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Exponentiated Quasi Lindley distribution

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Abstract: The Exponentiated Quasi Lindley (EQL) distribution which is an extension of the quasi Lindley Distribution is introduced and its properties are explored. This new distribution represents a more flexible model for the lifetime data. Some statistical properties of the proposed distribution including the shapes of the density and hazard rate functions, the moments and moment generating function, the distribution of the order statistics are given. The maximum likelihood estimation technique is used to estimate the model parameters and finally an application of the model with a real data set is presented for the illustration of the usefulness of the proposed distribution.

Key Words: *maximum likelihood, moment generating function, Quasi Lindley distribution*

1. INTRODUCTION

Lindley distribution was proposed by Lindley (1958) in the context of Bayesian statistics, as a counter example of fiducially statistics. However, due to the popularity of the exponential distribution in statistics especially in reliability theory, Lindley distribution has been

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overlooked in the literature. Recently, many authors have paid great attention to the Lindley distribution as a lifetime model. From different point of view, Ghitany et al. (2008) showed that Lindley distribution is a better lifetime model than exponential distribution. More so, in practice, it has been observed that many real life system models have increasing failure rate with time.Krishna and Kumar (2011) estimated the parameter of Lindley distribution with progressive Type-II censoring scheme. They also showed that it may fit better thanexponential, lognormal and gamma distributions in some real life situations.

Lindley (1958), introduced a one- parameter distribution known as Lindley distribution, given by its probability density function

$$g(x,\theta)=\frac{\theta^2}{\theta+1}(1+x)e^{-\theta x};x>0,\theta>0, \tag{1}$$

the cumulative distribution function (cdf) of Lindley distribution is obtained as

$$G(x,\theta)=1-\frac{\theta+1+\theta x}{\theta+1}e^{-\theta x},x>0,\theta>0, \tag{2}$$

Rama and Mishra (2013) introduced quasi Lindley distributionof which the Lindley distribution (LD) is a particular case. They studied several properties of the QLD, and shown that the QLD is more flexible than Lindley and exponential distributions. Quasi flexible than Lindley and exponential distributions. Quasi probability density function (p.d.f)

$$g(x,\theta,\alpha)=\frac{\theta}{\alpha+1}(\alpha+\theta x)e^{-\theta x}\ ;\ x>0,\theta>0,\alpha>-1. \tag{3}$$

It can easily be seen that at $\alpha=\theta$, the QLD (3) reduces to the Lindley distribution (1958) with probability density function and at $\alpha=0$, it reduces to the gamma distribution with parameters $(2,\theta)$. The p.d.f. (3) can be shown as a mixture of exponential (θ) and gamma $(2,\theta)$ distributions as follows

$$g(x,\theta,\alpha)=pg_1(x)+(1-p)g_2(x)\ ,$$

where

$$p=\frac{1}{\alpha+1},g_1(x)=\theta e^{-\theta x}\text{ and }g_2(x)=\theta^2 e^{-\theta x}\quad ,$$

The cumulative distribution function (cdf) of QLD is obtained as

$$G(x,\theta,\alpha)=1-e^{-\theta x}\left[1+\frac{\theta x}{\alpha+1}\right],x>0,\theta>0,\alpha>-1, \tag{4}$$

where θ is scale parameter.

The exponentiated distributions have been widely studied in statistics and numerous authors have developed various classes of these distributions. A good review of some of these models is presented by Pham and Lai (2007). The exponentiation of distributions is a mechanism that makes the model more flexible, Nadarajah and Kotz (2006) introduce four more exponentiated type distributions: the exponentiated Gamma, exponentiated Weibull, exponentiated Gumbel and the Exponentiated Fréchet distribution. We also, several authors

presented exponentiated distributions, such as Mudholkar and Srivastava (1993) with the exponentiation of the Weibull distribution. Elbatal (2011) introduced Exponentiated modified Weibull distribution.
This paper offers new distribution with three parameters called exponentiated quasi Lindley distribution, this article is organized as follows. In Section 2, we define the Exponentiated Quasi Lindley distribution, the expansion for the cumulative and density functions of the EQL distribution and some special cases. Quantile function, moments, moment generating function are discussed in Section 3. In Section 4 included the distribution of the order statistics. Maximumlikelihood estimation is performed in Section 5Finally, some applications of the distribution in section 6.

2. EXPONENTIATED QUASI LINDLEY DISTRIBUTION

In this section, we introduce the three − parameter Exponentiated Quasi Lindley (EQL) distribution, the cdf of the EQL distribution can be written as

$$F_{EQL}(x,\theta,\alpha,\beta)=\left[1-e^{-\theta x}\left[1+\frac{\theta x}{\alpha+1}\right]\right]^{\beta}. \tag{5}$$

The corresponding pdf, survival function, hazard function and reverse hazard rate function respectively,

$$f_{EQL}(x,\theta,\alpha,\beta)=\frac{\beta\theta}{\alpha+1}(\alpha+\theta x)e^{-\theta x}\left[1-e^{-\theta x}\left[1+\frac{\theta x}{\alpha+1}\right]\right]^{\beta-1}. \tag{6}$$

$$R(x,\theta,\alpha,\beta)=1-\left[1-e^{-\theta x}\left[1+\frac{\theta x}{\alpha+1}\right]\right]^{\beta},$$

$$h(x,\theta,\alpha,\beta)=\frac{f(x,\theta,\alpha,\beta)}{\overline{F}(x,\theta,\alpha,\beta)}=\frac{\frac{\beta\theta}{\alpha+1}(\alpha+\theta x)e^{-\theta x}\left[1-e^{-\theta x}\left[1+\frac{\theta x}{\alpha+1}\right]\right]^{\beta-1}}{1-\left[1-e^{-\theta x}\left[1+\frac{\theta x}{\alpha+1}\right]\right]^{\beta}},$$

and

$$\tau(x,\theta,\alpha,\beta)=\frac{f(x,\theta,\alpha,\beta)}{F(x,\theta,\alpha,\beta)}=\frac{\frac{\beta\theta}{\alpha+1}(\alpha+\theta x)e^{-\theta x}}{\left[1-e^{-\theta x}\left[1+\frac{\theta x}{\alpha+1}\right]\right]}.$$

Figures 1, 2, 3, 4 and 5 illustrate some of the possible shapes of the pdf, cdf, survival function, hazard rate function and reversed hazard rate function of the EQL distribution for selected values of the parameters θ,α and β, respectively.

