\alpha-1}\left[1-\left(1-e^{-(\frac{\theta}{x})^{\lambda}}\right)^{\alpha}\right]^{a(i+1)-1}, \tag{13}$$

again, by using Equation (11) in the last factor of each summand in (13) we obtain

$$f_{X\mid\{a,b,\alpha,\theta,\lambda\}}(x)=\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}(-1)^{i+j}\binom{b-1}{i}\binom{a(i+1)-1}{j}ab\alpha\lambda\theta^{\lambda}x^{-(1+\lambda)}e^{-(\frac{\theta}{x})^{\lambda}}\left(1-e^{-(\frac{\theta}{x})^{\lambda}}\right)^{\alpha(j+1)-1}, \tag{14}$$

again, by using Equation (11) in the last factor of each summand in (14) we obtain

$$\begin{aligned}f_{X\mid\{a,b,\alpha,\theta,\lambda\}}(x)&=\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}(-1)^{i+j+k}\binom{b-1}{i}\binom{a(i+1)-1}{j}\binom{\alpha(j+1)-1}{k}ab\alpha\lambda\theta^{\lambda}x^{-(1+\lambda)}e^{-(k+1)(\frac{\theta}{x})^{\lambda}}\\&=C_{i,j,k}ab\alpha\lambda\theta^{\lambda}x^{-(1+\lambda)}e^{-(k+1)(\frac{\theta}{x})^{\lambda}},\end{aligned} \tag{15}$$

where

$$C_{i,j,k}=\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}(-1)^{i+j+k}\binom{b-1}{i}\binom{a(i+1)-1}{j}\binom{\alpha(j+1)-1}{k}.$$

3. TATISTICAL PROPERTIES

This section is devoted to studying statistical properties of the (KEF) distribution, specifically quantile function, moments, moment generating function and incomplete moment.

3.1 Quantile function and simulation

The quantile function corresponding to Eq. is $F(x_q)=P(X\le x_q)$ where $(x_q)_{(KEF)}=F^{-1}(u)$, is given by the following relation

$$\left(1-e^{-(\frac{\theta}{x})^{\lambda}}\right)^{\alpha}=1-\left[1-(1-q)^{\frac{1}{b}}\right]^{\frac{1}{a}} \tag{16}$$

$$\left(\frac{\theta}{x}\right)^{\lambda}=-\ln\left[1-\left(1-\left[1-(1-u)^{\frac{1}{b}}\right]^{\frac{1}{a}}\right)^{\frac{1}{\alpha}}\right]$$

Simulating the KEF random variable is straightforward. Let U be a uniform variate on the unit interval (0, 1). Thus, by means of the inverse transformation method, we consider the random variable X given by the relation

$$x=\frac{\theta}{\left[-\ln\left[1-\left(1-\left[1-(1-u)^{\frac{1}{b}}\right]^{\frac{1}{a}}\right)^{\frac{1}{\alpha}}\right]\right]^{\frac{1}{\lambda}}}. \tag{17}$$

3.2 Moments

In this subsection we discuss the r_{th} moment for KEF distribution. Moments are necessary and important in any statistical analysis, especially in applications. It can be used to study the most important features and characteristics of a distribution (e.g., tendency, dispersion, skewness and kurtosis).

Theorem 1.

If X has $KEF\ (\Phi,x)$,$\Phi=(a,b,\alpha,\theta,\lambda)$ then the r_{th} moment of X is given by the following

$$\mu_r(x)=C_{i,j,k}ab\alpha\theta^{r}\left(k+1\right)^{\frac{r}{\lambda}-1}\Gamma(1-\frac{r}{\lambda}), \tag{18}$$

where

$$C_{i,j,k}=\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{\infty}(-1)^{i+j+k}\binom{b-1}{i}\binom{a(i+1)-1}{j}\binom{\alpha(j+1)-1}{k}.$$

Proof:

Let X be a random variable with density function (14). The r_{th} ordinary moment of the (KEF) distribution is given by

$$\begin{aligned}\mu_r(x)&=E(X^{r})=\int_0^{\infty}x^{r}f(x,\Phi)dx\\&=C_{i,j,k}ab\alpha\lambda\theta^{\lambda}\int_0^{\infty}x^{r-\lambda-1}e^{-(k+1)(\frac{\theta}{x})^{\lambda}}dx\end{aligned} \tag{19}$$

let $(k+1)(\frac{\theta}{x})^{\lambda}=t$ then

$$\mu_r(x) = C_{i,j,k} ab\alpha\theta^r \int_0^\infty t^{\frac{-r}{\lambda}} e^{-t} dt$$
$$= C_{i,j,k} ab\alpha\theta^r (k+1)^{\frac{r}{\lambda}-1} \Gamma(1-\frac{r}{\lambda}). \tag{20}$$

which completes the proof .

