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Design of bivariate step-stress partially accelerated degradation test plan using copula and gamma process

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Abstract: Many mechanical, electrical and electronic products have more than one performance characteristics (PCs). For example the performance degradation of rubidium discharge lamps can be characterized by the rubidium consumption or the decreasing intensity the lamp. The product may degrade due to all the PCs which may be independent or dependent. This paper deals with the design of optimal bivariate step-stress partially accelerated degradation test (PADT) with degradation paths modelled by gamma process. The dependency between PCs has been modelled through Frank copula function. In partial step-stress loading, the unit is tested at usual stress for some time, and then the stress is accelerated. This helps in preventing over-stressing of the test specimens. Failure occurs when the performance characteristic crosses the critical value the first time. Under the constraint of total experimental cost, the optimal test duration and the optimal number of inspections at each intermediate stress level are obtained using variance optimality criterion.

Key Words: *accelerated degradation test, accelerated step-stress test, copula function, gamma process, partially variance optimality criterion*

1. INTRODUCTION

Modern products have become increasingly more sophisticated in the light of rapid technological advances. Each piece of equipment is composed of numerous elementary components and (or) sub-systems that work as a unit either to achieve specific objectives or to perform a variety of functions. It is difficult to assess the reliability of these products with traditional life tests or accelerated life tests that record only failure times. In such a situation, degradation analysis serves as an alternate source

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of reliability analysis. Accelerated Degradation Testing is carried out to obtain reliability information of highly reliable products.

In an ADT a performance characteristic of the product whose degradation over time can be related to reliability, is measured at a higher than nominal stress level. For obtaining reliability information about the product, the degradation data so obtained is extrapolated by means of an appropriate model for performance degradation to a certain use-stress level. Here failure is defined in terms of performance characteristic of the product exceeding its critical (threshold) value. For example, decreasing light output from a fluorescent light bulb where in failure may be defined as performance characteristic exceeding a specified level of degradation such as 60% of initial output. Such a failure is called a soft failure (Tseng, Hamada and Chiao (1995)), i.e., degradation of a product's performance to an unacceptable level. An Accelerated Degradation Test can be conducted under fully accelerated or partially accelerated environmental conditions. In fully ADT, all the test units are run at accelerated conditions, while in partially accelerated degradation test (PADT) they are run at both normal and accelerated conditions.

Complex structure and multifunctional attributes of modern day products in manufacturing industry have necessitated the study of simultaneous effect of degradation of two or more than two performance characteristics or component failure mechanisms on their reliability. For example, with different purpose of lighting (i.e., illumination and colour), the LED system may generate multiple degradation mechanism that leads to failure (Fan, J. J., Yung, K.C. and Pecht, M. (2011), Hao and Su (2014)). The dependency amongst two or more degraded PCs can be modelled through a copula function. Escarela and Carriere (2003) have used bivariate Frank copula to fit a prostate cancer data set. Sari, Newby, Brombacher, Tang (2009) has modelled the degradation data with generalized linear model and described the dependency of degradation paths depicted by PCs using a copula function. Pan, Balakrishnan and Sun (2011) have proposed bivariate constant-stress accelerated degradation test (CSADT) model by describing the dependence between the two PCs using Frank copula. They have assumed that the copula parameter is a function of the stress level that can be described by a logistic function. Pan, Balakrishnan, Sun, and Zhou (2013) have modelled bivariate degradation based on a Wiener process with a time-scale transformation using Gaussian, Frank, Gumbel and Clayton copulas to describe the dependency between the two PCs, and compared them. Hao and Su (2014) have described the dependency between two PCs by the Frank copula function with each PC governed by a random effect nonlinear diffusion process.

However, there does not seem to exist any work in the literature on optimal design of ADT plan using step-stress loading under partially accelerated environmental condition involving more than one PC. In this paper, two PCs each of whose degradation is governed by gamma process are considered. The optimal plan consists in finding out optimal number of inspections and test duration by minimizing the asymptotic variance of reliability function at normal use condition with given mission time and cost constraint. The method developed has been explained using a numerical example and sensitivity analysis carried out.

2. THE MODEL

Acronyms

ADT	accelerated degradation test
Avar	asymptotic variance
cdf	cumulative distribution function
MLE	maximum likelihood estimator
PC	performance characteristic
pdf	probability distribution function
SSPADT	step-stress partially ADT
TC	total cost

2.1 Assumptions

- The total number of test units, n, is given.
- At each inspection time, units are measured simultaneously.
- The test units have two-dimensional PCs, and each PC is governed by gamma process.
- The threshold vector is $\tilde{d} = (d_1, d_2)$. In an ADT, inspection would be stopped on a particular unit if any PC crosses its corresponding threshold (critical value) where the failure is defined.
- The two PCs are dependent on each other and the dependency can be described by the Frank copula function.

2.2 Test procedure

In the SSPADT,

a) n units are put to test;

b) They are inspected under normal operating conditions up to time t_1 and l_1 measurements are recorded;

c) Then, the surviving units are inspected under accelerated condition a_1 up to time t_2 and l_2 measurements are recorded, and so on; and

d) The test duration is $t = \sum_{i=1}^{r} l_i f \Delta t$.

2.3 Gamma process and first passage time distribution

A gamma process with shape parameter v and scale parameter u is a continuous time stochastic process $\{G(t), t \geq 0\}$ with the following three properties:

a) $G(0) = 0$ with probability one,

b) $G(t)$ has independent increments,

c) G(t) − G(s) follows gamma distribution with v(t-s) and u as shape and scale parameter, respectively for all $t > s \geq 0$.

Assume that the degradation path of a unit is governed by gamma process. If T denotes the lifetime of the unit with the threshold value d, then T can be defined as
$T = \inf\{t | G(t) \geq d\}$.
Pan and BalaKrishnan (2011) have shown that when degradation paths follow gamma process then the lifetime of the unit, T, follows Birnbaum-Saunders (BS) distribution. Therefore, the cdf and, the corresponding pdf of a two-parameter Birnbaum-Saunders random variable T, for $\alpha > 0$ and, $\beta > 0$ can be expressed as

$$F_T(t;\alpha,\beta) = \Phi\left[\frac{1}{\alpha}\left(\sqrt{\frac{t}{\beta}} - \sqrt{\frac{\beta}{t}}\right)\right], t > 0, \tag{1}$$

$$f_T(t;\alpha,\beta) = \frac{1}{2\sqrt{2\pi}\alpha\beta}\left[\left(\frac{\beta}{t}\right)^{1/2} + \left(\frac{\beta}{t}\right)^{3/2}\right]\exp\left[-\frac{1}{2\alpha^2}\left(\frac{t}{\beta} - 2 + \frac{\beta}{t}\right)\right], t > 0, \tag{2}$$

where $\alpha = \frac{1}{\sqrt{d/u}}$ and $\beta = \frac{d}{vu}$.

2.4 Model with two dependent PCs based on gamma process

Let G(t) denote the degradation path of the unit under normal operating condition. Consider an experiment in which 'n' units operating independently are put on test at time t_0, $(t_0 = 0)$. In the first interval $[0, t_1)$ the units are tested under usual (normal operating) conditions, and then after, the surviving units are tested under accelerated condition with a new process $G(t_1 + a_1(t - t_1))$, for $[t_1, t_2)$ starting out at the point $(t_1, G(t_1))$, and so on. As the process changes from one gamma process to another at the stress change points, the decay rate correspondingly changes. Here, a_i (> 1), i = 1, 2, … r, is referred to as an acceleration factor and is assumed to be known. Thus, the degradation path, $G_0(t)$, of an SSPADT experiment can be expressed as:

$$G_0(t) = \begin{cases} G(t), & 0 \leq t < t_1 \\ G(t_1 + a_1(t - t_1)), & t_1 \leq t < t_2 \\ \vdots & \\ G(t_{i-1} + a_{i-1}(t - t_{i-1})), & t_{i-1} \leq t < t_i \\ \vdots & \\ G(t_{r-1} + a_{r-1}(t - t_{r-1})), & t_{r-1} \leq t < t_r. \end{cases} \tag{3}$$

2.5 Copula function

Copula is a function that couples marginal distributions to their multivariate distribution functions (Nelson (2006)).

Let X_1 and X_2 be the random variables with $G_1(x_1)$ and $G_2(x_2)$ as their marginal cdfs, respectively. Let $H(x_1, x_2)$ be their joint distribution function. Then, according to Sklar's Theorem, there exists a copula C such that for all (x_1, x_2) in the defined range,

$$H(x_1, x_2) = C(G_1(x_1), G_2(x_2)), \tag{4}$$

where $C(\cdot,\cdot)$ is cdf of a copula function.

One of the popular Archimedean copulas is Frank copula defined as

$$C(s_1, s_2) = -\frac{1}{\alpha}\ln\left[1 + \frac{(\exp(-\alpha s_1) - 1)(\exp(-\alpha s_2) - 1)}{(\exp(-\alpha) - 1)}\right], \tag{5}$$

where $\alpha \in (-\infty, \infty) \setminus \{0\}$.

Its generator is

$$\varphi_\alpha(t) = -\ln\left[\frac{(\exp(-\alpha t) - 1)}{(\exp(-\alpha) - 1)}\right], \tag{6}$$

where α which is the parameter of Frank copula can be estimated from bivariate data as s_1 and s_2 are the cdfs of the marginal PCs. Frank copula is one-parametric and symmetrical.

The relationship between Kendall's tau τ and parameter α of Frank copula is given by

$$\frac{1 - D_1(\alpha)}{\alpha} = \frac{1 - \tau}{4}, \tag{7}$$

where $D_1(\alpha) = \frac{1}{\alpha}\int_0^\alpha \frac{t}{\exp(t) - 1}\,dt$ is a Debye function of first kind.

Let x_{ijkp} be the degradation measurement of k^{th} unit under i^{th} stress at j^{th} time y_j corresponding to p^{th} PC where $1 \le k \le n$, $1 \le i \le r$, $\zeta_{i-1} \le j \le \zeta_i, 1 \le p \le 2$, and where ζ_i is cumulative number of measurements up to i^{th} stress level. Let f be the measurement frequency, then we have $t_i = f\zeta_i \Delta t$.

By the independent increment property of gamma process, we have independent but non-identical random variables

$$\Delta x_{ijkp} \sim \text{Gamma Distribution}(v_p(y_j - y_{j-1})a_{i-1}, u_p). \tag{8}$$

So the pdf of Δx_{ijkp} is

$$g(\Delta x_{ijkp}) = \frac{(u_p)^{-v_p a_{i-1}\Delta y_{j-1}}}{\Gamma(v_p\, a_{i-1}\Delta y_{j-1})}(\Delta x_{ijkp})^{v_p a_{i-1}\Delta y_{j-1}-1}\exp\left[-\frac{\Delta x_{ijkp}}{u_p}\right] I_{(0,\infty)}(\Delta x_{ijkp}), \quad (9)$$

where $1 \le i \le r$, $\zeta_{i-1}+1 \le j \le \zeta_i$, $1 \le k \le n$, $1 \le p \le 2$, and a_i is the stress factor with $a_0 = 1$, and $I_{(0,\infty)}(\Delta x_{ijkp})$ is the indicator function defined as

$$I_{(0,\infty)}(\Delta x_{ijkp}) = \begin{cases} 1 & \Delta x_{ijkp} > 0 \\ 0 & \text{otherwise} \end{cases}.$$

Thus, using (4) $H(\Delta x_{ijk1}, \Delta x_{ijk2}) = C(G_1(\Delta x_{ijk1}), G_2(\Delta x_{ijk2}))$.

Let $c(\cdot,\cdot)$ be the pdf of $C(\cdot,\cdot)$.Then, using (4), (5) and (9), the likelihood function of SSPADT model for a bivariate gamma process is given by

$$l(\theta) = \prod_{k=1}^{n}\prod_{i=1}^{r}\prod_{j=\zeta_{i-1}+1}^{\zeta_i} c(G_1(\Delta x_{ijk1}), G_2(\Delta x_{ijk2})).\prod_{p=1}^{2} g_p(\Delta x_{ijkp}) \text{ (see Appendix 2)} \quad (10)$$

where $\zeta_i = l_1 + l_2 + \ldots + l_i$ is the cumulative number of measurements up to i^{th} stress level, $\zeta_0 = 0$.

By using Eq. (10), the log-likelihood function is:

$$L(\theta) = \sum_{k=1}^{n}\sum_{i=1}^{r}\sum_{j=\zeta_{i-1}+1}^{\zeta_i}\left[\ln c(G_1(\Delta x_{ijk1}), G_2(\Delta x_{ijk2})) + \sum_{p=1}^{2}\ln g_p(\Delta x_{ijkp})\right]$$

$$= \sum_{k=1}^{n}\sum_{i=1}^{r}\sum_{j=\zeta_{i-1}+1}^{\zeta_i}\{\ln(-\alpha) + \ln(e^{-\alpha}-1) - \alpha(G_1(\Delta x_{ijk1}) + G_2(\Delta x_{ijk2})) - 2\ln[e^{-\alpha} + e^{-\alpha(G_1(\Delta x_{ijk1})+G_2(\Delta x_{ijk2}))}$$
$$- e^{-\alpha G_1(\Delta x_{ijk1})} - e^{-\alpha G_2(\Delta x_{ijk2})}] + \sum_{p=1}^{2}[-\frac{\Delta x_{ijkp}}{u_p} - v_p a_{i-1}\Delta y_{j-1}\ln(u_p) + (v_p\, a_{i-1}\Delta y_{j-1} - 1)\ln(\Delta x_{ijkp})$$
$$-\ln(\Gamma(v_p a_{i-1}\Delta y_{j-1}))]\}. \quad (11)$$

The MLEs of $\alpha, u_1, u_2, v_1,$ and v_2 are obtained by using *NMaximize* option of *Mathematica* 9.0.