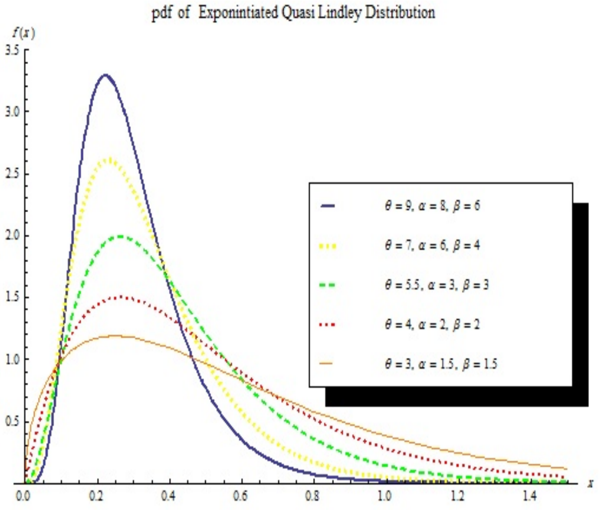

Figure 1. The pdf of EQL distribution for various values of parameters

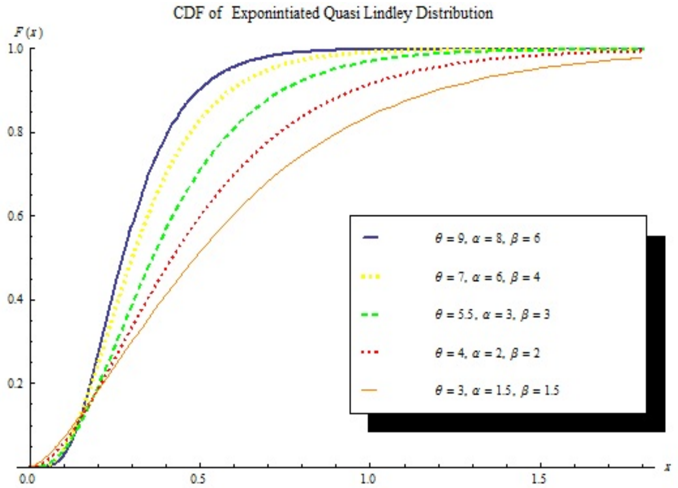

Figure 2. The cdf of EQL distribution for various values of parameters

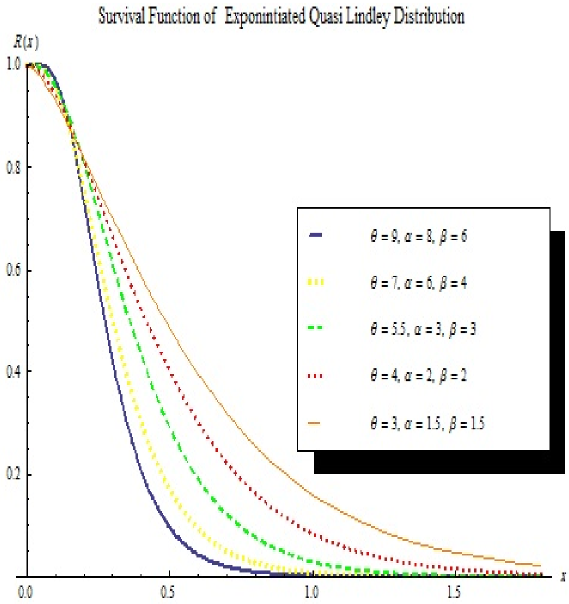

Figure 3. The survival function of EQL distribution for various values of parameters

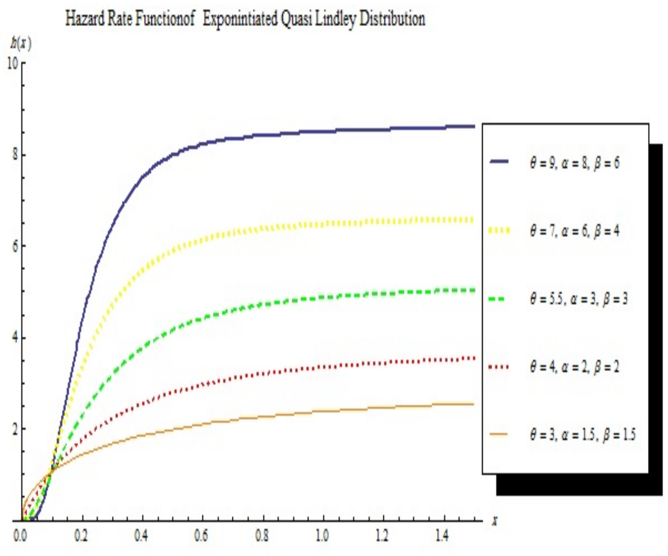

Figure 4. The hazard rate function of EQL distribution for various values of parameters