The central moments μ_r and cumulants k_r of the KEF distribution can be determined from expression (18) as

$$\mu_r = \sum_{m=0}^{\infty} \binom{r}{m} (-1)^m \mu_1^{lm} \mu_{r-m}^l \text{ and } k_r = \mu_r^l - \sum_{m=1}^{r-1} \binom{r-1}{m-1} k_m \mu_{r-m}^l \text{ respectively,}$$

where

$$k_1 = \mu_1^l, k_2 = \mu_2^l - \mu_1^{l\,2}, k_3 = \mu_3^l - 3\mu_2^l \mu_1^l + 2\mu_1^{l\,3},$$

and

$$k_4 = \mu_4^l - 4\mu_1^l \mu_3^l - 3\mu_2^{l\,2} + 12\mu_2^l \mu_1^{l\,2} - 6\mu_1^{l\,4}, etc$$

Additionally, the skewness and kurtosis can be calculated from the third and forth standarized cumulants in the forms SK= $\frac{K_3}{\sqrt{K_2^3}}$ and KU= $\frac{K_4}{K_2^2}$, respectively.

Theorem 2. If X has KEF distribution, then the moment generating function $M_X(t)$ has the following form

$$M_X(t) = \sum_{r=0}^{\infty} \frac{t^r}{r!} C_{i,j,k} ab\alpha\theta^r (k+1)^{\frac{r}{\lambda}-1} \Gamma(1-\frac{r}{\lambda}). \tag{21}$$

Proof.

We start with the well known definition of the moment generating function given by $M_X(t) = E(e^{tX}) = \int_0^\infty e^{tx} f_{KEF}(x,\Phi)dx,$ since $\sum_{r=0}^{\infty} \frac{t^r}{r!} x^r f(x)$ converges and each term is integrable for all t close to 0, then we can we rewrite the moment generating function as $M_X(t) = \sum_{r=0}^{\infty} \frac{t^r}{r!} E(X^r)$ by replacing $E(X^r)$. Hence using Equation (18) the MGF of KEF distribution is given by

$$M_X(t) = \sum_{r=0}^{\infty} \frac{t^r}{r!} C_{i,j,k} \left[\frac{\alpha\Gamma(\frac{r+\gamma k}{\theta}+1)}{(\alpha(j+1))^{\frac{r+\gamma k}{\theta}+1}} + \frac{\beta\gamma\Gamma(\frac{r+\gamma(k+1)}{\theta})}{\theta(\alpha(j+1))^{\frac{r+\gamma(k+1)}{\theta}}} \right].$$

This completes the proof.

Similarly, the characteristic function of the KEF distribution becomes $\phi_x(t) = M_X(it)$ where

$i=\sqrt{-1}$ is the unite imaginary number.

Theorem 3. If X has KEF distribution, then the conditional moments for KEF distribution is given by

$$E(X^s \mid X>t)=C_{i,j,k}ab\alpha\theta^s(k+1)^{\frac{s}{\lambda}-1}\Gamma(1-\frac{s}{\lambda},(k+1)(\frac{\theta}{t})^{\lambda}). \tag{22}$$

Proof.

$$E(X^s \mid X>t)=\int_t^{\infty}x^s f(x,\varphi)dx$$
$$=C_{i,j,k}ab\alpha\lambda\theta^{\lambda}\int_t^{\infty}x^{s-\lambda-1}e^{-(k+1)(\frac{\theta}{x})^{\lambda}}dx$$

Let $(k+1)(\frac{\theta}{x})^{\lambda}=u,$ the above integral can be written as

$$E(X^s \mid X>t)=C_{i,j,k}ab\alpha\theta^s(k+1)^{\frac{s}{\lambda}-1}\int_{(k+1)(\frac{\theta}{t})^{\lambda}}^{\infty}u^{-\frac{s}{\lambda}}e^{-u}du$$
$$=C_{i,j,k}ab\alpha\theta^s(k+1)^{\frac{s}{\lambda}-1}\Gamma(1-\frac{s}{\lambda},(k+1)(\frac{\theta}{t})^{\lambda}).$$