2.6 Case of independent PCs based on gamma process

If two PCs are independent then joint distribution function and likelihood function are given by

$$H(\Delta x_{ijk1}, \Delta x_{ijk2}) = G_1(\Delta x_{ijk1}) \cdot G_2(\Delta x_{ijk2})$$

and

$$l(\theta)=\prod_{k=1}^{n}\prod_{i=1}^{r}\prod_{j=\zeta_{i-1}+1}^{\zeta_i}\prod_{p=1}^{2}g_p(\Delta x_{ijkp})$$

2.7 Parameter estimation

The first and second order partial derivatives of (11) with respect to the model parameters are given in the Appendix 3.

2.8 Fisher information matrix

The elements of *Fisher Information Matrix* are negative expectations of the second order partial derivatives of L (log-likelihood function) with respect to model parameters (Lawless (1982)) (see Appendix 3**)**.
Thus, the Fisher information matrix F is:

$$\begin{bmatrix} E\left(-\frac{\partial^2 \ln L}{\partial\alpha^2}\right) & E\left(-\frac{\partial^2 \ln L}{\partial\alpha\partial u_1}\right) & E\left(-\frac{\partial^2 \ln L}{\partial\alpha\partial u_2}\right) & E\left(-\frac{\partial^2 \ln L}{\partial\alpha\partial v_1}\right) & E\left(-\frac{\partial^2 \ln L}{\partial\alpha\partial v_2}\right) \\ E\left(-\frac{\partial^2 \ln L}{\partial u_1\partial\alpha}\right) & E\left(-\frac{\partial^2 \ln L}{\partial u_1^2}\right) & E\left(-\frac{\partial^2 \ln L}{\partial u_1\partial u_2}\right) & E\left(-\frac{\partial^2 \ln L}{\partial u_1\partial v_1}\right) & E\left(-\frac{\partial^2 \ln L}{\partial u_1\partial v_2}\right) \\ E\left(-\frac{\partial^2 \ln L}{\partial u_2\partial\alpha}\right) & E\left(-\frac{\partial^2 \ln L}{\partial u_2\partial u_1}\right) & E\left(-\frac{\partial^2 \ln L}{\partial u_2^2}\right) & E\left(-\frac{\partial^2 \ln L}{\partial u_2\partial v_1}\right) & E\left(-\frac{\partial^2 \ln L}{\partial u_2\partial v_2}\right) \\ E\left(-\frac{\partial^2 \ln L}{\partial v_1\partial\alpha}\right) & E\left(-\frac{\partial^2 \ln L}{\partial v_1\partial u_1}\right) & E\left(-\frac{\partial^2 \ln L}{\partial v_1\partial u_2}\right) & E\left(-\frac{\partial^2 \ln L}{\partial v_1^2}\right) & E\left(-\frac{\partial^2 \ln L}{\partial v_1\partial v_2}\right) \\ E\left(-\frac{\partial^2 \ln L}{\partial v_2\partial\alpha}\right) & E\left(-\frac{\partial^2 \ln L}{\partial v_2\partial u_1}\right) & E\left(-\frac{\partial^2 \ln L}{\partial v_2\partial u_2}\right) & E\left(-\frac{\partial^2 \ln L}{\partial v_2\partial v_1}\right) & E\left(-\frac{\partial^2 \ln L}{\partial v_2^2}\right) \end{bmatrix}. \tag{12}$$

3. OPTIMAL TEST PLAN

The optimization problem consists of two parts:
1) the computation of approximate variance of R(t), and,
2) the total cost of experimentation.

3.1 Computation of asymptotic variance

Let T_1 and T_2 be the first passage times of gamma processes G_1 and G_2 with the threshold values d_1 and d_2, respectively. Assume that the measurement frequency is constant. The failure time T of a unit with two PCs is denoted as

$$T = \min\{T_1, T_2\}.$$

With the non-decreasing property of the degradation function the corresponding reliability function can be written as

$$R(t) = P[T_1 \geq t, T_2 \geq t] = P\left[\sup_{s\leq t} G_1(s) \leq d_1, \sup_{s\leq t} G_2(s) \leq d_2\right]$$
$$= C(F_{T1}(\alpha_1, 1/\beta_1), F_{T2}(\alpha_2, 1/\beta_2)), \qquad (13)$$

where $\alpha_1 = \frac{1}{\sqrt{d_1/u_1}}, \beta_1 = \frac{d_1}{v_1 u_1}, \alpha_2 = \frac{1}{\sqrt{d_2/u_2}}$, and $\beta_2 = \frac{d_2}{v_2 u_2}$.

Thus, the asymptotic variance of $R(t)$ at normal conditions is:

$$\text{Avar}(R(t)) = h^t F^{-1} h,$$

where

$$h^t = \left(\frac{\partial R(t)}{\partial \alpha}, \frac{\partial R(t)}{\partial u_1}, \frac{\partial R(t)}{\partial u_2}, \frac{\partial R(t)}{\partial v_1}, \frac{\partial R(t)}{\partial v_2}\right),$$

h^t denotes the transpose of h, and the variance-covariance matrix F^{-1} is the inverse of the *Fisher Information Matrix* F.

3.2 Cost function

The total cost TC $(f, \{\zeta_i\}_{i=1}^r)$ involves three parts **(**Yu and Tseng (1999),Yu and Chiao (2002),Wu and Chang (2002)) :

1) *operating cost*, which mainly comprises the operator's salary, power bills, depreciation of testing equipments, can be expressed as

$$C_o f \zeta_r,$$

where C_o is salary of operator per unit of time.

2) *measurement cost*, which includes the cost of using measuring equipments, and the expense of testing materials, can be expressed as

$$C_m n \zeta_r,$$

where C_m is the cost per measurement per device.

3) *sample cost*, which is related to number of samples or testing devices, and can be formulated as

$$C_d n,$$

where C_d is the price of an individual device.

So, the *total cost* of testing (TC) is:

$$TC(f, \{\zeta_i\}_{i=1}^r) = C_o f \zeta_r + C_m n \zeta_r + C_d n. \qquad (14)$$

3.3 Optimization model

The problem can be formulated as:

$$\text{Minimize} \quad Avar(R(t)) = h^t F^{-1} h$$

Subject to

$$\text{Total Cost}(\{\zeta_i\}_{i=1}^{r}, f) \leq \text{Pre-defined budget}, \ (\{\zeta_i\}_{i=1}^{r}, f) \in \mathbb{N}^{r+1}. \tag{15}$$

4. CONFIDENCE INTERVAL

The MLEs $\hat{\alpha}, \hat{u}_1, \hat{u}_2, \hat{v}_1,$ and $\hat{v}_2$ are approximately normally distributed in large samples, so $(\hat{\alpha}, \hat{u}_1, \hat{u}_2, \hat{v}_1, \hat{v}_2) \sim N((\alpha, u_1, u_2, v_1, v_2), F^{-1})$. The two-sided $100(1-\alpha_1)$ % approximate confidence interval for the parameter u_1 is given by $\hat{u}_1 \pm z_{\alpha_1/2}\sqrt{\hat{var}(\hat{u}_1)}$, where $z_{\alpha_1/2}$ is the $(1-\alpha_1/2)^{th}$ quantile of a standard normal distribution , and $\sqrt{\hat{var}(\hat{u}_1)}$ is obtained by taking the square root of the first diagonal element of F^{-1}. Similarly, two-sided $100(1-\alpha_1)$ % approximate confidence intervals for the parameters $\alpha, u_2, v_1,$ and v_2 can be obtained.

5. NUMERICAL EXAMPLE

In this section the proposed method is illustrated using one acceleration factor. Assume that an alloy product has the two dependent fatigue cracks. For illustrative purpose, the following hypothetical data set is used for dependent case $\theta = \{\alpha, u_1, u_2, v_1, v_2\} = \{0.73, 9.297\times10^{-3}, 6.152\times10^{-3}, 658.9, 622.5\}$, and for independent case $\theta = \{u_1, u_2, v_1, v_2\} = \{8.271\times10^{-3}, 5.288\times10^{-3}, 733.5, 715.5\}$, with cost factors as C_o = 13 per unit time, C_m = 2 per measurement, and C_d = 750 per unit.

Here, $d_1 = 0.9$, $d_2 = 1$, $\Delta t = 0.01$ million cycles and the mission time is 0.2 million cycles. Fig. 1 and Fig. 2 show the marginal pdf and the marginal reliability for both dependent and independent case.

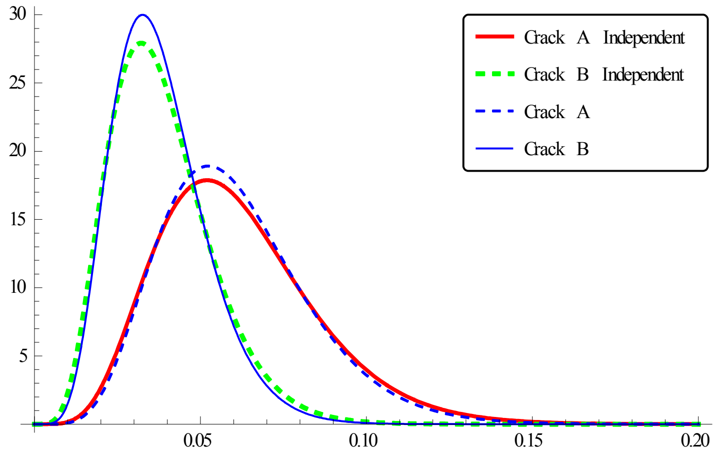

Figure. 1. Comparison of unit's marginal pdf considering both dependent and independent case

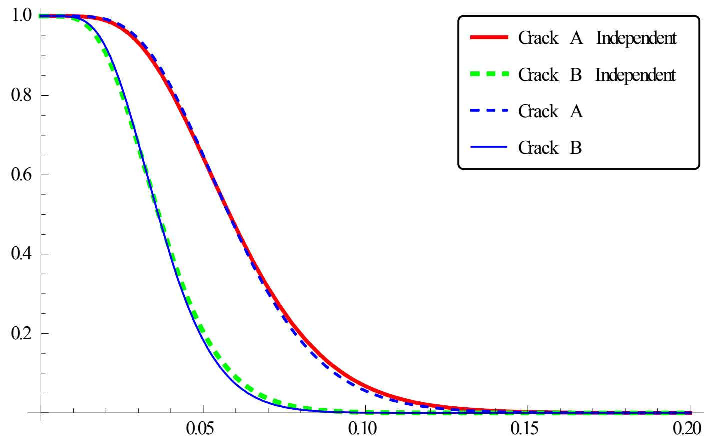

Figure 2. Comparison of unit's marginal reliability considering both dependent and independent case

5.1 Optimal plan

Optimal values are obtained using the *NMinimize* option of *Mathematica* 9.0. Table I depicts the optimal plans for one acceleration factor viz., $a_1 = 1.2$ with pre-defined budget of 10,000 monetary units.
Table 1 shows that total cost and the stress change point, $t_1 = l_1 \times f \times \Delta t$, is smaller when the PCs are dependent than when they are independent.