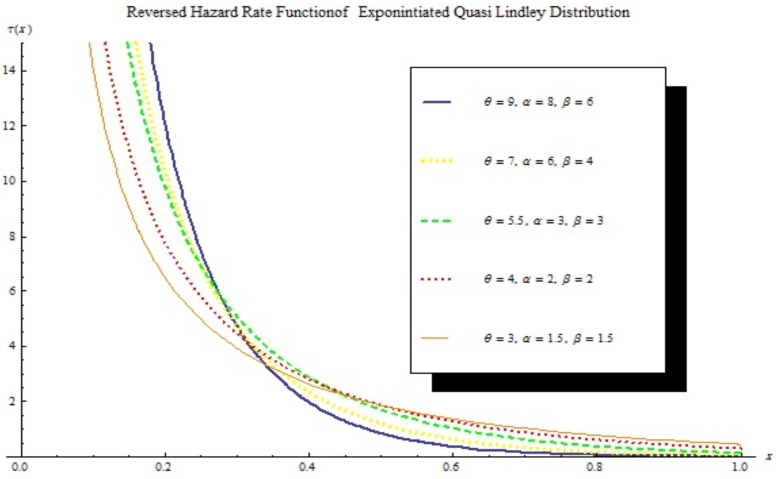

Figure 5. The reversed hazard rate function of EQL distribution for various values of parameters

Special Cases of the EQL Distribution

The Exponentiated Quasi Lindley is very flexible model that approaches to different distributions when its parameters are changed. The EQL distribution contains as special-models the following well known distributions. If X is a random variable with cdf of the EQL distribution, then we have the following cases**.**

-If $\beta = 1$, then we have Quasi Lindley distribution which is introduced by Rama and Mishra(2013).

- If $\alpha = \theta$ we get the Generalized Lindley distribution which is introduced by Nadarajah et al . (2011).
- If $\alpha = 0$ we get the generalized gamma distribution with parameters $(2,\theta)$.
- If $\beta = 1$, and $\alpha = \theta$ we get the Lindley distribution Lindley (1958).
- If $\beta = 1$, and $\alpha = \theta$ we get the gamma distribution with parameters $(2,\theta)$.

2.1 Expansion for the cumulative and density functions.

In this subsection we present some representations of cdf, pdf of exponentiated quasi Lindley distribution. The mathematical relation given below will be useful in this subsection. By using the generalized binomial theorem if β is a positive and $|z| < 1$, then

$$\left(1-z\right)^{\beta-1}=\sum_{j=0}^{\infty}(-1)^{j}\binom{\beta-1}{j}z^{j}, \tag{7}$$

Then, equation (4) become

$$f_{EQL}(x,\theta,\alpha,\beta)=\sum_{j=0}^{\infty}(-1)^{j}\binom{\beta-1}{j}\frac{\beta\theta}{\alpha+1}(\alpha+\theta x)e^{-\theta(j+1)x}\left[1+\frac{\theta x}{\alpha+1}\right]^{j} \tag{8}$$

where

$$\left(1+z\right)^{j}=\sum_{j=0}^{j}\binom{j}{k}z^{k} \tag{9}$$

Now using(9) in Equation (8), we obtain

$$\begin{aligned} f_{EQL}(x,\theta,\alpha,\beta) &= \sum_{j=0}^{\infty}\sum_{k=0}^{j}((-1)^{j}\binom{\beta-1}{j}\binom{j}{k}\beta\left(\frac{\theta}{\alpha+1}\right)^{k+1}x^{k}(\alpha+\theta x)e^{-\theta(j+1)x} \\ &= w_{j,k}x^{k}(\alpha+\theta x)e^{-\theta(j+1)x}, \end{aligned} \tag{10}$$

where

$$w_{j,k}=\sum_{j=0}^{\infty}\sum_{k=0}^{j}(-1)^{j}\beta\binom{\beta-1}{j}\binom{j}{k}\left(\frac{\theta}{\alpha+1}\right)^{k+1}$$

3. STATISTICAL PROPERTIES

This section is devoted to studying statistical properties of the EQL distribution, specifically quantile function, moments, and moment generating function.

3.1 Quantile function

The EQL quantile function, say $Q(u)=F^{-1}(u)$, is straightforward to be computed by inverting (2.1), we have

$$e^{-\theta x}\left[1+\frac{\theta x}{\alpha+1}\right]=1-q^{\frac{1}{\beta}} \tag{11}$$

We can easily generate X by taking u as a uniform random variable in $(0,1)$.

3.2 Moments

In this subsection we discuss the r_{th} non-central moment for EQL distribution. Moments are necessary and important in any statistical analysis, especially in applications. It can be used to study the most important features and characteristics of a distribution (e.g., tendency, dispersion, skewness and kurtosis).

Theorem 1.
If X has $EQL(x,\varphi)$,$\phi=(\alpha,\theta,\beta)$ then the r_{th} non-central moment of X is given by the following

$$\mu_r^{'}(x)=E(X^r)=w_{j,k}\left[\frac{\alpha\Gamma(r+k+1)}{(\theta(j+1))^{r+k+1}}+\frac{\theta\Gamma(r+k+2)}{(\theta(j+1))^{r+k+2}}\right]. \quad (12)$$

Where

$$w_{j,k}=\sum_{j=0}^{\infty}\sum_{k=0}^{j}((-1)^j\beta\binom{\beta-1}{j}\binom{j}{k}\left(\frac{\theta}{\alpha+1}\right)^{k+1}$$

Proof.
Let X be a random variable EQL distribution with density function from Equation (6). The r_{th} non-central moment of the EQL distribution is given by

$$\mu_r^{'}(x)=E(X^r)=\int_0^\infty x^r f(x,\varphi)dx=w_{j,k}\int_0^\infty x^{r+k}(\alpha+\theta x)e^{-\theta(j+1)x}dx \quad (13)$$

Then

$$\mu_r^{'}(x)=E(X^r)w_{j,k}\left[\alpha\int_0^\infty x^{r+k}e^{-\theta(j+1)x}dx+\theta\int_0^\infty x^{r+k}e^{-\theta(j+1)x}dx\right] \quad (14)$$

$$\mu_r^{'}(x)=E(X^r)=w_{j,k}\left[\frac{\alpha\Gamma(r+k+1)}{(\theta(j+1))^{r+k+1}}+\frac{\theta\Gamma(r+k+2)}{(\theta(j+1))^{r+k+2}}\right]. \quad (15)$$

where

$$w_{j,k}=\sum_{j=0}^{\infty}\sum_{k=0}^{j}((-1)^j\binom{\beta-1}{j}\binom{j}{k}\beta\left(\frac{\theta}{\alpha+1}\right)^{k+1}$$

Which completes the proof .