Where $\Gamma(s,t)=\int_t^{\infty}x^{s-1}e^{-x}dx$ is the upper incomplete gamma function. The mean residual lifetime function is given by

$$\mu(t)=E(X \mid X>t)-t=\left\{C_{i,j,k}ab\alpha\theta(k+1)^{\frac{1}{\lambda}-1}\Gamma(1-\frac{1}{\lambda},(k+1)(\frac{\theta}{t})^{\lambda})\right\}-t.$$

The importance of the MRL function is due to its uniquely determination of the lifetime distribution as well as the failure rate (FR) function. Lifetime can exhibit IMRL (increasing MRL) or DMRL (decreasing MRL). MRL function that first decreases (increases) and then increases (decreases) are usually called bathtub (upside-down as bathtub) shaped, BMRL (UMRL). Many authors such as Mi (1995), Park (1985) and Tang et al (1999) have been studied the relationship between the behaviors of the MRL and FR functions of a distribution.

4. DISTRIBUTION OF THE ORDER STATISTICS

In this Section, we derive closed form expressions for the pdfs of the r_{th} order statistic of the KEF distribution, also, the measures of skewness and kurtosis of the distribution of the r_{th} order statistic in a sample of size n for different choices of $n;r$ are presented in this Section. Let $X_1,X_2,...,X_n$ be a simple random sample from KEF distribution with pdf

and cdf given by Equation (8) and Equation (12), respectively.
Let $X_1, X_2, ..., X_n$ denote the order statistics obtained from this sample. We now give the probability density function of $X_{r:n}$, say $f_{r:n}(x,\Phi)$ and the moments of $X_{r:n}$ $, r = 1,2,...,$. Therefore, the measures of skewness and kurtosis of the distribution of the $X_{r:n}$ are presented. The probability density function of $X_{r:n}$ is given by

$$f_{r:n}(x,\Phi) = \frac{1}{B(r,n-r+1)}\left[F(x,\Phi)\right]^{r-1}\left[1-F(x,\Phi)\right]^{n-r} f(x,\Phi), \tag{23}$$

where $F(x,\Phi)$ and $f(x,\Phi)$ are the cdf and pdf of the KEF distribution given by (7), (8), respectively, and since $0 < F(x,\Phi) < 1$, for $x > 0$, by using the binomial series expansion of $\left[1-F(x,\Phi)\right]^{n-r}$, given by

$$\left[1-F(x,\Phi)\right]^{n-r} = \sum_{j=0}^{n-r}(-1)^j \binom{n-r}{j}\left[F(x,\Phi)\right]^j, \tag{24}$$

we have

$$f_{r:n}(x,\Phi) = \sum_{j=0}^{n-r}(-1)^j \binom{n-r}{j}\left[F(x,\Phi)\right]^{r+j-1} f(x,\Phi), \tag{25}$$

substituting from Equation (7) and (8) into Equation (25), we can express the k_{th} ordinary moment of the r_{th} order statistics $X_{r:n}$ say $E(X_{r:n}^k)$ as a liner combination of the k_{th} moments of the KEF distribution with different shape parameters. Therefore, the measures of skewness and kurtosis of the distribution of $X_{r:n}$ can be calculated.

5. LEAST SQUARES AND WEIGHTED LEAST SQUARES ESTIMATORS

In this Section we provide the regression based method estimators of the unknown parameters of the Kumaraswamy exponentiated Lomax, which was originally suggested by Swain, Venkatraman and Wilson (1988) to estimate the parameters of beta distributions. It can be used some other cases also. Suppose $Y_1, ..., Y_n$ is a random sample of size n from a distribution function $G(\cdot)$ and suppose $Y_{(i)}$; $i = 1,2,...,n$ denotes the ordered sample. The proposed method uses the distribution of $G(Y_{(i)})$. For a sample of size n, we have

$$E\left(G(Y_{(j)})\right) = \frac{j}{n+1}, V\left(G(Y_{(j)})\right) = \frac{j(n-j+1)}{(n+1)^2(n+2)}$$

and

$$Cov\left(G(Y_{(j)}),G(Y_{(k)})\right)=\frac{j(n-k+1)}{(n+1)^2(n+2)};\text{for}\, j<k,$$

see Johnson, Kotz and Balakrishnan (1995). Using the expectations and the variances, two variants of the least squares methods can be used.