Table 1. Optimal plans corresponding to one acceleration factor: $a_1 = 1.2$

Cases	r	$Avar(\hat{t}_{q,0})$	l_1	l_2	f	t_1 (millions cycles)	Total cycles (millions)	Total budget	Optimal Cost
Dependent	2	1.75552×10^{-7}	14	6	1	0.140	0.20	10,000	8,160
ndependent	2	6.83934×10^{-6}	16	7	1	0.160	0.230	10,000	8,259

Table 2. Simulated data set based on the degradation increments of fatigue size of crack A

Time	Unit 1	Unit 2	Unit 3	Unit 4	Unit 5	Unit 6	Unit 7	Unit 8	Unit 9	Unit 10
0.01	0.07571	0.03073	0.07645	0.05804	0.02448	0.09534	0.027894	0.05903	0.080835	0.067065
0.02	0.1024	0.02811	0.0568	0.050115	0.052682	0.10951	0.05675	0.02773	0.026085	0.06554
0.03	0.0641	0.04651	0.04866	0.04267	0.04158	0.05068	0.05503	0.057632	0.04562	0.035315
0.04	0.0345	0.08961	0.038767	0.06389	0.03703	0.05927	0.049513	0.078238	0.0338	0.039846
0.05	0.0762	0.07701	0.0338	0.09624	0.04872	0.03142	0.045521	0.042255	0.128256	0.060083
0.06	0.0785	0.03418	0.04308	0.03894	0.0357	0.1088	0.06542	0.070271	0.045548	0.026132
0.07	0.0395	0.07752	0.12015	0.05029	0.054397	0.06055	0.06208	0.06803	0.041026	0.047147
0.08	0.0252	0.05398	0.09766	0.08789	0.02337	0.038555	0.04073	0.05205	0.030168	0.039735
0.09	0.03658	0.04014	0.145896	0.030389	0.03789	0.081129	0.043154	0.079451	0.07437	0.048065
0.10	0.03299	0.04286	0.075061	0.0487	0.0836	0.048005	0.047004	0.035843	0.13106	0.038167
0.11	0.0409	0.051408	0.047231	0.0738	0.03645	0.07737	0.032348	0.050996	0.048358	0.060488
0.12	0.0653	0.066642	0.01678	0.0458	0.08456	0.06686	0.03597	0.069204	0.047378	0.062381
0.13	0.03857	0.061347	0.0301	0.06803	0.094755	0.046004	0.06807	0.05972	0.033397	0.042417
0.14	0.057834	0.05009	0.114826	0.057602	0.047179	0.089231	0.06867	0.03724	0.067306	0.076878
AFTER ACCELERATION										
Time	Unit 1	Unit 2	Unit 3	Unit 4	Unit 5	Unit 6	Unit 7	Unit 8	Unit 9	Unit 10
0.15	0.06785	0.08302	0.06341	0.07296	0.124133	0.057203	0.03682	0.060161	0.066946	0.032382
0.16	0.17065	0.07188	0.062247	0.070857	0.051937	0.09395	0.0989	0.062505	0.053449	0.036519
0.17	0.03497	0.15602	0.091898	0.109265	0.06626	0.15451	0.10037	0.06702	0.017518	0.050046
0.18	0.10088	0.05918	0.106455	0.092898	0.12697	0.07134	0.061861	0.098518	0.04464	0.060373
0.19	0.068274	0.057891	0.052489	0.065295	0.0251	0.095373	0.076234	0.05806	0.0285	0.093585
0.20	0.065759	0.06096	0.099216	0.075903	0.042655	0.03537	0.082666	0.04718	0.060311	0.061888

5.2 Simulated data

To validate the proposed method, we run a simulation procedure to generate the degradation of a unit with two PCs. Degradation increments of fatigue size of crack A and crack B are displayed in Table II and Table III, respectively. Fig. 3 and Fig. 4 show the cumulative degradation of the fatigue crack size A and B, respectively.

Table 3. Simulated data set based on the degradation increments of fatigue size of crack B

Time	Unit 1	Unit 2	Unit 3	Unit 4	Unit 5	Unit 6	Unit 7	Unit 8	Unit 9	Unit 10
0.01	0.0589	0.030533	0.062127	0.03397	0.059876	0.05086	0.052178	0.029831	0.023739	0.032227
0.02	0.03697	0.01397	0.043043	0.028733	0.024888	0.022376	0.023106	0.026136	0.033552	0.039835
0.03	0.040656	0.031655	0.04	0.09397	0.04559	0.027859	0.042376	0.058446	0.026049	0.03549
0.04	0.03081	0.02339	0.040914	0.027581	0.055248	0.03806	0.058525	0.043649	0.038497	0.030124
0.05	0.05757	0.024488	0.022758	0.059425	0.021352	0.029876	0.027381	0.034737	0.038403	0.036859
0.06	0.034888	0.02073	0.031897	0.025443	0.035307	0.036401	0.041977	0.06279	0.031704	0.018744
0.07	0.02227	0.022226	0.04896	0.022351	0.064956	0.037832	0.01655	0.062075	0.039405	0.020374
0.08	0.09361	0.045817	0.02628	0.025047	0.033075	0.06033	0.021692	0.04846	0.052213	0.035726
0.09	0.02895	0.025674	0.01717	0.039181	0.02409	0.039492	0.0341	0.042291	0.022683	0.060429
0.10	0.03533	0.043799	0.06655	0.034153	0.028186	0.06033	0.03646	0.040301	0.040574	0.02664
0.11	0.0365	0.030639	0.050489	0.045096	0.041782	0.035829	0.018699	0.019122	0.033477	0.020549
0.12	0.057169	0.05025	0.021686	0.032637	0.019269	0.058063	0.043059	0.057722	0.049881	0.017143
0.13	0.052178	0.069799	0.028614	0.057395	0.050165	0.018339	0.047012	0.031809	0.05281	0.020295
0.14	0.067589	0.025859	0.037855	0.04466	0.036675	0.034027	0.048692	0.02135	0.049067	0.062143
AFTER ACCELERATION										
Time	Unit 1	Unit 2	Unit 3	Unit 4	Unit 5	Unit 6	Unit 7	Unit 8	Unit 9	Unit 10
0.15	0.056132	0.032342	0.044771	0.086063	0.054754	0.028383	0.063913	0.053737	0.062687	0.02474
0.16	0.03631	0.04966	0.02935	0.027395	0.057953	0.01231	0.072401	0.06459	0.03876	0.020396
0.17	0.0511	0.047811	0.055467	0.028461	0.040799	0.079127	0.057592	0.064589	0.051559	0.034577
0.18	0.0553	0.052578	0.03088	0.029946	0.045763	0.06787	0.039352	0.04038	0.04692	0.047002
0.19	0.02453	0.066142	0.040044	0.031612	0.058044	0.0691	0.020611	0.02896	0.068898	0.049776
0.20	0.029059	0.057176	0.034286	0.066353	0.048619	0.04593	0.039781	0.028594	0.027496	0.032149

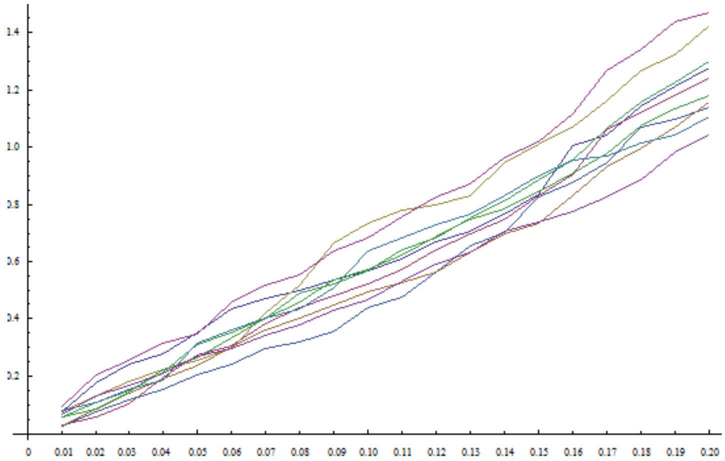

Figure 3. Cumulative degradation of fatigue crack size A

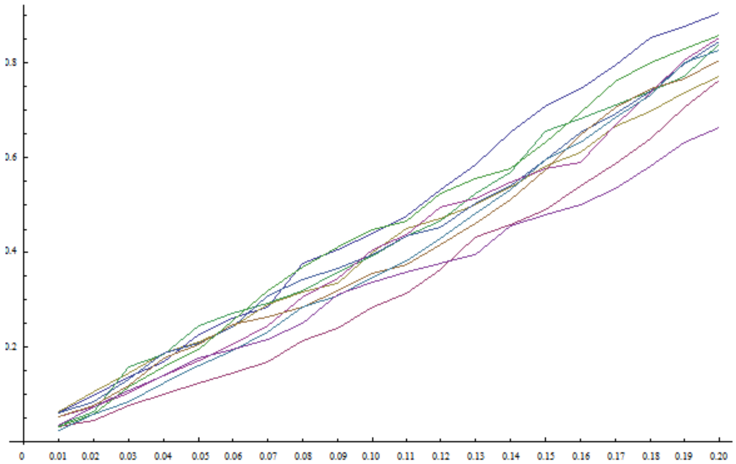

Figure 4. Cumulative degradation of fatigue crack size B

5.3 ML Estimates of the design parameters

The ML estimates of the design parameters obtained using simulated data in Section V-B are $\hat{\alpha} = 0.647345, \hat{u}_1 = 0.00955709, \hat{u}_2 = 0.00616001, \hat{v}_1 = 632.603,$ and $\hat{v}_2 = 627.823$.
These estimates are obtained using the *NMaximize* option of *Mathematica* 9.0.

5.4 Confidence intervals

$100(1-\alpha_1)\%$ confidence intervals for the parameters are obtained using the inverse of the observed Fisher information matrix $\hat{F}^{-1}$, and is given by

$$\hat{F}^{-1} = \begin{bmatrix} 1.49873 & 3.81658\times10^{-4} & -6.73634\times10^{-5} & 48.9861 & 0.442989 \\ 3.81658\times10^{-4} & 7.0127\times10^{-6} & -9.04306\times10^{-9} & -0.417808 & 8.92762\times10^{-4} \\ -6.73634\times10^{-5} & -9.04306\times10^{-9} & 6.26351\times10^{-9} & 0.00426381 & -0.0549578 \\ 48.9861 & -0.417808 & 0.00426381 & 34288.7 & 416.297 \\ 0.442989 & 8.92762\times10^{-4} & -0.0549578 & 416.297 & 9878.41 \end{bmatrix}.$$

The observed value of F^{-1}, that is $\hat{F}^{-1}$, is determined by substituting the estimated parameters $\hat{\alpha}, \hat{u}_1, u_2, v_1,$ and $\hat{v}_2$ for the true parameters in the asymptotic covariance matrix. The square root of a diagonal element of $\hat{F}^{-1}$ gives the standard error of an estimator. Thus, the 95% approximate confidence intervals for $\alpha, u_1, u_2, v_1,$ and v_2, respectively, are $-1.75 \le \alpha \le 3.0468$, $0.004 \le u_1 \le 0.0147$, $0.00461 \le u_2 \le 0.00771$, $269.67 \le v_1 \le 995.54$, and $433.02 \le v_2 \le 822.62$.

Table 4. Sensitivity analysis

Parameter	% change	Optimal plan (l_1, l_2, f, t_1)	Avar	Z^{**} (PD %)
α	+2.5%	(14,6,1,0.14)	1.75661×10^{-7}	0.06208
α	−2.5%	(14,6,1,0.14)	1.75449×10^{-7}	0.05867
u_1	+2.5%	(14,6,1,0.14)	1.74496×10^{-7}	0.6015311
u_1	−2.5%	(10,9,1,0.10)	1.25257×10^{-7}	28.64963
v_1	+2.5%	(10,9,1,0.10)	1.29547×10^{-7}	26.2059
v_1	−2.5%	(10,9,1,0.10)	1.25525×10^{-7}	28.49697
u_2	+2.5%	(12,8,1,0.12)	1.56869×10^{-6}	793.5
u_2	−2.5%	(8,10,1,0.08)	1.91183×10^{-9}	98.91
v_2	+2.5%	(10,9,1,0.10)	1.2975×10^{-6}	639.09
v_2	−2.5%	(10,9,1,0.10)	9.54053×10^{-9}	94.5
No change		(14,6,1,0.14)	1.75552×10^{-7}	0

5.5 Sensitivity analysis

To use an optimum test plan, one needs estimates of the design parameters $\theta = \{\alpha, u_1, u_2, v_1, v_2\}$. These estimates sometimes may significantly affect the values of the resulting decision variables; therefore, their incorrect choice may give a poor estimate. Hence, it is important to conduct sensitivity analysis to evaluate the robustness of the resulting SSPADT plan.

The percentage deviations of the optimal settings are measured by $\text{PD} = (|Z^{**} - Z^{*}|/Z^{*}) \times 100$, where Z^{*} is the setting obtained with the given design parameters, and Z^{**} is the one obtained when the parameter is mis-specified (Srivastava and Mittal (14)). Table 4 presents the optimal test plans, and the corresponding asymptotical variance for various deviations from the design parameter estimates.

6. CONCLUDING REMARKS

Life test under accelerated environmental conditions may be fully accelerated or partially accelerated. In fully accelerated testing all the test units are run at accelerated conditions; while in partially accelerated testing they are run at both normal and accelerated conditions. Unlike in ALT where failure occurs if the unit stops working, in ADT a failure occurs if performance characteristic of the unit crosses a pre-specified threshold. PADT helps in preventing over-stressing of the test units. This paper focuses on the design of partially accelerated degradation test under step-stress loading using gamma process for modelling degradation paths of two PCs. The dependency between the two PCs has been modelled through Frank copula function. The optimal plan consists in determining the optimal test duration and number of inspections at each intermediate stress level, and is obtained using variance optimality criterion. Under the constraint of total experimental cost; the variance optimality criterion consists in minimizing asymptotic variance of reliability function at normal operating condition with mission time pre-specified. The method developed has been explained using an example and sensitivity analysis carried out.