Substitution in the Equation (15) by $r=1,2,3,4$ we get the first four moments of $EQLD$ as:
at $r=1,2,3$ and 4

$$\mu_1'(x)=w_{j,k}\left[\frac{\alpha\Gamma(\mathrm{k}+2)}{(\theta(\mathrm{j}+1))^{k+2}}+\frac{\theta\Gamma(\mathrm{k}+3)}{(\theta(\mathrm{j}+1))^{k+3}}\right],$$

$$\mu_2'(x)=w_{j,k}\left[\frac{\alpha\Gamma(\mathrm{k}+3)}{(\theta(\mathrm{j}+1))^{k+3}}+\frac{\theta\Gamma(\mathrm{k}+4)}{(\theta(\mathrm{j}+1))^{k+4}}\right],$$

$$\mu_3'(x)=w_{j,k}\left[\frac{\alpha\Gamma(\mathrm{k}+4)}{(\theta(\mathrm{j}+1))^{k+4}}+\frac{\theta\Gamma(\mathrm{k}+5)}{(\theta(\mathrm{j}+1))^{k+5}}\right],$$

and

$$\mu_4'(x) = w_{j,k}\left[\frac{\alpha\Gamma(\mathrm{k}+5)}{(\theta(\mathrm{j}+1))^{k+5}} + \frac{\theta\Gamma(\mathrm{k}+6)}{(\theta(\mathrm{j}+1))^{k+6}}\right].$$

Based on the first four moments of the EQL distribution, the measures of skewness $A(\Phi)$ and kurtosis $k(\Phi)$ of the EQL distribution can obtained as

$$A(\Phi) = \frac{\mu_3(\theta) - 3\mu_1(\theta)\mu_2(\theta) + 2\mu_1^3(\theta)}{\left[\mu_2(\theta) - \mu_1^2(\theta)\right]^{\frac{3}{2}}},$$

and

$$k(\Phi) = \frac{\mu_4(\theta) - 4\mu_1(\theta)\mu_3(\theta) + 6\mu_1^2(\theta)\mu_2(\theta) - 3\mu_1^4(\theta)}{\left[\mu_2(\theta) - \mu_1^2(\theta)\right]^2}.$$

3.3 Moment generating function

In this subsection we derived the moment generating function of EQL distribution.

Theorem 2. If X has EQL distribution, then the moment generating function $M_X(t)$ has the following form

$$M_X(t) = w_{j,k}\left[\frac{\alpha\Gamma(k+1)}{(\theta(j+1)-t)^{k+1}} + \frac{\theta\Gamma(k+2)}{(\theta(j+1)-t)^{k+2}}\right]. \tag{16}$$

Proof.
We start with the well known definition of the moment generating function given by

$$\begin{aligned} M_X(t) = E(e^{tX}) = \int_0^\infty e^{tx} f_{EQL}(x,\varphi)dx = w_{j,k}\int_0^\infty x^k(\alpha+\theta x)e^{-x(\theta(j+1)-t)}dx \\ = w_{j,k}\left[\frac{\alpha\Gamma(k+1)}{(\theta(j+1)-t)^{k+1}} + \frac{\theta\Gamma(k+2)}{(\theta(j+1)-t)^{k+2}}\right]. \end{aligned} \tag{17}$$

This completes the proof.

In the same way, the characteristic function of the EQL distribution becomes $\varphi_X(t) = M_X(it)$ where $i = \sqrt{-1}$ is the unit imaginary number.

$$\phi_X(t) = E(e^{itX}) = w_{j,k}\left[\frac{\alpha\Gamma(k+1)}{(\theta(j+1)-it)^{k+1}} + \frac{\theta\Gamma(k+2)}{(\theta(j+1)-it)^{k+2}}\right].$$

Laplace and Fourier transforms as calculated as:

$$L_X(t) = E(e^{-tX}) = w_{j,k}\left[\frac{\alpha\Gamma(k+1)}{(\theta(j+1)+t)^{k+1}} + \frac{\theta\Gamma(k+2)}{(\theta(j+1)+t)^{k+2}}\right].$$

$$FO_X(t) = E(e^{-itX}) = w_{j,k}\left[\frac{\alpha\Gamma(k+1)}{(\theta(j+1)+it)^{k+1}} + \frac{\theta\Gamma(k+2)}{(\theta(j+1)+it)^{k+2}}\right].$$

4. DISTRIBUTION OF THE ORDER STATISTICS

In this section, we derive the pdfs of the j^{th} order statistic of the EQL distribution, also, the first, largest and joint of order statistic are obtained .The distribution of the r^{th} moment of the j^{th} order statistic are presented.

Let $X_{1:n} < X_{2:n} < \cdots < X_{n:n}$ denote the order statistics obtained from this sample. The probability density function of the j^{th} order statistic, say $f_{j:n}(x,\phi)$ is given by

$$f_{j:n}(x,\Phi) = \frac{1}{B(j,n-j+1)}\left[F(x,\varphi)\right]^{j-1}\left[1-F(x,\varphi)\right]^{n-j} f(x,\varphi), \quad (18)$$

where $f(x,\phi)$ and $F(x,\phi)$ are the pdf and cdf of the EQL distribution given by Equation (6) and Equation (5), respectively, and B (.,.) is the beta function, since $0 < F(x,\phi) < 1$, for $x > 0$, by using the binomial series expansion of $\left[1-F(x,\phi)\right]^{n-j}$, we get

$$\left[1-F(x,\varphi)\right]^{n-j} = \sum_{i=0}^{n-j}(-1)^j \binom{n-j}{i}\left[F(x,\varphi)\right]^i, \quad (19)$$

Thus

$$f_{j:n}(x,\varphi) = \sum_{i=0}^{n-j}(-1)^j \binom{n-j}{i}\left[F(x,\varphi)\right]^{j+i-1} f(x,\varphi), \quad (20)$$

substituting from Equation (5) and Equation (6) in Equation (20), we can express the k^{th} ordinary moment of the r^{th} order statistics $X_{r:n}$ say $E(X_{r:n}^k)$ as a liner combination of the k^{th} moments of the EQL distribution with different shape parameters.