Method 1. (Least Squares Estimators) Obtain the estimators by minimizing

$$\sum_{j=1}^{n}\left(G(Y_{(j)}-\frac{j}{n+1}\right)^2, \tag{26}$$

with respect to the unknown parameters. Therefore in case of KEF distribution the least squares estimators of $a,b,\alpha,\theta,\ and.\lambda$ say $\widehat{a}_{LSE}$, $\widehat{b}_{LSE},\widehat{\alpha}_{LSE},\widehat{\theta}_{LSE}$, and $\widehat{\lambda}_{LSE}$ respectively, can be obtained by minimizing

$$\sum_{j=1}^{n}\left[1-\left[1-\left(1-\left(1-e^{-(\frac{\theta}{x})^{\lambda}}\right)^{\alpha}\right)^{a}\right]^{b}-\frac{j}{n+1}\right]^2$$

with respect to $a,b,\alpha,\theta,\ and.\lambda.$

Method 2. (Weighted Least Squares Estimators) The weighted least squares estimators can be obtained by minimizing

$$\sum_{j=1}^{n}w_j\left(G(Y_{(j)}-\frac{j}{n+1}\right)^2, \tag{27}$$

with respect to the unknown parameters, where

$$w_j=\frac{1}{V\left(G(Y_{(j)})\right)}=\frac{(n+1)^2(n+2)}{j(n-j+1)}.$$

Therefore, in case of KEF distribution the weighted least squares estimators of $a,b,\alpha,\theta,\ and.\lambda$ say $\widehat{a}_{WLSE}$, $\widehat{b}_{WLSE},\widehat{\alpha}_{WLSE},\widehat{\theta}_{WLSE}$, and $\widehat{\lambda}_{WLSE}$ respectively, can be obtained by minimizing

$$\sum_{j=1}^{n}w_j\left[1-\left[1-\left(1-\left(1-e^{-(\frac{\theta}{x})^{\lambda}}\right)^{\alpha}\right)^{a}\right]^{b}-\frac{j}{n+1}\right]^2$$

with respect to the unknown parameters only.

6. ESTIMATION AND INFERENCE

In this Section, we determine the maximum likelihood estimates (MLEs) of the parameters of the KEF distribution from complete samples only. Let $X_1, X_2, ..., X_n$ be a random sample of size n from $KTF(\lambda,\theta,a,b)$. The likelihood function for the vector of parameters $\Phi=(a,b,\alpha,\theta,\lambda)$ can be written as

$$Lf(x_{(i)},\Phi)=\Pi_{i=1}^{n} f(x_{(i)},\Phi)$$

$$=\left(ab\alpha\lambda\theta^{\lambda}\right)^{n}\ \Pi_{i=1}^{n} x_i^{-(1+\lambda)}\ e^{-\sum_{i=1}^{n}(\frac{\theta}{x_i})^{\lambda}}\ \Pi_{i=1}^{n}\left[\left(1-e^{-(\frac{\theta}{x})^{\lambda}}\right)^{\alpha-1}\right] \tag{28}$$

$$\times\Pi_{i=1}^{n}\left[1-\left[1-e^{-(\frac{\theta}{x})^{\lambda}}\right]^{\alpha}\right]^{a-1}\times\Pi_{i=1}^{n}\left[1-\left[1-\left[1-e^{-(\frac{\theta}{x})^{\lambda}}\right]^{\alpha}\right]^{a}\right]^{b-1}$$

Taking the log-likelihood function for the vector of parameters $\Phi=(a,b,\alpha,\theta,\lambda)$ we get

$$\log\ L=n\log a+n\log b+n\log\alpha+n\log\lambda+n\lambda\log\theta-(\lambda+1)\sum_{i=1}^{n}\log(x_i)$$

$$-\sum_{i=1}^{n}(\frac{\theta}{x_i})^{\lambda}+(\alpha-1)\sum_{i=1}^{n}\log\left[1-e^{-(\frac{\theta}{x})^{\lambda}}\right]+(a-1)\sum_{i=1}^{n}\log\left[1-\left[1-e^{-(\frac{\theta}{x})^{\lambda}}\right]^{\alpha}\right] \tag{29}$$

$$+(b-1)\sum_{i=1}^{n}\log\left[1-\left[1-\left[1-e^{-(\frac{\theta}{x})^{\lambda}}\right]^{\alpha}\right]^{a}\right],$$