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APPENDIX 1

Notation

r	number of stress levels
n	sample size
f	measurement frequency
p	number of PCs
F_T	cdf of Birnbaum-Saunders distribution
f_T	pdf of Birnbaum-Saunders distribution
F	Fisher information matrix
x_{ijkp}	degradation measurements of k^{th} unit under i^{th} stress at j^{th} time y_j corresponding to p^{th} PC, $1 \le i \le r$, $\zeta_{-} + 1 \le j \le \zeta$, $1 \le k \le n$, $1 \le p \le 2$.
l_i	total number of measurements under the i^{th} stress level
t_i	duration time of degradation test under the i^{th} stress level
ζ_i	cumulative number of measurements up to i^{th} stress level (with, $\zeta_0 \equiv 0$)
d_p	critical value corresponding to p^{th} PC
C_o	unit cost of operation
C_m	unit cost of measurement
C_d	unit cost of the device
TC	total cost of conducting a SSPADT experiment
θ	parameters of SSPADT model for a Gamma process
$l(\theta)$	likelihood function of SSPADT model for a Gamma process
$\hat{\theta}$	the MLE of θ
$R(\cdot)$	reliability function

APPENDIX 2

Calculations of pdf $c(\cdot,\cdot)$ of $C(\cdot,\cdot)$ are as below:

$$C(G_1(\Delta x_{ijk1}), G_2(\Delta x_{ijk2})) = -\frac{1}{\alpha}\ln\left[1+\frac{\left(\exp(-\alpha G_1(\Delta x_{ijk1}))-1\right)\left(\exp(-\alpha G_2(\Delta x_{ijk2}))-1\right)}{\left(\exp(-\alpha)-1\right)}\right]$$

Taking derivative w.r.t Δx_{ijk1}

$$\frac{\exp(-\alpha G_1(\Delta x_{ijk1}))\left(\exp(-\alpha G_2(\Delta x_{ijk2}))-1\right)}{\left(\exp(-\alpha)-1\right)+\left(\exp(-\alpha G_1(\Delta x_{ijk1}))-1\right)\left(\exp(-\alpha G_2(\Delta x_{ijk2}))-1\right)} g_1(\Delta x_{ijk1})$$

Taking derivative w.r.t Δx_{ijk2}

$$-\frac{\alpha\left(\exp(-\alpha)-1\right)\exp(-\alpha(G_1(\Delta x_{ijk1})+G_2(\Delta x_{ijk2})))}{\left[\left(\exp(-\alpha)-1\right)+\left(\exp(-\alpha G_1(\Delta x_{ijk1}))-1\right)\left(\exp(-\alpha G_2(\Delta x_{ijk2}))-1\right)\right]^2} g_1(\Delta x_{ijk1}) g_2(\Delta x_{ijk2})$$

is the pdf of Frank copula distribution.

APPENDIX 3

Calculations of derivatives of the log-likelihood and the elements of Fisher information matrix given in section 2.7 and 2.8 respectively, are as below:
The first order partial derivatives are,

$$\frac{\partial L}{\partial u_1}=\sum_{k=1}^{n}\sum_{i=1}^{r}\sum_{j=\zeta_{i-1}+1}^{\zeta_{i-1}}\left[\frac{\Delta x_{ijk1}}{u_1^2}-a_{i-1}\Delta y_{j-1}\frac{v_1}{u_1}-\alpha\frac{\partial G_1(\Delta x_{ijk1})}{\partial u_1}\left(\frac{e^{-\alpha}-e^{-\alpha(G_1(\Delta x_{ijk1})+G_2(\Delta x_{ijk2}))}+e^{-\alpha G_1(\Delta x_{ijk1})}-e^{-\alpha G_2(\Delta x_{ijk2})}}{e^{-\alpha}+e^{-\alpha(G_1(\Delta x_{ijk1})+G_2(\Delta x_{ijk2}))}-e^{-\alpha G_1(\Delta x_{ijk1})}-e^{-\alpha G_2(\Delta x_{ijk2})}}\right)\right]$$

$$\frac{\partial L}{\partial u_2}=\sum_{k=1}^{n}\sum_{i=1}^{r}\sum_{j=\zeta_{i-1}+1}^{\zeta_{i-1}}\left[\frac{\Delta x_{ijk2}}{u_2^2}-a_{i-1}\Delta y_{j-1}\frac{v_2}{u_2}-\alpha\frac{\partial G_2(\Delta x_{ijk2})}{\partial u_2}\left(\frac{e^{-\alpha}-e^{-\alpha(G_1(\Delta x_{ijk1})+G_2(\Delta x_{ijk2}))}-e^{-\alpha G_1(\Delta x_{ijk1})}+e^{-\alpha G_2(\Delta x_{ijk2})}}{e^{-\alpha}+e^{-\alpha(G_1(\Delta x_{ijk1})+G_2(\Delta x_{ijk2}))}-e^{-\alpha G_1(\Delta x_{ijk1})}-e^{-\alpha G_2(\Delta x_{ijk2})}}\right)\right]$$

$$\frac{\partial L}{\partial v_1}=\sum_{k=1}^{n}\sum_{i=1}^{r}\sum_{j=\zeta_{i-1}+1}^{\zeta_{i-1}}\left[a_{i-1}\Delta y_{j-1}\left(\ln\left(\frac{\Delta x_{ijk1}}{u_1}\right)-\text{PolyGamma}[0,a_{i-1}\Delta y_{j-1}v_1]\right)-\right.$$
$$\left.\alpha\frac{\partial G_1(\Delta x_{ijk1})}{\partial v_1}\left(\frac{e^{-\alpha}-e^{-\alpha(G_1(\Delta x_{ijk1})+G_2(\Delta x_{ijk2}))}+e^{-\alpha G_1(\Delta x_{ijk1})}-e^{-\alpha G_2(\Delta x_{ijk2})}}{e^{-\alpha}+e^{-\alpha(G_1(\Delta x_{ijk1})+G_2(\Delta x_{ijk2}))}-e^{-\alpha G_1(\Delta x_{ijk1})}-e^{-\alpha G_2(\Delta x_{ijk2})}}\right)\right]$$

$$\frac{\partial L}{\partial v_2}=\sum_{k=1}^{n}\sum_{i=1}^{r}\sum_{j=\zeta_{i-1}+1}^{\zeta_{i-1}}\left[a_{i-1}\Delta y_{j-1}\left(\ln\left(\frac{\Delta x_{ijk2}}{u_2}\right)-\text{PolyGamma}[0,a_{i-1}\Delta y_{j-1}v_2]\right)-\right.$$
$$\left.\alpha\frac{\partial G_2(\Delta x_{ijk2})}{\partial v_2}\left(\frac{e^{-\alpha}-e^{-\alpha(G_1(\Delta x_{ijk1})+G_2(\Delta x_{ijk2}))}-e^{-\alpha G_1(\Delta x_{ijk1})}+e^{-\alpha G_2(\Delta x_{ijk2})}}{e^{-\alpha}+e^{-\alpha(G_1(\Delta x_{ijk1})+G_2(\Delta x_{ijk2}))}-e^{-\alpha G_1(\Delta x_{ijk1})}-e^{-\alpha G_2(\Delta x_{ijk2})}}\right)\right]$$

$$\frac{\partial L}{\partial \alpha}=\sum_{k=1}^{n}\sum_{i=1}^{r}\sum_{j=\zeta_{i-1}+1}^{\zeta_{i-1}}\left[\frac{1}{\alpha}-\frac{e^{-\alpha}}{e^{-\alpha}-1}-(G_1(\Delta x_{ijk1})+G_2(\Delta x_{ijk2}))+\right.$$
$$\left.2\left(\frac{e^{-\alpha}+(G_1(\Delta x_{ijk1})+G_2(\Delta x_{ijk2}))e^{-\alpha(G_1(\Delta x_{ijk1})+G_2(\Delta x_{ijk2}))}-G_1(\Delta x_{ijk1})e^{-\alpha G_1(\Delta x_{ijk1})}-G_2(\Delta x_{ijk2})e^{-\alpha G_2(\Delta x_{ijk2})}}{e^{-\alpha}+e^{-\alpha(G_1(\Delta x_{ijk1})+G_2(\Delta x_{ijk2}))}-e^{-\alpha G_1(\Delta x_{ijk1})}-e^{-\alpha G_2(\Delta x_{ijk2})}}\right)\right]$$

The second order partial derivatives are,

$$\frac{\partial^2 L}{\partial u_1^2}=\sum_{k=1}^{n}\sum_{i=1}^{r}\sum_{j=\zeta_{i-1}+1}^{\zeta_{i-1}}\left[\frac{-2\Delta x_{ijk1}}{u_1^3}+a_{i-1}\Delta y_{j-1}\frac{v_1}{u_1^2}-\alpha\frac{\partial^2 G_1(\Delta x_{ijk1})}{\partial u_1^2}\left(\frac{e^{-\alpha}-e^{-\alpha(G_1(\Delta x_{ijk1})+G_2(\Delta x_{ijk2}))}+e^{-\alpha G_1(\Delta x_{ijk1})}-e^{-\alpha G_2(\Delta x_{ijk2})}}{e^{-\alpha}+e^{-\alpha(G_1(\Delta x_{ijk1})+G_2(\Delta x_{ijk2}))}-e^{-\alpha G_1(\Delta x_{ijk1})}-e^{-\alpha G_2(\Delta x_{ijk2})}}\right)\right.$$
$$\left.+2\left(\alpha\frac{\partial G_1(\Delta x_{ijk1})}{\partial u_1}\right)^2\left(\frac{e^{-\alpha(G_1(\Delta x_{ijk1})+G_2(\Delta x_{ijk2}))}-e^{-\alpha G_1(\Delta x_{ijk1})}}{e^{-\alpha}+e^{-\alpha(G_1(\Delta x_{ijk1})+G_2(\Delta x_{ijk2}))}-e^{-\alpha G_1(\Delta x_{ijk1})}-e^{-\alpha G_2(\Delta x_{ijk2})}}\right)^2\right]$$

$$\frac{\partial^2 L}{\partial u_1\partial v_1}=\sum_{k=1}^{n}\sum_{i=1}^{r}\sum_{j=\zeta_{i-1}+1}^{\zeta_{i-1}}\left[-\frac{a_{i-1}\Delta y_{j-1}}{u_1}-\alpha\frac{\partial^2 G_1(\Delta x_{ijk1})}{\partial u_1\partial v_1}\left(\frac{e^{-\alpha}-e^{-\alpha(G_1(\Delta x_{ijk1})+G_2(\Delta x_{ijk2}))}+e^{-\alpha G_1(\Delta x_{ijk1})}-e^{-\alpha G_2(\Delta x_{ijk2})}}{e^{-\alpha}+e^{-\alpha(G_1(\Delta x_{ijk1})+G_2(\Delta x_{ijk2}))}-e^{-\alpha G_1(\Delta x_{ijk1})}-e^{-\alpha G_2(\Delta x_{ijk2})}}\right)\right.$$
$$\left.+\alpha^2\frac{\partial G_1(\Delta x_{ijk1})}{\partial u_1}\frac{\partial G_1(\Delta x_{ijk1})}{\partial v_1}\left(\frac{e^{-\alpha(G_1(\Delta x_{ijk1})+G_2(\Delta x_{ijk2}))}-e^{-\alpha G_1(\Delta x_{ijk1})}}{e^{-\alpha}+e^{-\alpha(G_1(\Delta x_{ijk1})+G_2(\Delta x_{ijk2}))}-e^{-\alpha G_1(\Delta x_{ijk1})}-e^{-\alpha G_2(\Delta x_{ijk2})}}\right)^2\right]$$

$$\frac{\partial^2 L}{\partial u_1\partial u_2}=\sum_{k=1}^{n}\sum_{i=1}^{r}\sum_{j=\zeta_{i-1}+1}^{\zeta_{i-1}}\left[-2\alpha^2\frac{\partial G_1(\Delta x_{ijk1})}{\partial u_1}\frac{\partial G_2(\Delta x_{ijk2})}{\partial u_2}\frac{e^{-\alpha(G_1(\Delta x_{ijk1})+G_2(\Delta x_{ijk2}))}(e^{-\alpha}-1)}{\left(e^{-\alpha}+e^{-\alpha(G_1(\Delta x_{ijk1})+G_2(\Delta x_{ijk2}))}-e^{-\alpha G_1(\Delta x_{ijk1})}-e^{-\alpha G_2(\Delta x_{ijk2})}\right)^2}\right]$$

$$\frac{\partial^2 L}{\partial u_1\partial v_2}=\sum_{k=1}^{n}\sum_{i=1}^{r}\sum_{j=\zeta_{i-1}+1}^{\zeta_{i-1}}\left[-2\alpha^2\frac{\partial G_1(\Delta x_{ijk1})}{\partial u_1}\frac{\partial G_2(\Delta x_{ijk2})}{\partial v_2}\frac{e^{-\alpha(G_1(\Delta x_{ijk1})+G_2(\Delta x_{ijk2}))}(e^{-\alpha}-1)}{\left(e^{-\alpha}+e^{-\alpha(G_1(\Delta x_{ijk1})+G_2(\Delta x_{ijk2}))}-e^{-\alpha G_1(\Delta x_{ijk1})}-e^{-\alpha G_2(\Delta x_{ijk2})}\right)^2}\right]$$