Therefore, the measures of skewness and kurtosis of the distribution of $X_{r:n}$ can be calculated.

$$f_{X_{(j)}}(x_{(j)})=\frac{n!}{(j-1)!(n-j)!}f(x_{(j)})[F(x_{(j)})]^{j-1}[1-F(x_{(j)})]^{n-j}$$

$$=\frac{n!}{(j-1)!(n-j)!}\frac{\beta\theta(\alpha+\theta x_{(j)})e^{-\theta x_{(j)}}}{1+\alpha}\left[1-e^{-\theta x_{(j)}}\left(1+\frac{\theta}{1+\alpha}x_{(j)}\right)\right]^{\beta-1}$$

$$\times\left[1-e^{-\theta x_{(j)}}\left(1+\frac{\theta}{1+\alpha}x_{(j)}\right)\right]^{\beta(j-1)}\left[1-\left[1-e^{-\theta x_{(j)}}\left(1+\frac{\theta}{1+\alpha}x_{(j)}\right)\right]^{\beta}\right]^{(n-j)}$$

$$f_{X_{(j)}}(x)=\frac{n!}{(j-1)!(n-j)!}\times\frac{\beta\theta(\alpha+\theta x_{(j)})e^{-\theta x_{(j)}}}{1+\alpha}\times\left[1-e^{-\theta x_{(j)}}\left(1+\frac{\theta}{1+\alpha}x_{(j)}\right)\right]^{\beta j-1}$$
$$\times\left[1-\left[1-e^{-\theta x_{(j)}}\left(1+\frac{\theta}{1+\alpha}x_{(j)}\right)\right]^{\beta}\right]^{n-j} \quad (21)$$

The probability density function $f_{X_{(1)}}(x_{(1)})$ of the first order statistic is given by

$$f_{X_{(1)}}(x_{(1)})=n[1-F(x_{(1)})]^{n-1}\,\mathrm{f}\,(x_{(1)})$$
$$=\frac{n\beta\theta(\alpha+\theta x_{(1)})e^{-\theta x_{(1)}}}{1+\alpha}\left[1-e^{-\theta x_{(1)}}\left(1+\frac{\theta}{1+\alpha}x_{(1)}\right)\right]^{\beta-1}.\left[1-\left[1-e^{-\theta x_{(1)}}\left(1+\frac{\theta}{1+\alpha}x_{(1)}\right)\right]^{a}\right]^{n-1},$$

$$(22)$$

The probability density function $f_{X_{(n)}}(x)$ of the largest order statistic is given by

$$f_{X_{(n)}}(x_{(\mathrm{n})})=n[F(x_{(\mathrm{n})})]^{n-1}f(x_{(\mathrm{n})})=\frac{n\beta\theta(\alpha+\theta x_{(\mathrm{n})})e^{-\theta x_{(\mathrm{n})}}}{1+\alpha}\left[1-e^{-\theta x_{(\mathrm{n})}}\left(1+\frac{\theta}{1+\alpha}x_{(\mathrm{n})}\right)\right]^{\beta n-1}. \quad (23)$$

The joint p.d.f of $x_{(j)}$ and $x_{(k)}$for$x_{(j)} < x_{(k)}$is given by:

$$f_{X_{(j)},X_{(k)}(x_{(j)},x_{(k)})}=\frac{n!}{(j-1)!(k-j-1)!(n-k)!}[F(x_{(j)})]^{j-1}[F(x_{(k)})-F(x_{(j)})]^{k-j-1}[1-F(x_{(k)})]^{n-k}f(x_{(j)}).f(x_{(k)})$$
$$=\frac{n!\beta^2\theta^2(\alpha+\theta x_{(j)})(\alpha+\theta x_{(k)})e^{-\theta(x_{(j)}+x_{(k)})}}{(j-1)!(k-j-1)!(n-k)!(1+\alpha)^2}\left[1-\left[1-e^{-\theta x_{(k)}}\left(1+\frac{\theta}{1+\alpha}x_{(k)}\right)\right]^{\beta}\right]^{n-k}$$
$$\times\left[1-e^{-\theta x_{(j)}}\left(1+\frac{\theta}{1+\alpha}x_{(j)}\right)\right]^{\beta j-1}\left[1-e^{-\theta x_{(k)}}\left(1+\frac{\theta}{1+\alpha}x_{(k)}\right)\right]^{a-1}$$
$$\times\left\{\left[1-e^{-\theta x_{(\mathrm{k})}}\left(1+\frac{\theta}{1+\alpha}x_{(\mathrm{k})}\right)\right]^{\beta}-\left[1-e^{-\theta x_{(\mathrm{j})}}\left(1+\frac{\theta}{1+\alpha}x_{(\mathrm{j})}\right)\right]^{\beta}\right\}^{k-j-1}$$
(24)

The r^{th} moment of the j^{th} order statistic is denoted by:

$$E(X_{(j)}^r)=\int_{-\infty}^{\infty}x_{(j)}^r f(x_{(j)})dx_{(j)}=\int_0^{\infty}\frac{n!}{(j-1)!(n-j)!}\times\frac{\beta\theta(\alpha+\theta x_{(j)})x_{(\mathrm{j})}^r e^{-\theta x_{(j)}}}{1+\alpha}\times\left[1-e^{-\theta x_{(j)}}\left(1+\frac{\theta}{1+\alpha}x_{(j)}\right)\right]^{\beta j-1}$$
$$\times\left[1-\left[1-e^{-\theta x_{(j)}}\left(1+\frac{\theta}{1+\alpha}x_{(j)}\right)\right]^{\beta}\right]^{n-j}dx_{(j)},$$
(25)