The log-likelihood can be maximized either directly or by solving the nonlinear likelihood equations obtained by differentiating (29). The components of the score vector are given by

$$\frac{\partial\log\ L}{\partial a}=\frac{n}{a}+\sum_{i=1}^{n}\log\left[1-\left[1-e^{-(\frac{\theta}{x})^{\lambda}}\right]^{\alpha}\right]$$

$$-(b-1)\sum_{i=1}^{n}\frac{\left[1-\left[1-e^{-(\frac{\theta}{x})^{\lambda}}\right]^{\alpha}\right]^{a-1}\log\left[1-\left[1-e^{-(\frac{\theta}{x})^{\lambda}}\right]^{\alpha}\right]}{\left[1-\left[1-\left[1-e^{-(\frac{\theta}{x})^{\lambda}}\right]^{\alpha}\right]^{a-1}\right]}, \tag{30}$$

$$\frac{\partial\log\ L}{\partial b}=\frac{n}{b}+\sum_{i=1}^{n}\log\left[1-\left[1-\left[1-e^{-(\frac{\theta}{x})^{\lambda}}\right]^{\alpha}\right]^{a-1}\right], \tag{31}$$

$$\frac{\partial \log\ L}{\partial \alpha}=\frac{n}{\alpha}-(\alpha-1)\sum_{i=1}^{n}\frac{\left(e^{-(\frac{\theta}{x})^{\lambda}}\right)^{\alpha-1}\log\left[e^{-(\frac{\theta}{x})^{\lambda}}\right]}{\left(1-\left(e^{-(\frac{\theta}{x})^{\lambda}}\right)^{\alpha-1}\right)}$$

$$-(a-1)\sum_{i=1}^{n}\frac{\left[1-e^{-(\frac{\theta}{x})^{\lambda}}\right]^{\alpha}\log\left[1-e^{-(\frac{\theta}{x})^{\lambda}}\right]}{\left(1-\left[1-e^{-(\frac{\theta}{x})^{\lambda}}\right]^{\alpha}\right)} \tag{32}$$

$$+(b-1)\sum_{i=1}^{n}\frac{(-1+a)\left[1-e^{-(\frac{\theta}{x})^{\lambda}}\right]^{\alpha}\left(1-\left[1-e^{-(\frac{\theta}{x})^{\lambda}}\right]^{\alpha}\right)^{a-2}\log\left[1-e^{-(\frac{\theta}{x})^{\lambda}}\right]}{x\left(1-\left[1-e^{-(\frac{\theta}{x})^{\lambda}}\right]^{\alpha-1}\right)}$$

$$+\sum_{i=1}^{n}\log\left[1-\left(e^{-(\frac{\theta}{x})^{\lambda}}\right)^{\alpha-1}\right],$$

$$\frac{\partial \log\ L}{\partial \theta}=\frac{n\lambda}{\theta}-\sum_{i=1}^{n}\frac{\lambda(\frac{\theta}{x})^{\lambda-1}}{x}+(\alpha-1)^{2}\lambda\sum_{i=1}^{n}\frac{\left(e^{-(\frac{\theta}{x})^{\lambda}}\right)^{\alpha-1}(\frac{\theta}{x})^{\lambda-1}}{\left(1-\left[e^{-(\frac{\theta}{x})^{\lambda}}\right]^{\alpha-1}\right)x}$$

$$-(a-1)\alpha\lambda\sum_{i=1}^{n}\frac{\left(e^{-(\frac{\theta}{x})^{\lambda}}\right)\left[1-e^{-(\frac{\theta}{x})^{\lambda}}\right]^{\alpha-1}(\frac{\theta}{x})^{\lambda-1}}{(1-\left[1-e^{-(\frac{\theta}{x})^{\lambda}}\right]^{\alpha})x}+(a-1)(b-1)\lambda\alpha \tag{33}$$

$$\times\sum_{i=1}^{n}\frac{e^{-(\frac{\theta}{x})^{\lambda}}\left(1-e^{-(\frac{\theta}{x})^{\lambda}}\right)^{\alpha-1}\left(1-\left(1-e^{-(\frac{\theta}{x})^{\lambda}}\right)^{\alpha}\right)^{a-2}(\frac{\theta}{x})^{\lambda-1}}{\left(1-\left(1-\left(1-e^{-(\frac{\theta}{x})^{\lambda}}\right)^{\alpha}\right)^{a-1}\right)x},$$

and

$$\frac{\partial \log L}{\partial \lambda}=\frac{n}{\lambda}+n\log\theta-\sum_{i=1}^{n}\log x-\sum_{i=1}^{n}\log(\frac{\theta}{x})\left(\frac{\theta}{x}\right)^{\lambda}+(\alpha-1)^{2}\sum_{i=1}^{n}\frac{\left(e^{-(\frac{\theta}{x})^{\lambda}}\right)^{\alpha-1}\log(\frac{\theta}{x})\left(\frac{\theta}{x}\right)^{\lambda}}{\left(1-\left(e^{-(\frac{\theta}{x})^{\lambda}}\right)^{\alpha-1}\right)}$$