$$\frac{\partial^2 L}{\partial u_1\partial \alpha}=\sum_{k=1}^{n}\sum_{i=1}^{r}\sum_{j=\zeta_{i-1}+1}^{\zeta_{i-1}}\left[-\frac{\partial G_1(\Delta x_{ijk1})}{\partial u_1}\left(\frac{e^{-\alpha}-e^{-\alpha(G_1(\Delta x_{ijk1})+G_2(\Delta x_{ijk2}))}+e^{-\alpha G_1(\Delta x_{ijk1})}-e^{-\alpha G_2(\Delta x_{ijk2})}}{e^{-\alpha}+e^{-\alpha(G_1(\Delta x_{ijk1})+G_2(\Delta x_{ijk2}))}-e^{-\alpha G_1(\Delta x_{ijk1})}-e^{-\alpha G_2(\Delta x_{ijk2})}}\right)\right.$$
$$\left.+\alpha e^{-\alpha G_1(\Delta x_{ijk1})}\frac{\partial G_1(\Delta x_{ijk1})}{\partial u_1}\left(\frac{(G_1(\Delta x_{ijk1})-e^{-\alpha})+(e^{-\alpha}-2(G_1(\Delta x_{ijk1})+G_2(\Delta x_{ijk2})))e^{-\alpha G_2(\Delta x_{ijk2})}+(G_1(\Delta x_{ijk1})+2G_2(\Delta x_{ijk2}))e^{-2\alpha G_2(\Delta x_{ijk2}))}}{\left(e^{-\alpha}+e^{-\alpha(G_1(\Delta x_{ijk1})+G_2(\Delta x_{ijk2}))}-e^{-\alpha G_1(\Delta x_{ijk1})}-e^{-\alpha G_2(\Delta x_{ijk2})}\right)^2}\right)\right]$$

$$\frac{\partial^2 L}{\partial u_2^2}=\sum_{k=1}^{n}\sum_{i=1}^{r}\sum_{j=\zeta_{i-1}+1}^{\zeta_{i-1}}\left[\frac{-2\Delta x_{ijk2}}{u_2^3}+a_{i-1}\Delta y_{j-1}\frac{v_2}{u_2^2}-\alpha\frac{\partial^2 G_2(\Delta x_{ijk2})}{\partial u_2^2}\left(\frac{e^{-\alpha}-e^{-\alpha(G_1(\Delta x_{ijk1})+G_2(\Delta x_{ijk2}))}-e^{-\alpha G_1(\Delta x_{ijk1})}+e^{-\alpha G_2(\Delta x_{ijk2})}}{e^{-\alpha}+e^{-\alpha(G_1(\Delta x_{ijk1})+G_2(\Delta x_{ijk2}))}-e^{-\alpha G_1(\Delta x_{ijk1})}-e^{-\alpha G_2(\Delta x_{ijk2})}}\right)\right.$$
$$\left.-2\left(\alpha\frac{\partial G_2(\Delta x_{ijk2})}{\partial u_2}\right)^2\left(\frac{e^{-\alpha(G_1(\Delta x_{ijk1})+G_2(\Delta x_{ijk2}))}-e^{-\alpha G_2(\Delta x_{ijk2})}}{e^{-\alpha}+e^{-\alpha(G_1(\Delta x_{ijk1})+G_2(\Delta x_{ijk2}))}-e^{-\alpha G_1(\Delta x_{ijk1})}-e^{-\alpha G_2(\Delta x_{ijk2})}}\right)^2\right]$$

$$\frac{\partial^2 L}{\partial u_2\partial v_1}=\sum_{k=1}^{n}\sum_{i=1}^{r}\sum_{j=\zeta_{i-1}+1}^{\zeta_{i-1}}\left[-2\alpha^2\frac{\partial G_2(\Delta x_{ijk2})}{\partial u_2}\frac{\partial G_1(\Delta x_{ijk1})}{\partial v_1}\frac{e^{-\alpha(G_1(\Delta x_{ijk1})+G_2(\Delta x_{ijk2}))}(e^{-\alpha}-1)}{\left(e^{-\alpha}+e^{-\alpha(G_1(\Delta x_{ijk1})+G_2(\Delta x_{ijk2}))}-e^{-\alpha G_1(\Delta x_{ijk1})}-e^{-\alpha G_2(\Delta x_{ijk2})}\right)^2}\right]$$

$$\frac{\partial^2 L}{\partial u_2\partial v_2}=\sum_{k=1}^{n}\sum_{i=1}^{r}\sum_{j=\zeta_{i-1}+1}^{\zeta_{i-1}}\left[-\frac{a_{i-1}\Delta y_{j-1}}{u_2}-\alpha\frac{\partial^2 G_2(\Delta x_{ijk2})}{\partial u_2\partial v_2}\left(\frac{e^{-\alpha}-e^{-\alpha(G_1(\Delta x_{ijk1})+G_2(\Delta x_{ijk2}))}-e^{-\alpha G_1(\Delta x_{ijk1})}+e^{-\alpha G_2(\Delta x_{ijk2})}}{e^{-\alpha}+e^{-\alpha(G_1(\Delta x_{ijk1})+G_2(\Delta x_{ijk2}))}-e^{-\alpha G_1(\Delta x_{ijk1})}-e^{-\alpha G_2(\Delta x_{ijk2})}}\right)\right.$$
$$\left.+\alpha^2\frac{\partial G_2(\Delta x_{ijk2})}{\partial u_2}\frac{\partial G_2(\Delta x_{ijk2})}{\partial v_2}\left(\frac{e^{-\alpha(G_1(\Delta x_{ijk1})+G_2(\Delta x_{ijk2}))}-e^{-\alpha G_2(\Delta x_{ijk2})}}{e^{-\alpha}+e^{-\alpha(G_1(\Delta x_{ijk1})+G_2(\Delta x_{ijk2}))}-e^{-\alpha G_1(\Delta x_{ijk1})}-e^{-\alpha G_2(\Delta x_{ijk2})}}\right)^2\right]$$

$$\frac{\partial^2 L}{\partial u_2\partial\alpha}=\sum_{k=1}^{n}\sum_{i=1}^{r}\sum_{j=\zeta_{i-1}+1}^{\zeta_{i-1}}\left[-\frac{\partial G_2(\Delta x_{ijk2})}{\partial u_2}\left(\frac{e^{-\alpha}-e^{-\alpha(G_1(\Delta x_{ijk1})+G_2(\Delta x_{ijk2}))}-e^{-\alpha G_1(\Delta x_{ijk1})}+e^{-\alpha G_2(\Delta x_{ijk2})}}{e^{-\alpha}+e^{-\alpha(G_1(\Delta x_{ijk1})+G_2(\Delta x_{ijk2}))}-e^{-\alpha G_1(\Delta x_{ijk1})}-e^{-\alpha G_2(\Delta x_{ijk2})}}\right)\right.$$
$$\left.+\alpha e^{-\alpha G_2(\Delta x_{ijk2})}\frac{\partial G_2(\Delta x_{ijk2})}{\partial u_2}\left(\frac{(G_2(\Delta x_{ijk2})-e^{-\alpha})+(e^{-\alpha}-2(G_1(\Delta x_{ijk1})+G_2(\Delta x_{ijk2})))e^{-\alpha G_1(\Delta x_{ijk1})}+(2G_1(\Delta x_{ijk1})+G_2(\Delta x_{ijk2}))e^{-2\alpha G_1(\Delta x_{ijk1}))}}{\left(e^{-\alpha}+e^{-\alpha(G_1(\Delta x_{ijk1})+G_2(\Delta x_{ijk2}))}-e^{-\alpha G_1(\Delta x_{ijk1})}-e^{-\alpha G_2(\Delta x_{ijk2})}\right)^2}\right)\right]$$

$$\frac{\partial^2 L}{\partial v_1^2}=\sum_{k=1}^{n}\sum_{i=1}^{r}\sum_{j=\zeta_{i-1}+1}^{\zeta_{i-1}}\Bigg[\left(a_{i-1}\Delta y_{j-1}\right)^2 \text{PolyGamma}[1,a_{i-1}\Delta y_{j-1}v_1]-\alpha\frac{\partial^2 G_1(\Delta x_{ijk1})}{\partial v_1^2}\left(\frac{e^{-\alpha}-e^{-\alpha(G_1(\Delta x_{ijk1})+G_2(\Delta x_{ijk2}))}+e^{-\alpha G_1(\Delta x_{ijk1})}-e^{-\alpha G_2(\Delta x_{ijk2})}}{e^{-\alpha}+e^{-\alpha(G_1(\Delta x_{ijk1})+G_2(\Delta x_{ijk2}))}-e^{-\alpha G_1(\Delta x_{ijk1})}-e^{-\alpha G_2(\Delta x_{ijk2})}}\right)$$
$$-2\left(\alpha\frac{\partial G_1(\Delta x_{ijk1})}{\partial v_1}\right)^2\left(\frac{e^{-\alpha(G_1(\Delta x_{ijk1})+G_2(\Delta x_{ijk2}))}-e^{-\alpha G_1(\Delta x_{ijk1})}}{e^{-\alpha}+e^{-\alpha(G_1(\Delta x_{ijk1})+G_2(\Delta x_{ijk2}))}-e^{-\alpha G_1(\Delta x_{ijk1})}-e^{-\alpha G_2(\Delta x_{ijk2})}}\right)^2\Bigg]$$

$$\frac{\partial^2 L}{\partial v_1\partial v_2}=\sum_{k=1}^{n}\sum_{i=1}^{r}\sum_{j=\zeta_{i-1}+1}^{\zeta_{i-1}}\left[-2\alpha^2\frac{\partial G_1(\Delta x_{ijk1})}{\partial v_1}\frac{\partial G_2(\Delta x_{ijk2})}{\partial v_2}\frac{e^{-\alpha(G_1(\Delta x_{ijk1})+G_2(\Delta x_{ijk2}))}(e^{-\alpha}-1)}{\left(e^{-\alpha}+e^{-\alpha(G_1(\Delta x_{ijk1})+G_2(\Delta x_{ijk2}))}-e^{-\alpha G_1(\Delta x_{ijk1})}-e^{-\alpha G_2(\Delta x_{ijk2})}\right)^2}\right]$$

$$\frac{\partial^2 L}{\partial v_1\partial\alpha}=\sum_{k=1}^{n}\sum_{i=1}^{r}\sum_{j=\zeta_{i-1}+1}^{\zeta_{i-1}}\Bigg[-\frac{\partial G_1(\Delta x_{ijk1})}{\partial v_1}\left(\frac{e^{-\alpha}-e^{-\alpha(G_1(\Delta x_{ijk1})+G_2(\Delta x_{ijk2}))}+e^{-\alpha G_1(\Delta x_{ijk1})}-e^{-\alpha G_2(\Delta x_{ijk2})}}{e^{-\alpha}+e^{-\alpha(G_1(\Delta x_{ijk1})+G_2(\Delta x_{ijk2}))}-e^{-\alpha G_1(\Delta x_{ijk1})}-e^{-\alpha G_2(\Delta x_{ijk2})}}\right)$$
$$+\alpha e^{-\alpha G_1(\Delta x_{ijk1})}\frac{\partial G_1(\Delta x_{ijk1})}{\partial v_1}\left(\frac{(G_1(\Delta x_{ijk1})-e^{-\alpha})+(e^{-\alpha}-2(G_1(\Delta x_{ijk1})+G_2(\Delta x_{ijk2})))e^{-\alpha G_2(\Delta x_{ijk2})}+(G_1(\Delta x_{ijk1})+2G_2(\Delta x_{ijk2}))e^{-2\alpha G_2(\Delta x_{ijk2}))}}{\left(e^{-\alpha}+e^{-\alpha(G_1(\Delta x_{ijk1})+G_2(\Delta x_{ijk2}))}-e^{-\alpha G_1(\Delta x_{ijk1})}-e^{-\alpha G_2(\Delta x_{ijk2})}\right)^2}\right)\Bigg]$$

$$\frac{\partial^2 L}{\partial v_2^2}=\sum_{k=1}^{n}\sum_{i=1}^{r}\sum_{j=\zeta_{i-1}+1}^{\zeta_{i-1}}\Bigg[\left(a_{i-1}\Delta y_{j-1}\right)^2 \text{PolyGamma}[1,a_{i-1}\Delta y_{j-1}v_2]-\alpha\frac{\partial^2 G_2(\Delta x_{ijk2})}{\partial v_2^2}\left(\frac{e^{-\alpha}-e^{-\alpha(G_1(\Delta x_{ijk1})+G_2(\Delta x_{ijk2}))}-e^{-\alpha G_1(\Delta x_{ijk1})}+e^{-\alpha G_2(\Delta x_{ijk2})}}{e^{-\alpha}+e^{-\alpha(G_1(\Delta x_{ijk1})+G_2(\Delta x_{ijk2}))}-e^{-\alpha G_1(\Delta x_{ijk1})}-e^{-\alpha G_2(\Delta x_{ijk2})}}\right)$$
$$-2\left(\alpha\frac{\partial G_2(\Delta x_{ijk2})}{\partial v_2}\right)^2\left(\frac{e^{-\alpha(G_1(\Delta x_{ijk1})+G_2(\Delta x_{ijk2}))}-e^{-\alpha G_2(\Delta x_{ijk2})}}{e^{-\alpha}+e^{-\alpha(G_1(\Delta x_{ijk1})+G_2(\Delta x_{ijk2}))}-e^{-\alpha G_1(\Delta x_{ijk1})}-e^{-\alpha G_2(\Delta x_{ijk2})}}\right)^2\Bigg]$$