By using the fact that

$$\left[1-\left[1-e^{-\theta x_{(j)}}\left(1+\frac{\theta}{1+\alpha}x_{(j)}\right)\right]^{\beta}\right]^{n-j}=\sum_{u=0}^{\infty}(-1)^u\binom{n-j}{u}\left[1-e^{-\theta x_{(j)}}\left(1+\frac{\theta}{1+\alpha}x_{(j)}\right)\right]^{\beta u}. \quad (26)$$

Substituting from Equation (4.9) into Equation (4.8), we get

$$E(X_{(j)}^r)=\frac{n!\beta\theta}{(j-1)!(n-j)!(1+\alpha)}\sum_{u=0}^{\infty}(-1)^u\binom{n-j}{u}$$
$$\int_0^{\infty}(\alpha+\theta x)x^r e^{-\theta x}\left[1-e^{-\theta x}\left(1+\frac{\theta}{1+\alpha}x\right)\right]^{\beta(j+u)-1}dx\ . \quad (27)$$

Applying the binomial expansion in last term of Equation (4.10), we have

$$E(X_{(j)}^{r}) = \frac{n!\beta\theta}{(j-1)!(n-j)!(1+\alpha)} \sum_{u=0}^{\infty}\sum_{z=0}^{\infty}(-1)^{u+z}\binom{n-j}{u}\binom{\beta(\mathrm{u}+j)-1}{z}\int_{0}^{\infty} x_{(j)}^{r}(\alpha+\theta x_{(j)})$$

$$\times e^{-(z+1)\theta x_{(j)}}\left(1+\frac{\theta}{1+\alpha}x_{(j)}\right)^{z} dx_{(j)}$$

$$= \frac{n!\beta\theta}{(j-1)!(n-j)!(1+\alpha)} \sum_{u=0}^{\infty}\sum_{z=0}^{\infty}\sum_{t=0}^{z}(-1)^{u+z}\binom{n-j}{u}\binom{\beta(\mathrm{u}+j)-1}{z}\binom{z}{t}\left(\frac{\theta}{1+\alpha}\right)^{t}$$

$$\times \int_{0}^{\infty} x_{(j)}^{r+t}(\alpha+\theta x_{(j)})e^{-(z+1)\theta x_{(j)}} dx_{(j)}$$

$$= \sum_{u=0}^{\infty}\sum_{z=0}^{\infty}\sum_{t=0}^{z} M\int_{0}^{\infty} x_{(j)}^{r+t}(\alpha+\theta x_{(j)})e^{-(z+1)\theta x_{(j)}} dx_{(j)}$$

$$= \sum_{u=0}^{\infty}\sum_{z=0}^{\infty}\sum_{t=0}^{z} M\int_{0}^{\infty}\left[\alpha\int_{0}^{\infty} x_{(j)}^{r+t}e^{-(z+1)\theta x_{(j)}} dx_{(j)} + \theta\int_{0}^{\infty} x_{(j)}^{r+t+1}e^{-(z+1)\theta x_{(j)}} dx_{(j)}\right] \tag{28}$$

$$= \sum_{u=0}^{\infty}\sum_{z=0}^{\infty}\sum_{t=0}^{z} M\left[\frac{\alpha\Gamma(r+t+1)}{[(z+1)\theta]^{r+t+1}} + \frac{\theta\Gamma(r+t+2)}{[(z+1)\theta]^{r+t+2}}\right].$$

where

$$M = \frac{n!\beta\theta}{(j-1)!(n-j)!(1+\alpha)}(-1)^{u+z}\binom{n-j}{u}\binom{\beta(\mathrm{u}+j)-1}{z}\binom{z}{t}\left(\frac{\theta}{1+\alpha}\right)^{t}$$

5. ESTIMATION AND INFERENCE

In this section, we determine the maximum likelihood estimates (MLEs) of the parameters of the EQL distribution from complete samples only. Let $X_1, X_2, \ldots, X_n$ be a random sample of size n from $EQL(x,\varphi)$. The likelihood function for the vector of parameters $\phi = (\alpha,\theta,\beta)$ can be written as

$$L(x_i,\Phi) = \Pi_{i=1}^{n} f(x_i,\varphi) = \left(\frac{\beta\theta}{\alpha+1}\right)^{n} \Pi_{i=1}^{n}(\alpha+\theta x_i)e^{-\theta\sum_{i=1}^{n}x_i}\Pi_{i=1}^{n}\left[1-e^{-\theta x_i}\left[1+\frac{\theta x_i}{\alpha+1}\right]\right]^{\beta-1}. \tag{29}$$

Taking the log-likelihood function for the vector of parameters $\phi = (\alpha,\theta,\beta)$ we get

$$\log\ L = n\log\beta + n\log\theta - n\log(1+\alpha) + \sum_{i=1}^{n}\log(\alpha+\theta x_i)$$
$$-\theta\sum_{i=1}^{n}x_i + (\beta-1)\sum_{i=1}^{n}\log\left[1-e^{-\theta x_i}\left[1+\frac{\theta x_i}{\alpha+1}\right]\right]. \tag{30}$$

The log-likelihood can be maximized either directly or by solving the nonlinear likelihood equations obtained by differentiating Equation (30). The components of the score vector are given by

$$\frac{\partial\log\ L}{\partial\beta} = \frac{n}{\beta} + \sum_{i=1}^{n}\log\left[1-e^{-\theta x_i}\left[1+\frac{\theta x_i}{\alpha+1}\right]\right], \tag{31}$$

$$\frac{\partial\log\ L}{\partial\theta} = \frac{n}{\theta} + \sum_{i=1}^{n}\frac{x_i}{(\alpha+\theta x_i)} - \sum_{i=1}^{n}x_i + (\beta-1)\sum_{i=1}^{n}\frac{x_i(1+\frac{\theta x_i}{\alpha+1}-\frac{1}{\alpha+1})(e^{-\theta x_i})}{\left[1-e^{-\theta x_i}\left[1+\frac{\theta x_i}{\alpha+1}\right]\right]}, \tag{32}$$

and

$$\frac{\partial\log\ L}{\partial\alpha} = \frac{-n}{\alpha+1} - \sum_{i=1}^{n}\frac{1}{(\alpha+\theta x_i)} + (\beta-1)\sum_{i=1}^{n}\frac{\theta x_i e^{-\theta x_i}}{(\alpha+1)^2\left[1-e^{-\theta x_i}\left[1+\frac{\theta x_i}{\alpha+1}\right]\right]}, \tag{33}$$

We can find the estimates of the unknown parameters by maximum likelihood method by setting these above non-linear Equations (31), (32) and (33)to zero and solve them simultaneously. Therefore, we have to use mathematical package to get the MLE of the unknown parameters.