$$-\alpha(a-1)\sum_{i=1}^{n}\frac{e^{-(\frac{\theta}{x})^{\lambda}}\left(1-e^{-(\frac{\theta}{x})^{\lambda}}\right)^{\alpha-1}\log(\frac{\theta}{x})\left(\frac{\theta}{x}\right)^{\lambda}}{\left(1-\left(1-e^{-(\frac{\theta}{x})^{\lambda}}\right)^{\alpha-1}\right)}+(a-1)(b-1)\alpha \tag{34}$$

$$\times\sum_{i=1}^{n}\frac{e^{-(\frac{\theta}{x})^{\lambda}}\left(1-e^{-(\frac{\theta}{x})^{\lambda}}\right)^{\alpha-1}\left(1-\left(1-e^{-(\frac{\theta}{x})^{\lambda}}\right)^{\alpha}\right)^{a-2}\log(\frac{\theta}{x})\left(\frac{\theta}{x}\right)^{\lambda}}{\left(1-\left(1-\left(1-e^{-(\frac{\theta}{x})^{\lambda}}\right)^{\alpha}\right)^{a-1}\right)}.$$

We can find the estimates of the unknown parameters by maximum likelihood method by setting these above non-linear equations (30)- (34) to zero and solve them simultaneously. Therefore, we have to use mathematical package to get the MLE of the unknown parameters. Also, all the second order derivatives exist. Thus we have the inverse dispersion matrix is given by

$$\begin{pmatrix}\widehat{a}\\ \widehat{b}\\ \widehat{\alpha}\\ \widehat{\theta}\\ \widehat{\lambda}\end{pmatrix}\sim N\left[\begin{pmatrix}a\\ b\\ \alpha\\ \theta\\ \lambda\end{pmatrix},\begin{pmatrix}\widehat{V_{aa}} & \widehat{V_{ab}} & \widehat{V_{a\theta}} & \widehat{V_{a\theta}} & \widehat{V_{a\lambda}}\\ \widehat{V_{ba}} & \widehat{V_{bb}} & \widehat{V_{b\alpha}} & \widehat{V_{b\theta}} & \widehat{V_{b\lambda}}\\ \widehat{V_{\alpha a}} & \widehat{V_{\alpha b}} & \widehat{V_{\alpha\alpha}} & \widehat{V_{\alpha\theta}} & \widehat{V_{\alpha\lambda}}\\ \widehat{V_{\theta a}} & \widehat{V_{\theta b}} & \widehat{V_{\theta\alpha}} & \widehat{V_{\theta\theta}} & \widehat{V_{\theta\lambda}}\\ \widehat{V_{\lambda a}} & \widehat{V_{\lambda b}} & \widehat{V_{\lambda\alpha}} & \widehat{V_{\lambda\theta}} & \widehat{V_{\lambda\lambda}}\end{pmatrix}\right]. \tag{35}$$

$$V^{-1}=-E\begin{bmatrix}V_{aa} & V_{ab} & V_{a\theta} & V_{a\theta} & V_{a\lambda}\\ V_{ba} & V_{bb} & V_{b\alpha} & V_{b\theta} & V_{b\lambda}\\ V_{\alpha a} & V_{\alpha b} & V_{\alpha\alpha} & V_{\alpha\theta} & V_{\alpha\lambda}\\ V_{\theta a} & V_{\theta b} & V_{\theta\alpha} & V_{\theta\theta} & V_{\theta\lambda}\\ V_{\lambda a} & V_{\lambda b} & V_{\lambda\alpha} & V_{\lambda\theta} & V_{\lambda\lambda}\end{bmatrix}$$

where

$$V_{\lambda\lambda}=\frac{\partial^{2}L}{\partial\lambda^{2}},V_{\theta\theta}=\frac{\partial^{2}L}{\partial\theta^{2}},V_{aa}=\frac{\partial^{2}L}{\partial a^{2}},V_{a\lambda}=\frac{\partial^{2}L}{\partial a\partial\lambda},V_{ab}=\frac{\partial^{2}L}{\partial a\partial b},V_{a\theta}=\frac{\partial^{2}L}{\partial a\partial\theta}.$$