$$\frac{\partial^2 L}{\partial v_2\partial\alpha}=\sum_{k=1}^{n}\sum_{i=1}^{r}\sum_{j=\zeta_{i-1}+1}^{\zeta_{i-1}}\Bigg[-\frac{\partial G_2(\Delta x_{ijk2})}{\partial v_2}\left(\frac{e^{-\alpha}-e^{-\alpha(G_1(\Delta x_{ijk1})+G_2(\Delta x_{ijk2}))}-e^{-\alpha G_1(\Delta x_{ijk1})}+e^{-\alpha G_2(\Delta x_{ijk2})}}{e^{-\alpha}+e^{-\alpha(G_1(\Delta x_{ijk1})+G_2(\Delta x_{ijk2}))}-e^{-\alpha G_1(\Delta x_{ijk1})}-e^{-\alpha G_2(\Delta x_{ijk2})}}\right)$$
$$+\alpha e^{-\alpha G_2(\Delta x_{ijk2})}\frac{\partial G_2(\Delta x_{ijk2})}{\partial v_2}\left(\frac{(G_2(\Delta x_{ijk2})-e^{-\alpha})+(e^{-\alpha}-2(G_1(\Delta x_{ijk1})+G_2(\Delta x_{ijk2})))e^{-\alpha G_1(\Delta x_{ijk1})}+(2G_1(\Delta x_{ijk1})+G_2(\Delta x_{ijk2}))e^{-2\alpha G_1(\Delta x_{ijk1}))}}{\left(e^{-\alpha}+e^{-\alpha(G_1(\Delta x_{ijk1})+G_2(\Delta x_{ijk2}))}-e^{-\alpha G_1(\Delta x_{ijk1})}-e^{-\alpha G_2(\Delta x_{ijk2})}\right)^2}\right)\Bigg]$$

$$\frac{\partial^2 L}{\partial \alpha^2}=\sum_{k=1}^{n}\sum_{i=1}^{r}\sum_{j=\zeta_{i-1}+1}^{\zeta_{i-1}}\left[-\frac{1}{\alpha^2}-\frac{e^{-\alpha}}{\left(e^{-\alpha}-1\right)^2}\right.$$

$$-2\left(\frac{\left(e^{-\alpha}+e^{-\alpha(G_1(\Delta x_{ijk1})+G_2(\Delta x_{ijk2}))}-e^{-\alpha G_1(\Delta x_{ijk1})}-e^{-\alpha G_2(\Delta x_{ijk2})}\right)\left(e^{-\alpha}+(G_1(\Delta x_{ijk1})+G_2(\Delta x_{ijk2}))^2e^{-\alpha(G_1(\Delta x_{ijk1})+G_2(\Delta x_{ijk2}))}-G_1(\Delta x_{ijk1})^2e^{-\alpha G_1(\Delta x_{ijk1})}-G_2(\Delta x_{ijk2})^2e^{-\alpha G_2(\Delta x_{ijk2})}\right)}{\left(e^{-\alpha}+e^{-\alpha(G_1(\Delta x_{ijk1})+G_2(\Delta x_{ijk2}))}-e^{-\alpha G_1(\Delta x_{ijk1})}-e^{-\alpha G_2(\Delta x_{ijk2})}\right)^2}\right.$$

$$\left.\left.-\frac{\left(e^{-\alpha}+(G_1(\Delta x_{ijk1})+G_2(\Delta x_{ijk2}))e^{-\alpha(G_1(\Delta x_{ijk1})+G_2(\Delta x_{ijk2}))}-G_1(\Delta x_{ijk1})e^{-\alpha G_1(\Delta x_{ijk1})}-G_2(\Delta x_{ijk2})e^{-\alpha G_2(\Delta x_{ijk2})}\right)^2}{\left(e^{-\alpha}+e^{-\alpha(G_1(\Delta x_{ijk1})+G_2(\Delta x_{ijk2}))}-e^{-\alpha G_1(\Delta x_{ijk1})}-e^{-\alpha G_2(\Delta x_{ijk2})}\right)^2}\right)\right]$$

The elements of the Fisher information matrix for an observation are the negative s-expectations of these second partial derivatives:

$$E\left[-\frac{\partial^2 L}{\partial u_1^2}\right]=\sum_{k=1}^{n}\sum_{i=1}^{r}\sum_{j=\zeta_{i-1}+1}^{\zeta_{i-1}}\int_0^\infty\int_0^\infty\left[\left(\left(\frac{2\Delta x_{ijk1}}{u_1^3}-a_{i-1}\Delta y_{j-1}\frac{v_1}{u_1^2}+\alpha\frac{\partial^2 G_1(\Delta x_{ijk1})}{\partial u_1^2}\right)\left(\frac{e^{-\alpha}-e^{-\alpha(G_1(\Delta x_{ijk1})+G_2(\Delta x_{ijk2}))}+e^{-\alpha G_1(\Delta x_{ijk1})}-e^{-\alpha G_2(\Delta x_{ijk2})}}{e^{-\alpha}+e^{-\alpha(G_1(\Delta x_{ijk1})+G_2(\Delta x_{ijk2}))}-e^{-\alpha G_1(\Delta x_{ijk1})}-e^{-\alpha G_2(\Delta x_{ijk2})}}\right)\right.\right.$$

$$\left.-2\left(\alpha\frac{\partial G_1(\Delta x_{ijk1})}{\partial u_1}\right)^2\left(\frac{e^{-\alpha(G_1(\Delta x_{ijk1})+G_2(\Delta x_{ijk2}))}-e^{-\alpha G_1(\Delta x_{ijk1})}}{e^{-\alpha}+e^{-\alpha(G_1(\Delta x_{ijk1})+G_2(\Delta x_{ijk2}))}-e^{-\alpha G_1(\Delta x_{ijk1})}-e^{-\alpha G_2(\Delta x_{ijk2})}}\right)^2\right)\times$$

$$\left.\prod_{p=1}^{2}\frac{\left(u_p\right)^{-v_p a_{i-1}\Delta y_{j-1}}}{\Gamma(v_p\, a_{i-1}\Delta y_{j-1})}\left(\Delta x_{ijkp}\right)^{v_p a_{i-1}\Delta y_{j-1}-1}\exp\left[-\frac{\Delta x_{ijkp}}{u_p}\right]\right]d\Delta x_{ijk1}d\Delta x_{ijk2}$$

$$E\left[-\frac{\partial^2 L}{\partial u_1\partial v_1}\right]=\sum_{k=1}^{n}\sum_{i=1}^{r}\sum_{j=\zeta_{i-1}+1}^{\zeta_{i-1}}\int_0^\infty\int_0^\infty\left[\left(\left(\frac{a_{i-1}\Delta y_{j-1}}{u_1}+\alpha\frac{\partial^2 G_1(\Delta x_{ijk1})}{\partial u_1\partial v_1}\right)\left(\frac{e^{-\alpha}-e^{-\alpha(G_1(\Delta x_{ijk1})+G_2(\Delta x_{ijk2}))}+e^{-\alpha G_1(\Delta x_{ijk1})}-e^{-\alpha G_2(\Delta x_{ijk2})}}{e^{-\alpha}+e^{-\alpha(G_1(\Delta x_{ijk1})+G_2(\Delta x_{ijk2}))}-e^{-\alpha G_1(\Delta x_{ijk1})}-e^{-\alpha G_2(\Delta x_{ijk2})}}\right)\right.\right.$$

$$\left.-\alpha^2\frac{\partial G_1(\Delta x_{ijk1})}{\partial u_1}\frac{\partial G_1(\Delta x_{ijk1})}{\partial v_1}\left(\frac{e^{-\alpha(G_1(\Delta x_{ijk1})+G_2(\Delta x_{ijk2}))}-e^{-\alpha G_1(\Delta x_{ijk1})}}{e^{-\alpha}+e^{-\alpha(G_1(\Delta x_{ijk1})+G_2(\Delta x_{ijk2}))}-e^{-\alpha G_1(\Delta x_{ijk1})}-e^{-\alpha G_2(\Delta x_{ijk2})}}\right)^2\right)\times$$

$$\left.\prod_{p=1}^{2}\frac{\left(u_p\right)^{-v_p a_{i-1}\Delta y_{j-1}}}{\Gamma(v_p\, a_{i-1}\Delta y_{j-1})}\left(\Delta x_{ijkp}\right)^{v_p a_{i-1}\Delta y_{j-1}-1}\exp\left[-\frac{\Delta x_{ijkp}}{u_p}\right]\right]d\Delta x_{ijk1}d\Delta x_{ijk2}$$

$$E\left[-\frac{\partial^2 L}{\partial u_1 \partial u_2}\right]=\sum_{k=1}^{n}\sum_{i=1}^{r}\sum_{j=\zeta_{i-1}+1}^{\zeta_{i-1}}\int_0^\infty\int_0^\infty\left[\left(2\alpha^2\frac{\partial G_1(\Delta x_{ijk1})}{\partial u_1}\frac{\partial G_2(\Delta x_{ijk2})}{\partial u_2}\frac{e^{-\alpha(G_1(\Delta x_{ijk1})+G_2(\Delta x_{ijk2}))}(e^{-\alpha}-1)}{\left(e^{-\alpha}+e^{-\alpha(G_1(\Delta x_{ijk1})+G_2(\Delta x_{ijk2}))}-e^{-\alpha G_1(\Delta x_{ijk1})}-e^{-\alpha G_2(\Delta x_{ijk2})}\right)^2}\right)\times\right.$$

$$\left.\prod_{p=1}^{2}\frac{(u_p)^{-v_p a_{i-1}\Delta y_{j-1}}}{\Gamma(v_p\, a_{i-1}\Delta y_{j-1})}\left(\Delta x_{ijkp}\right)^{v_p a_{i-1}\Delta y_{j-1}-1}\exp\left[-\frac{\Delta x_{ijkp}}{u_p}\right]\right]d\Delta x_{ijk1}d\Delta x_{ijk2}$$

$$E\left[-\frac{\partial^2 L}{\partial u_1 \partial v_2}\right]=\sum_{k=1}^{n}\sum_{i=1}^{r}\sum_{j=\zeta_{i-1}+1}^{\zeta_{i-1}}\int_0^\infty\int_0^\infty\left[\left(2\alpha^2\frac{\partial G_1(\Delta x_{ijk1})}{\partial u_1}\frac{\partial G_2(\Delta x_{ijk2})}{\partial v_2}\frac{e^{-\alpha(G_1(\Delta x_{ijk1})+G_2(\Delta x_{ijk2}))}(e^{-\alpha}-1)}{\left(e^{-\alpha}+e^{-\alpha(G_1(\Delta x_{ijk1})+G_2(\Delta x_{ijk2}))}-e^{-\alpha G_1(\Delta x_{ijk1})}-e^{-\alpha G_2(\Delta x_{ijk2})}\right)^2}\right)\times\right.$$

$$\left.\prod_{p=1}^{2}\frac{(u_p)^{-v_p a_{i-1}\Delta y_{j-1}}}{\Gamma(v_p\, a_{i-1}\Delta y_{j-1})}\left(\Delta x_{ijkp}\right)^{v_p a_{i-1}\Delta y_{j-1}-1}\exp\left[-\frac{\Delta x_{ijkp}}{u_p}\right]\right]d\Delta x_{ijk1}d\Delta x_{ijk2}$$

$$E\left[-\frac{\partial^2 L}{\partial u_1 \partial \alpha}\right]=\sum_{k=1}^{n}\sum_{i=1}^{r}\sum_{j=\zeta_{i-1}+1}^{\zeta_{i-1}}\int_0^\infty\int_0^\infty\left[\left(\frac{\partial G_1(\Delta x_{ijk1})}{\partial u_1}\left(\frac{e^{-\alpha}-e^{-\alpha(G_1(\Delta x_{ijk1})+G_2(\Delta x_{ijk2}))}+e^{-\alpha G_1(\Delta x_{ijk1})}-e^{-\alpha G_2(\Delta x_{ijk2})}}{e^{-\alpha}+e^{-\alpha(G_1(\Delta x_{ijk1})+G_2(\Delta x_{ijk2}))}-e^{-\alpha G_1(\Delta x_{ijk1})}-e^{-\alpha G_2(\Delta x_{ijk2})}}\right)\right.\right.$$

$$\left.\left.-\alpha e^{-\alpha G_1(\Delta x_{ijk1})}\frac{\partial G_1(\Delta x_{ijk1})}{\partial u_1}\left(\frac{(G_1(\Delta x_{ijk1})-e^{-\alpha})+(e^{-\alpha}-2(G_1(\Delta x_{ijk1})+G_2(\Delta x_{ijk2})))e^{-\alpha G_2(\Delta x_{ijk2})}+(G_1(\Delta x_{ijk1})+2G_2(\Delta x_{ijk2}))e^{-2\alpha G_2(\Delta x_{ijk2})}}{\left(e^{-\alpha}+e^{-\alpha(G_1(\Delta x_{ijk1})+G_2(\Delta x_{ijk2}))}-e^{-\alpha G_1(\Delta x_{ijk1})}-e^{-\alpha G_2(\Delta x_{ijk2})}\right)^2}\right)\right)\times\right.$$

$$\left.\prod_{p=1}^{2}\frac{(u_p)^{-v_p a_{i-1}\Delta y_{j-1}}}{\Gamma(v_p\, a_{i-1}\Delta y_{j-1})}\left(\Delta x_{ijkp}\right)^{v_p a_{i-1}\Delta y_{j-1}-1}\exp\left[-\frac{\Delta x_{ijkp}}{u_p}\right]\right]d\Delta x_{ijk1}d\Delta x_{ijk2}$$