For the observed information matrix of the parameters (α,θ,β). we find the second partial derivatives of L as

$$\frac{\partial^2\log L}{\partial\alpha^2} = \frac{n}{(1+\alpha)^2} + \sum_{i=1}^{n}\frac{1}{(\alpha+\theta x_i)^2} - (\beta-1)\sum_{i=1}^{n}\frac{C_i A_i - D_i^2}{A_i^2},$$

$$\frac{\partial^2\log L}{\partial\beta^2} = \frac{-n}{\beta^2},$$

$$\frac{\partial^2\log L}{\partial\theta^2} = \frac{-n}{\theta^2} + \sum_{i=1}^{n}\frac{x_i^2}{(\alpha+\theta x_i)^2} + (\beta-1)\sum_{i=1}^{n}\left\{\frac{T_i^2+T_i-K_i}{A_i} + \frac{(F_i-H_i)F_i}{A_i^2}\right\},$$

$$\frac{\partial^2\log L}{\partial\alpha\partial\beta} = \sum_{i=1}^{n}\frac{D_i}{A_i},$$

$$\frac{\partial^2\log L}{\partial\alpha\partial\theta} = -\sum_{i=1}^{n}\frac{x_i}{(\alpha+\theta x_i)^2} + (\beta-1)\sum_{i=1}^{n}\left\{\frac{G_i-D_i}{A_i} - \frac{D_i(F_i+H_i)}{A_i^2}\right\},$$

$$\frac{\partial^2 \log L}{\partial\theta\partial\beta} = \sum_{i=1}^{n} \frac{F_i - H_i}{A_i} ,$$

where

$$A_i = \left[1 - e^{-\theta x_i}\left(1 + \frac{\theta}{1+\alpha} x_i\right)\right] , \quad B_i = \left[1 - \left[1 - e^{-\theta x_i}\left(1 + \frac{\theta}{1+\alpha} x_i\right)\right]^{\beta}\right],$$

$$C_i = \frac{2\theta}{(1+\alpha)^3} x_i e^{-\theta x_i} , \qquad D_i = \frac{\theta}{(1+\alpha)^2} x_i e^{-\theta x_i} .$$

$$F_i = x_i e^{-\theta x_i}\left(1 + \frac{\theta}{1+\alpha} x_i\right) , \ H_i = \frac{1}{(1+\alpha)} x_i e^{-\theta x_i} \quad , \quad G_i = \frac{1}{(1+\alpha)^2} x_i e^{-\theta x_i} .$$

$$K_i = x_i^{\,2} e^{-\theta x_i}\left(1 + \frac{\theta}{1+\alpha} x_i\right) \quad , \quad T_i = \left(\frac{1}{1+\alpha}\right)^2 x_i^{\,2} e^{-\theta x_i} .$$

where

$$V_{\alpha\alpha} = \frac{\partial^2 L}{\partial\alpha^2}, V_{\theta\theta} = \frac{\partial^2 L}{\partial\theta^2}, V_{\beta\beta} = \frac{\partial^2 L}{\partial\beta^2}, V_{\alpha\theta} = \frac{\partial^2 L}{\partial\alpha\partial\theta}, V_{\beta\alpha} = \frac{\partial^2 L}{\partial\beta\partial\alpha}, V_{\beta\theta} = \frac{\partial^2 L}{\partial\beta\partial\theta} .$$

The inverse dispersion matrix is given by

$$\begin{pmatrix} \hat{\alpha} \\ \hat{\theta} \\ \hat{\beta} \end{pmatrix} \sim N\left[\begin{pmatrix} \alpha \\ \theta \\ \beta \end{pmatrix}, \begin{pmatrix} \hat{V}_{\alpha\alpha} & \hat{V}_{\alpha\theta} & \hat{V}_{\alpha\beta} \\ \hat{V}_{\theta\alpha} & \hat{V}_{\theta\theta} & \hat{V}_{\theta\beta} \\ \hat{V}_{\beta\alpha} & \hat{V}_{\beta\theta} & \hat{V}_{\beta\beta} \end{pmatrix}\right], \tag{33}$$

$$V^{-1} = -E\begin{bmatrix} V_{\alpha\alpha} & V_{\alpha\theta} & V_{\alpha\beta} \\ V_{\theta\alpha} & V_{\theta\theta} & V_{\theta\beta} \\ V_{\beta\alpha} & V_{\beta\theta} & V_{\beta\beta} \end{bmatrix} . \tag{34}$$

By solving this inverse dispersion matrix these solutions will yield asymptotic variance and covariances of these ML estimators for $\hat{\alpha}$, $\hat{\theta}$ and $\hat{\beta}$ using (5.6), we approximate $100(1-\gamma)\%$ confidence intervals for α, θ and β are determined ,respectively, as

$$\hat{\alpha} \pm z_{\frac{\gamma}{2}}\sqrt{\hat{V}_{\alpha\alpha}} \quad , \quad \hat{\theta} \pm z_{\frac{\gamma}{2}}\sqrt{\hat{V}_{\theta\theta}} \ \text{ and } \ \hat{\beta} \pm z_{\frac{\gamma}{2}}\sqrt{\hat{V}_{\beta\beta}}$$

where z_γ is the upper $100\gamma_{the}$ percentile of the standard normal distribution. Using MATHCAD we can easily compute the Hessian matrix and its inverse and asymptotic confidence intervals. Tables 1 represent the mean square error for some values of theparameters.