By solving this inverse dispersion matrix these solutions will yield asymptotic variance and co variances of these ML estimators for $\widehat{a}$, $\widehat{b}$,$\widehat{\alpha}$, $\widehat{\theta}$ and $\widehat{\lambda}$. Using equation (35), we approximate $100(1-\gamma)\%$ confidence intervals for a, b, α, θ and λ are determined respectively as,

$$\widehat{a} \pm z_{\frac{\gamma}{2}}\sqrt{\widehat{V_{aa}}},\ \widehat{b} \pm z_{\frac{\gamma}{2}}\sqrt{\widehat{V_{bb}}},\ \widehat{\alpha} \pm z_{\frac{\gamma}{2}}\sqrt{\widehat{V_{\alpha\alpha}}},\ \widehat{\theta} \pm z_{\frac{\gamma}{2}}\sqrt{\widehat{V_{\theta\theta}}}, \text{and } \widehat{\lambda} \pm z_{\frac{\gamma}{2}}\sqrt{\widehat{V_{\lambda\lambda}}},$$

where z_γ is the upper $100\gamma_{the}$ percentile of the standard normal distribution.
The following table represents the mean square error (MSEs) of the MLEs.

Table 1. Mean square errors of the MLEs

$KEF(a,b,\alpha,\theta,\lambda)$	n	$MSE(\widehat{a})$	$MSE(\widehat{b})$	$MSE(\widehat{\alpha})$	$MSE(\widehat{\theta})$	$MSE(\widehat{\lambda})$
	15	0.1610	0.706	0.0345	0.1489	0.3194
	25	0.1609	0.6626	0.0268	0.1338	0.3089
	35	0.1603	0.6504	0.0266	0.1065	0.3082
$KEF(1.15,1,1.25,0.75,0.65)$	45	0.159	0.6380	0.0251	0.0927	0.3068
	55	0.1509	0.6145	0.0214	0.0865	0.3012
	65	0.1467	0.603	0.0210	0.0765	0.2937
	75	0.1266	0.5921	0.0191	0.0652	0.2900
	15	0.2473	0.0686	0.1767	0.8281	0.2108
	25	0.2414	0.046	0.1687	0.5062	0.2073
	35	0.2091	0.0400	0.1590	0.4233	0.2003
$KEF(2.15,2,2.25,1.5,0.95)$	45	0.1560	0.0390	0.1567	0.302	0.1931
	55	0.1551	0.0371	0.1459	0.016	0.1814
	65	0.1050	0.0384	0.1454	0.010	0.1556
	75	0.086	0.0281	0.1438	0.080	0.1128
	15	0.702	0.229	0.0063	0.6003	0.1984
	25	0.341	0.1967	0.0059	0.5924	0.1929
	35	0.248	0.0974	0.0057	0.5869	0.1808
$KEF(3.25,3.5,1.75,1,1)$	45	0.088	0.0635	0.0044	0.5814	0.1713
	55	0.004	0.047	0.0035	0.5481	0.1693
	65	0.0029	0.029	0.0032	0.5319	0.1618
	75	0.0016	0.0145	0.001	0.50	0.157

We noticed from the above Table 1 that all MSEs decrease as the sample size increases, while they increase with increasing of the true parameter.

7. APPLICATION TO REAL DATA SET

In this Section we fit KEF to two real data sets and compare the fitness with Kumaraswamy Frechet distribution(KFD), kumaraswamy exponentiated inverse exponential distribution (KEIED), exponentiated Frechet distribution (EFD) distributions, kumaraswamy inverse exponential distribution(KIED) and Frechet distribution (FD). Specifically, we consider two data sets.

Table 2. Maximum-likelihood estimates, AIC, BIC and AICC values, and Kolmogorov-Smirnov statistics for the models based on data set 1

Model	estimators	K-S	-2logL	AIC	BIC	AICC
KEFD	$\hat{a}=2.1$ $\hat{b}=2.22$ $\hat{\alpha}=1.46$ $\hat{\theta}=1.28$ $\hat{\lambda}=0.94$	0.289	58.323	68.323	79.414	78.414
KFD	$\hat{a}=3.43$ $\hat{b}=2.5$ $\hat{\theta}=1.9$ $\hat{\lambda}=0.54$	0.15	183.296	193.296	204.679	203.387
KEIED	$\hat{a}=3.936$ $\hat{b}=3.824$ $\hat{\alpha}=0.68$ $\hat{\theta}=1.1$	0.122	162.211	172.211	183.94	182.302
EFD	$\hat{\alpha}=0.66$ $\hat{\theta}=0.85$ $\hat{\lambda}=0.68$	0.296	186.677	196.677	208.06	206.768
KIED	$\hat{a}=1.86$ $\hat{b}=1.71$ $\hat{\theta}=0.86$	0.156	114.076	124.076	135.46	134.167
FD	$\hat{\theta}=0.57$ $\hat{\lambda}=0.73$	0.324	204.312	214.312	225.695	224.403