$$E\left[-\frac{\partial^2 L}{\partial u_2^2}\right]=\sum_{k=1}^{n}\sum_{i=1}^{r}\sum_{j=\zeta_{i-1}+1}^{\zeta_{i-1}}\int_0^\infty\int_0^\infty\left[\left(\frac{2\Delta x_{ijk2}}{u_2^3}-a_{i-1}\Delta y_{j-1}\frac{v_2}{u_2^2}+\alpha\frac{\partial^2 G_2(\Delta x_{ijk2})}{\partial u_2^2}\left(\frac{e^{-\alpha}-e^{-\alpha(G_1(\Delta x_{ijk1})+G_2(\Delta x_{ijk2}))}-e^{-\alpha G_1(\Delta x_{ijk1})}+e^{-\alpha G_2(\Delta x_{ijk2})}}{e^{-\alpha}+e^{-\alpha(G_1(\Delta x_{ijk1})+G_2(\Delta x_{ijk2}))}-e^{-\alpha G_1(\Delta x_{ijk1})}-e^{-\alpha G_2(\Delta x_{ijk2})}}\right)\right.\right.$$

$$\left.+2\left(\alpha\frac{\partial G_2(\Delta x_{ijk2})}{\partial u_2}\right)^2\left(\frac{e^{-\alpha(G_1(\Delta x_{ijk1})+G_2(\Delta x_{ijk2}))}-e^{-\alpha G_2(\Delta x_{ijk2})}}{e^{-\alpha}+e^{-\alpha(G_1(\Delta x_{ijk1})+G_2(\Delta x_{ijk2}))}-e^{-\alpha G_1(\Delta x_{ijk1})}-e^{-\alpha G_2(\Delta x_{ijk2})}}\right)^2\right)\times$$

$$\left.\prod_{p=1}^{2}\frac{(u_p)^{-v_p a_{i-1}\Delta y_{j-1}}}{\Gamma(v_p a_{i-1}\Delta y_{j-1})}(\Delta x_{ijkp})^{v_p a_{i-1}\Delta y_{j-1}-1}\exp\left[-\frac{\Delta x_{ijkp}}{u_p}\right]\right]d\Delta x_{ijk1}d\Delta x_{ijk2}$$

$$E\left[-\frac{\partial^2 L}{\partial u_2\partial v_1}\right]=\sum_{k=1}^{n}\sum_{i=1}^{r}\sum_{j=\zeta_{i-1}+1}^{\zeta_{i-1}}\int_0^\infty\int_0^\infty\left[\left(2\alpha^2\frac{\partial G_2(\Delta x_{ijk2})}{\partial u_2}\frac{\partial G_1(\Delta x_{ijk1})}{\partial v_1}\frac{e^{-\alpha(G_1(\Delta x_{ijk1})+G_2(\Delta x_{ijk2}))}(e^{-\alpha}-1)}{\left(e^{-\alpha}+e^{-\alpha(G_1(\Delta x_{ijk1})+G_2(\Delta x_{ijk2}))}-e^{-\alpha G_1(\Delta x_{ijk1})}-e^{-\alpha G_2(\Delta x_{ijk2})}\right)^2}\right)\times\right.$$

$$\left.\prod_{p=1}^{2}\frac{(u_p)^{-v_p a_{i-1}\Delta y_{j-1}}}{\Gamma(v_p a_{i-1}\Delta y_{j-1})}(\Delta x_{ijkp})^{v_p a_{i-1}\Delta y_{j-1}-1}\exp\left[-\frac{\Delta x_{ijkp}}{u_p}\right]\right]d\Delta x_{ijk1}d\Delta x_{ijk2}$$

$$E\left[-\frac{\partial^2 L}{\partial u_2\partial v_2}\right]=\sum_{k=1}^{n}\sum_{i=1}^{r}\sum_{j=\zeta_{i-1}+1}^{\zeta_{i-1}}\int_0^\infty\int_0^\infty\left[\left(\frac{a_{i-1}\Delta y_{j-1}}{u_2}+\alpha\frac{\partial^2 G_2(\Delta x_{ijk2})}{\partial u_2\partial v_2}\left(\frac{e^{-\alpha}-e^{-\alpha(G_1(\Delta x_{ijk1})+G_2(\Delta x_{ijk2}))}-e^{-\alpha G_1(\Delta x_{ijk1})}+e^{-\alpha G_2(\Delta x_{ijk2})}}{e^{-\alpha}+e^{-\alpha(G_1(\Delta x_{ijk1})+G_2(\Delta x_{ijk2}))}-e^{-\alpha G_1(\Delta x_{ijk1})}-e^{-\alpha G_2(\Delta x_{ijk2})}}\right)\right.\right.$$

$$\left.-\alpha^2\frac{\partial G_2(\Delta x_{ijk2})}{\partial u_2}\frac{\partial G_2(\Delta x_{ijk2})}{\partial v_2}\left(\frac{e^{-\alpha(G_1(\Delta x_{ijk1})+G_2(\Delta x_{ijk2}))}-e^{-\alpha G_2(\Delta x_{ijk2})}}{e^{-\alpha}+e^{-\alpha(G_1(\Delta x_{ijk1})+G_2(\Delta x_{ijk2}))}-e^{-\alpha G_1(\Delta x_{ijk1})}-e^{-\alpha G_2(\Delta x_{ijk2})}}\right)^2\right)\times$$

$$\left.\prod_{p=1}^{2}\frac{(u_p)^{-v_p a_{i-1}\Delta y_{j-1}}}{\Gamma(v_p a_{i-1}\Delta y_{j-1})}(\Delta x_{ijkp})^{v_p a_{i-1}\Delta y_{j-1}-1}\exp\left[-\frac{\Delta x_{ijkp}}{u_p}\right]\right]d\Delta x_{ijk1}d\Delta x_{ijk2}$$

$$E\left[-\frac{\partial^2 L}{\partial u_2 \partial \alpha}\right]=\sum_{k=1}^{n}\sum_{i=1}^{r}\sum_{j=\zeta_{i-1}+1}^{\zeta_{i-1}}\left[\left(\frac{\partial G_2(\Delta x_{ijk2})}{\partial u_2}\left(\frac{e^{-\alpha}-e^{-\alpha(G_1(\Delta x_{ijk1})+G_2(\Delta x_{ijk2}))}-e^{-\alpha G_1(\Delta x_{ijk1})}+e^{-\alpha G_2(\Delta x_{ijk2})}}{e^{-\alpha}+e^{-\alpha(G_1(\Delta x_{ijk1})+G_2(\Delta x_{ijk2}))}-e^{-\alpha G_1(\Delta x_{ijk1})}-e^{-\alpha G_2(\Delta x_{ijk2})}}\right)\right.\right.$$

$$\left.\left.-\alpha e^{-\alpha G_2(\Delta x_{ijk2})}\frac{\partial G_2(\Delta x_{ijk2})}{\partial u_2}\left(\frac{(G_2(\Delta x_{ijk2})-e^{-\alpha})+(e^{-\alpha}-2(G_1(\Delta x_{ijk1})+G_2(\Delta x_{ijk2})))e^{-\alpha G_1(\Delta x_{ijk1})}+(2G_1(\Delta x_{ijk1})+G_2(\Delta x_{ijk2}))e^{-2\alpha G_1(\Delta x_{ijk1}))}}{\left(e^{-\alpha}+e^{-\alpha(G_1(\Delta x_{ijk1})+G_2(\Delta x_{ijk2}))}-e^{-\alpha G_1(\Delta x_{ijk1})}-e^{-\alpha G_2(\Delta x_{ijk2})}\right)^2}\right)\right)\times$$

$$\left.\prod_{p=1}^{2}\frac{\left(u_p\right)^{-v_p a_{i-1}\Delta y_{j-1}}}{\Gamma(v_p\, a_{i-1}\Delta y_{j-1})}\left(\Delta x_{ijkp}\right)^{v_p a_{i-1}\Delta y_{j-1}-1}\exp\left[-\frac{\Delta x_{ijkp}}{u_p}\right]\right]d\Delta x_{ijk1}d\Delta x_{ijk2}$$

$$E\left[-\frac{\partial^2 L}{\partial v_1^2}\right]=\sum_{k=1}^{n}\sum_{i=1}^{r}\sum_{j=\zeta_{i-1}+1}^{\zeta_{i-1}}\int_0^{\infty}\int_0^{\infty}\left[\left(-\left(a_{i-1}\Delta y_{j-1}\right)^2\text{PolyGamma}[1,a_{i-1}\Delta y_{j-1}v_1]+\alpha\frac{\partial^2 G_1(\Delta x_{ijk1})}{\partial v_1^2}\left(\frac{e^{-\alpha}-e^{-\alpha(G_1(\Delta x_{ijk1})+G_2(\Delta x_{ijk2}))}+e^{-\alpha G_1(\Delta x_{ijk1})}-e^{-\alpha G_2(\Delta x_{ijk2})}}{e^{-\alpha}+e^{-\alpha(G_1(\Delta x_{ijk1})+G_2(\Delta x_{ijk2}))}-e^{-\alpha G_1(\Delta x_{ijk1})}-e^{-\alpha G_2(\Delta x_{ijk2})}}\right)\right.\right.$$

$$\left.\left.+2\left(\alpha\frac{\partial G_1(\Delta x_{ijk1})}{\partial v_1}\right)^2\left(\frac{e^{-\alpha(G_1(\Delta x_{ijk1})+G_2(\Delta x_{ijk2}))}-e^{-\alpha G_1(\Delta x_{ijk1})}}{e^{-\alpha}+e^{-\alpha(G_1(\Delta x_{ijk1})+G_2(\Delta x_{ijk2}))}-e^{-\alpha G_1(\Delta x_{ijk1})}-e^{-\alpha G_2(\Delta x_{ijk2})}}\right)^2\right)\times$$

$$\left.\prod_{p=1}^{2}\frac{\left(u_p\right)^{-v_p a_{i-1}\Delta y_{j-1}}}{\Gamma(v_p\, a_{i-1}\Delta y_{j-1})}\left(\Delta x_{ijkp}\right)^{v_p a_{i-1}\Delta y_{j-1}-1}\exp\left[-\frac{\Delta x_{ijkp}}{u_p}\right]\right]d\Delta x_{ijk1}d\Delta x_{ijk2}$$

$$E\left[-\frac{\partial^2 L}{\partial v_1 \partial v_2}\right]=\sum_{k=1}^{n}\sum_{i=1}^{r}\sum_{j=\zeta_{i-1}+1}^{\zeta_{i-1}}\left[\left(2\alpha^2\frac{\partial G_1(\Delta x_{ijk1})}{\partial v_1}\frac{\partial G_2(\Delta x_{ijk2})}{\partial v_2}\frac{e^{-\alpha(G_1(\Delta x_{ijk1})+G_2(\Delta x_{ijk2}))}(e^{-\alpha}-1)}{\left(e^{-\alpha}+e^{-\alpha(G_1(\Delta x_{ijk1})+G_2(\Delta x_{ijk2}))}-e^{-\alpha G_1(\Delta x_{ijk1})}-e^{-\alpha G_2(\Delta x_{ijk2})}\right)^2}\right)\times\right.$$

$$\left.\prod_{p=1}^{2}\frac{\left(u_p\right)^{-v_p a_{i-1}\Delta y_{j-1}}}{\Gamma(v_p\, a_{i-1}\Delta y_{j-1})}\left(\Delta x_{ijkp}\right)^{v_p a_{i-1}\Delta y_{j-1}-1}\exp\left[-\frac{\Delta x_{ijkp}}{u_p}\right]\right]d\Delta x_{ijk1}d\Delta x_{ijk2}$$

$$E\left[-\frac{\partial^2 L}{\partial v_1 \partial \alpha}\right]=\sum_{k=1}^{n}\sum_{i=1}^{r}\sum_{j=\zeta_{i-1}+1}^{\zeta_{i-1}}\int_0^\infty\int_0^\infty\left[\left(\frac{\partial G_1(\Delta x_{ijk1})}{\partial v_1}\left(\frac{e^{-\alpha}-e^{-\alpha(G_1(\Delta x_{ijk1})+G_2(\Delta x_{ijk2}))}+e^{-\alpha G_1(\Delta x_{ijk1})}-e^{-\alpha G_2(\Delta x_{ijk2})}}{e^{-\alpha}+e^{-\alpha(G_1(\Delta x_{ijk1})+G_2(\Delta x_{ijk2}))}-e^{-\alpha G_1(\Delta x_{ijk1})}-e^{-\alpha G_2(\Delta x_{ijk2})}}\right)\right.\right.$$

$$\left.\left.-\alpha e^{-\alpha G_1(\Delta x_{ijk1})}\frac{\partial G_1(\Delta x_{ijk1})}{\partial v_1}\left(\frac{(G_1(\Delta x_{ijk1})-e^{-\alpha})+(e^{-\alpha}-2(G_1(\Delta x_{ijk1})+G_2(\Delta x_{ijk2})))e^{-\alpha G_2(\Delta x_{ijk2})}+(G_1(\Delta x_{ijk1})+2G_2(\Delta x_{ijk2}))e^{-2\alpha G_2(\Delta x_{ijk2}))}}{\left(e^{-\alpha}+e^{-\alpha(G_1(\Delta x_{ijk1})+G_2(\Delta x_{ijk2}))}-e^{-\alpha G_1(\Delta x_{ijk1})}-e^{-\alpha G_2(\Delta x_{ijk2})}\right)^2}\right)\right)\times$$

$$\left.\prod_{p=1}^{2}\frac{(u_p)^{-v_p a_{i-1}\Delta y_{j-1}}}{\Gamma(v_p\, a_{i-1}\Delta y_{j-1})}(\Delta x_{ijkp})^{v_p a_{i-1}\Delta y_{j-1}-1}\exp\left[-\frac{\Delta x_{ijkp}}{u_p}\right]\right]d\Delta x_{ijk1}d\Delta x_{ijk2}$$

$$E\left[-\frac{\partial^2 L}{\partial v_2^2}\right]=\sum_{k=1}^{n}\sum_{i=1}^{r}\sum_{j=\zeta_{i-1}+1}^{\zeta_{i-1}}\int_0^\infty\int_0^\infty\left[\left(-\left(a_{i-1}\Delta y_{j-1}\right)^2 \text{PolyGamma}[1,a_{i-1}\Delta y_{j-1}v_2]+\alpha\frac{\partial^2 G_2(\Delta x_{ijk2})}{\partial v_2^2}\left(\frac{e^{-\alpha}-e^{-\alpha(G_1(\Delta x_{ijk1})+G_2(\Delta x_{ijk2}))}-e^{-\alpha G_1(\Delta x_{ijk1})}+e^{-\alpha G_2(\Delta x_{ijk2})}}{e^{-\alpha}+e^{-\alpha(G_1(\Delta x_{ijk1})+G_2(\Delta x_{ijk2}))}-e^{-\alpha G_1(\Delta x_{ijk1})}-e^{-\alpha G_2(\Delta x_{ijk2})}}\right)\right.\right.$$

$$\left.+2\left(\alpha\frac{\partial G_2(\Delta x_{ijk2})}{\partial v_2}\right)^2\left(\frac{e^{-\alpha(G_1(\Delta x_{ijk1})+G_2(\Delta x_{ijk2}))}-e^{-\alpha G_2(\Delta x_{ijk2})}}{e^{-\alpha}+e^{-\alpha(G_1(\Delta x_{ijk1})+G_2(\Delta x_{ijk2}))}-e^{-\alpha G_1(\Delta x_{ijk1})}-e^{-\alpha G_2(\Delta x_{ijk2})}}\right)^2\right)\times$$

$$\left.\prod_{p=1}^{2}\frac{(u_p)^{-v_p a_{i-1}\Delta y_{j-1}}}{\Gamma(v_p\, a_{i-1}\Delta y_{j-1})}(\Delta x_{ijkp})^{v_p a_{i-1}\Delta y_{j-1}-1}\exp\left[-\frac{\Delta x_{ijkp}}{u_p}\right]\right]d\Delta x_{ijk1}d\Delta x_{ijk2}$$

$$E\left[-\frac{\partial^2 L}{\partial v_2 \partial \alpha}\right]=\sum_{k=1}^{n}\sum_{i=1}^{r}\sum_{j=\zeta_{i-1}+1}^{\zeta_{i-1}}\int_0^\infty\int_0^\infty\left[\left(\frac{\partial G_2(\Delta x_{ijk2})}{\partial v_2}\left(\frac{e^{-\alpha}-e^{-\alpha(G_1(\Delta x_{ijk1})+G_2(\Delta x_{ijk2}))}-e^{-\alpha G_1(\Delta x_{ijk1})}+e^{-\alpha G_2(\Delta x_{ijk2})}}{e^{-\alpha}+e^{-\alpha(G_1(\Delta x_{ijk1})+G_2(\Delta x_{ijk2}))}-e^{-\alpha G_1(\Delta x_{ijk1})}-e^{-\alpha G_2(\Delta x_{ijk2})}}\right)\right.\right.$$

$$\left.\left.-\alpha e^{-\alpha G_2(\Delta x_{ijk2})}\frac{\partial G_2(\Delta x_{ijk2})}{\partial v_2}\left(\frac{(G_2(\Delta x_{ijk2})-e^{-\alpha})+(e^{-\alpha}-2(G_1(\Delta x_{ijk1})+G_2(\Delta x_{ijk2})))e^{-\alpha G_1(\Delta x_{ijk1})}+(2G_1(\Delta x_{ijk1})+G_2(\Delta x_{ijk2}))e^{-2\alpha G_1(\Delta x_{ijk1}))}}{\left(e^{-\alpha}+e^{-\alpha(G_1(\Delta x_{ijk1})+G_2(\Delta x_{ijk2}))}-e^{-\alpha G_1(\Delta x_{ijk1})}-e^{-\alpha G_2(\Delta x_{ijk2})}\right)^2}\right)\right)\times$$

$$\left.\prod_{p=1}^{2}\frac{(u_p)^{-v_p a_{i-1}\Delta y_{j-1}}}{\Gamma(v_p\, a_{i-1}\Delta y_{j-1})}(\Delta x_{ijkp})^{v_p a_{i-1}\Delta y_{j-1}-1}\exp\left[-\frac{\Delta x_{ijkp}}{u_p}\right]\right]d\Delta x_{ijk1}d\Delta x_{ijk2}$$

$$E\left[-\frac{\partial^2 L}{\partial \alpha^2}\right]=\sum_{k=1}^{n}\sum_{i=1}^{r}\sum_{j=\zeta_{i-1}+1}^{\zeta_{i-1}}\int_0^\infty\int_0^\infty\left[\left(\frac{1}{\alpha^2}+\frac{e^{-\alpha}}{\left(e^{-\alpha}-1\right)^2}\right.\right.$$

$$+2\left(\frac{\left(e^{-\alpha}+e^{-\alpha(G_1(\Delta x_{ijk1})+G_2(\Delta x_{ijk2}))}-e^{-\alpha G_1(\Delta x_{ijk1})}-e^{-\alpha G_2(\Delta x_{ijk2})}\right)\left(e^{-\alpha}+(G_1(\Delta x_{ijk1})+G_2(\Delta x_{ijk2}))^2 e^{-\alpha(G_1(\Delta x_{ijk1})+G_2(\Delta x_{ijk2}))}-G_1(\Delta x_{ijk1})^2 e^{-\alpha G_1(\Delta x_{ijk1})}-G_2(\Delta x_{ijk2})^2 e^{-\alpha G_2(\Delta x_{ijk2})}\right)}{\left(e^{-\alpha}+e^{-\alpha(G_1(\Delta x_{ijk1})+G_2(\Delta x_{ijk2}))}-e^{-\alpha G_1(\Delta x_{ijk1})}-e^{-\alpha G_2(\Delta x_{ijk2})}\right)^2}\right.$$

$$\left.\left.-\left(\frac{\left(e^{-\alpha}+(G_1(\Delta x_{ijk1})+G_2(\Delta x_{ijk2}))e^{-\alpha(G_1(\Delta x_{ijk1})+G_2(\Delta x_{ijk2}))}-G_1(\Delta x_{ijk1})e^{-\alpha G_1(\Delta x_{ijk1})}-G_2(\Delta x_{ijk2})e^{-\alpha G_2(\Delta x_{ijk2})}\right)^2}{\left(e^{-\alpha}+e^{-\alpha(G_1(\Delta x_{ijk1})+G_2(\Delta x_{ijk2}))}-e^{-\alpha G_1(\Delta x_{ijk1})}-e^{-\alpha G_2(\Delta x_{ijk2})}\right)^2}\right)\right)\right)\times$$

$$\left.\prod_{p=1}^{2}\frac{\left(u_p\right)^{-v_p a_{i-1}\Delta y_{j-1}}}{\Gamma(v_p\, a_{i-1}\Delta y_{j-1})}\left(\Delta x_{ijkp}\right)^{v_p a_{i-1}\Delta y_{j-1}-1}\exp\left[-\frac{\Delta x_{ijkp}}{u_p}\right]\right]d\Delta x_{ijk1}d\Delta x_{ijk2}$$

where

$$\frac{\partial G_p(\Delta x_{ijkp})}{\partial u_p}=\frac{\Delta x_{ijkp}}{\left(u_p\right)^2\Gamma(v_p\, a_{i-1}\Delta y_{j-1})}\left[\frac{\Delta x_{ijkp}}{u_p}\right]^{v_p a_{i-1}\Delta y_{j-1}-1}\exp\left[-\frac{\Delta x_{ijkp}}{u_p}\right]I_{(0,\infty)}(\Delta x_{ijkp}),\ p=1,2,$$

$$\frac{\partial G_p(\Delta x_{ijkp})}{\partial v_p}=a_{i-1}\Delta y_{j-1}\left[G_p(\Delta x_{ijkp})\ln\left(\frac{\Delta x_{ijkp}}{u_p}\right)-\frac{\text{MeijerG}[\{\{\},\{1,1\}\},\{\{0,0,v_p\, a_{i-1}\Delta y_{j-1}\},\{\}\}}{\Gamma(v_p\, a_{i-1}\Delta y_{j-1})}\right.$$

$$\left.+G_p(\Delta x_{ijkp})\text{PolyGamma}[0,v_p\, a_{i-1}\Delta y_{j-1}]\,I_{(0,\infty)}(\Delta x_{ijkp})\right],\ p=1,2,$$

$$\frac{\partial^2 G_p(\Delta x_{ijkp})}{\partial u_p^2}=\frac{\left(1+v_p\, a_{i-1}\Delta y_{j-1}-\Delta x_{ijkp}/u_p\right)}{\left(u_p\right)^2\Gamma(v_p\, a_{i-1}\Delta y_{j-1})}\left[\frac{\Delta x_{ijkp}}{u_p}\right]^{v_p a_{i-1}\Delta y_{j-1}}\exp\left[-\frac{\Delta x_{ijkp}}{u_p}\right]I_{(0,\infty)}(\Delta x_{ijkp}),\ p=1,2,$$

$$\frac{\partial^2 G_p(\Delta x_{ijkp})}{\partial v_p^2} = \left(a_{i-1}\Delta y_{j-1}\right)^2 \text{PolyGamma}[0, v_p\, a_{i-1}\Delta y_{j-1}]\left[G_p(\Delta x_{ijkp})\left(\ln\left(\frac{\Delta x_{ijkp}}{u_p}\right) - \text{PolyGamma}[0, v_p\, a_{i-1}\Delta y_{j-1}]\right)\right.$$

$$\left. - \frac{\text{MeijerG}[\{\{\},\{1,1\}\},\{\{0,0,v_p\, a_{i-1}\Delta y_{j-1}\},\{\}\}]}{\Gamma(v_p\, a_{i-1}\Delta y_{j-1})}\right] - \frac{\left(a_{i-1}\Delta y_{j-1}\right)^2}{\Gamma(v_p\, a_{i-1}\Delta y_{j-1})}\left[\ln\left(\frac{\Delta x_{ijkp}}{u_p}\right)\text{MeijerG}[\{\{\},\{1,1\}\},\{\{0,0,v_p\, a_{i-1}\Delta y_{j-1}\},\{\}\}]\right.$$

$$+ 2\text{MeijerG}[\{\{\},\{1,1,1\}\},\{\{0,0,0,v_p\, a_{i-1}\Delta y_{j-1}\},\{\}\}] + \Gamma(v_p\, a_{i-1}\Delta y_{j-1}) G_p(\Delta x_{ijkp}) \text{PolyGamma}[0, v_p\, a_{i-1}\Delta y_{j-1}] \times$$

$$\left(\ln\left(\frac{\Delta x_{ijkp}}{u_p}\right) - \text{PolyGamma}[0, v_p\, a_{i-1}\Delta y_{j-1}]\right) + \Gamma(v_p\, a_{i-1}\Delta y_{j-1})\left(\ln\left(\frac{\Delta x_{ijkp}}{u_p}\right) - \text{PolyGamma}[0, v_p\, a_{i-1}\Delta y_{j-1}]\right) \times$$

$$\left(G_p(\Delta x_{ijkp})\left(\ln\left(\frac{\Delta x_{ijkp}}{u_p}\right) - \text{PolyGamma}[0, v_p\, a_{i-1}\Delta y_{j-1}]\right) + \frac{\text{MeijerG}[\{\{\},\{1,1\}\},\{\{0,0,v_p\, a_{i-1}\Delta y_{j-1}\},\{\}\}]}{\Gamma(v_p\, a_{i-1}\Delta y_{j-1})}\right)$$

$$\left. - \Gamma(v_p\, a_{i-1}\Delta y_{j-1}) G_p(\Delta x_{ijkp}) \text{PolyGamma}[0, v_p\, a_{i-1}\Delta y_{j-1}]\right],\ p = 1, 2,$$

$$\frac{\partial^2 G_p(\Delta x_{ijkp})}{\partial u_p \partial v_p} = \frac{a_{i-1}\Delta y_{j-1}}{u_p \Gamma(v_p\, a_{i-1}\Delta y_{j-1})}\left[\text{PolyGamma}[0, v_p\, a_{i-1}\Delta y_{j-1}] - \ln\left(\frac{\Delta x_{ijkp}}{u_p}\right)\right]\left[\frac{\Delta x_{ijkp}}{u_p}\right]^{v_p a_{i-1}\Delta y_{j-1}} \times$$

$$\exp\left[-\frac{\Delta x_{ijkp}}{u_p}\right] I_{(0,\infty)}(\Delta x_{ijkp}),\ p = 1, 2.$$