Data set 1. The following data represent the survival times (in days) of 72 guinea pigs infected with virulent tubercle bacilli, observed and reported by Bjerkedal (1960). The data are as follows:
0.1, 0.33, 0.44, 0.56, 0.59, 0.72, 0.74, 0.77, 0.92, 0.93, 0.96, 1, 1, 1.02, 1.05, 1.07, 07, .08, 1.08, 1.08, 1.09, 1.12, 1.13, 1.15, 1.16, 1.2, 1.21, 1.22, 1.22, 1.24, 1.3, 1.34, 1.36, 1.39, 1.44, 1.46, 1.53, 1.59, 1.6, 1.63, 1.63, 1.68, 1.71, 1.72, 1.76, 1.83, 1.95, 1.96, 1.97, 2.02, 2.13, 2.15, 2.16, 2.22, 2.3, 2.31, 2.4, 2.45, 2.51, 2.53, 2.54, 2.54, 2.78, 2.93, 3.27, 3.42, 3.47, 3.61, 4.02, 4.32, 4.58, 5.55.

Table 3. Maximum-likelihood estimates, AIC, BIC and AICC values, and Kolmogorov-Smirnov statistics for the models based on data set 2.

Model	estimators	K-S	-2logL	AIC	BIC	AICC
KEFD	$\hat{a}=2.0$ $\hat{b}=0.53$ $\hat{\alpha}=1.0$ $\hat{\theta}=1.18$ $\hat{\lambda}=0.85$	0.322	14.0	24.0	34.635	34.107
KFD	$\hat{a}=1.2$ $\hat{b}=1.93$ $\hat{\theta}=0.87$ $\hat{\lambda}=0.60$	0.409	102.175	112.175	122.811	122.282
KEIED	$\hat{a}=2.04$ $\hat{b}=1.2$ $\hat{\alpha}=0.65$ $\hat{\theta}=1.1$	0.122	162.211	172.211	183.94	182.302
EFD	$\hat{\alpha}=0.60$ $\hat{\theta}=0.90$ $\hat{\lambda}=0.66$	0. 534	163.619	173.619	184.254	183.726
KIED	$\hat{a}=1.0$ $\hat{b}=2.1$ $\hat{\theta}=0.47$	0. 512	216.45	226.45	237.085	236.557
FD	$\hat{\theta}=0.52$ $\hat{\lambda}=0.9$	0. 457	227.709	237.709	248.345	247.816

Data set 2. The data set is obtained from Smith and Naylor (1987). The data are the strengths of 1.5 cm glass fibres, measured at the National Physical Laboratory, England. Unfortunately, the units of measurement are not given in the paper. The data set is
0.55, 0.93, 1.25, 1.36, 1.49, 1.52, 1.58, 1.61, 1.64, 1.68, 1.73, 1.81, 2, 0.74, 1.04, 1.27, 1.39, 1.49, 1.53, 1.59, 1.61, 1.66, 1.68, 1.76, 1.82, 2.01, 0.77, 1.11, 1.28, 1.42, 1.5, 1.54, 1.6, 1.62, 1.66, 1.69, 1.76, 1.84, 2.24, 0.81, 1.13, 1.29, 1.48, 1.5, 1.55, 1.61, 1.62, 1.66, 1.7, 1.77, 1.84, 1.24, 1.3, 1.48, 1.51, 1.55, 1.61, 1.63, 1.67, 1.7, 1.78, 1.89.

In order to compare distributions, we consider the K_S (Kolmogorov-Smirnov) statistic, -2logL, AIC (Akaike Information Criterion), AICC (Akaike Information Criterion Corrected), BIC (Bayesian Information Criterion). The best distribution corresponds to lower -2logL, AIC, BIC, AICC statistics value.
Table 2 and Table 3 show parameter MLEs, the values of K_S, -2logL, AIC, BIC, AICC statistics for the three data set consecutively. From the above results, it is evident that the KEF distribution is the best distribution for fitting these data sets compared to other distributions considered here. And is a strong competitor to other distributions commonly used in literature for fitting lifetime data.

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