Table 1. Mean square errors of the MLEs

EQLD(α,θ,β)	n	$MSE(\widehat{\alpha})$	$MSE(\widehat{\theta})$	$MSE(\widehat{\beta})$
	15	0.0923	0.3452	0.3344
	25	0.0811	0.2545	0.1385
	35	0.0423	0.2354	0.0955
EQLD(0.85,1.25,1.5)	45	0.0350	0.2016	0.0868
	55	0.0245	0.1635	0.0805
	65	0.0231	0.1396	0.0579
	75	0.0185	0.1177	0.0452
	15	0.8493	0.1387	0.4542
	25	0.7371	0.0899	0.1769
	35	0.4760	0.06	0.0737
EQLD(2.0,1,1.25)	45	0.4642	0.0458	0.0613
	55	0.4267	0.0428	0.0583
	65	0.4141	0.0391	0.0536
	75	0.1944	0.0345	0.0409
	15	0.4178	0.2763	0.3959
	25	0.4063	0.2028	0.2589
	35	0.1859	0.1585	0.1163
EQLD(1.5,1.75,1.25)	45	0.1410	0.1377	0.0944
	55	0.1311	0.120	0.0763
	65	0.1095	0.1081	0.0584
	75	0.0823	0.0943	0.0408

We noticed from the above Table 1 that all MSEs decrease as the sample size increases, while they increase with increasing of the true parameter.

6. APPLICATIONS

In this section, we use two real data sets to show that the exponentiated quasi Lindley distribution can be a better model than one based on the Lindley distribution.

Data set 1. The following data represent the survival times (in days) of 72 guinea pigs infected with virulent tubercle bacilli, observed and reported by Bjerkedal (1960). The data are as follows:
0.1, 0.33, 0.44, 0.56, 0.59, 0.72, 0.74, 0.77, 0.92, 0.93, 0.96, 1, 1, 1.02, 1.05, 1.07, 07, .08, 1.08, 1.08, 1.09, 1.12, 1.13, 1.15, 1.16, 1.2, 1.21, 1.22, 1.22, 1.24, 1.3, 1.34, 1.36, 1.39, 1.44, 1.46, 1.53, 1.59, 1.6, 1.63, 1.63, 1.68, 1.71, 1.72, 1.76, 1.83, 1.95, 1.96, 1.97, 2.02, 2.13, 2.15, 2.16, 2.22, 2.3, 2.31, 2.4, 2.45, 2.51, 2.53, 2.54, 2.54, 2.78, 2.93, 3.27, 3.42, 3.47, 3.61, 4.02, 4.32, 4.58, 5.55.

Data set 2. The data set is obtained from Smith and Naylor (1987). The data are the strengths of 1.5 cm glass fibres, measured at the National Physical Laboratory, England. Unfortunately, the units of measurement are not given in the paper. The data set is
0.55, 0.93, 1.25, 1.36, 1.49, 1.52, 1.58, 1.61, 1.64, 1.68, 1.73, 1.81, 2, 0.74, 1.04, 1.27, 1.39, 1.49, 1.53, 1.59, 1.61, 1.66, 1.68, 1.76, 1.82, 2.01, 0.77, 1.11, 1.28, 1.42, 1.5, 1.54, 1.6, 1.62, 1.66, 1.69, 1.76, 1.84, 2.24, 0.81, 1.13, 1.29, 1.48, 1.5, 1.55, 1.61, 1.62, 1.66, 1.7, 1.77, 1.84, 1.24, 1.3, 1.48, 1.51, 1.55, 1.61, 1.63, 1.67, 1.7, 1.78, 1.89.

In order to compare the three distribution models, we consider criteria like $-2\ln L$, AIC (Akaike information criterion), CAIC (Corrected Akaike information criterion), BIC (Bayesian information criterion) and K-S (Kolmogorov-Smirnov test) for the data set. The better distribution corresponds to smaller $-2\ln L$, AIC and CAIC values:

$$AIC = 2k - 2\ln L, \quad CAIC = AIC + \frac{2k(k+1)}{n-k-1},$$

$$BIC = k * \ln(n) - 2\ln L \quad \text{and } K - S = \sup_x |F_n(x) - F(x)|$$

where $F_n(x) = \frac{1}{n}\sum_{i=1}^{n} I_{x_i \le x}$ is empirical distribution function, $F(x)$ is comulative distribution function, k is the number of parameters in the statistical model, n the sample size and .

Table 2. AIC, CAIC, BIC and K-S of the models based on data set 1

Model	Estimators	-2lnL	AIC	BIC	AICC	k-s
EQLD	$\hat{\theta}=1.328$ $\hat{\alpha}=3.284$ $\hat{\beta}=3.361$	188.24	194.24	201.07	200.298	1
QLD	$\hat{\theta}=0.641$ $\hat{\alpha}=7.007$	225.316	229.316	233.869	233.36	1.0126
LD	$\hat{\theta}=0.868$	213.857	215.857	218.134	217.885	1.026

Table 3. AIC, AICC, BIC and K-S of the models based on data set 2

Model	Estimators	-2lnL	AIC	BIC	AICC	K-S
EQLD	$\hat{\theta}=2.04$ $\hat{\alpha}=2.086$ $\hat{\beta}=8.189$	75.196	81.196	87.577	87.265	1
QLD	$\hat{\theta}=1.196$ $\hat{\alpha}=2.086$	155.541	159.541	163.795	163.591	1.0126
LD	$\hat{\theta}=0.99$	160.701	162.701	164.828	164.734	1.121

Tables 2 and 3 show the values of $-2\ln(L)$, AIC, AICC, BIC and K-S values for data set 1 and 2. The values in tables 1 and 2, indicate that the exponentiated quasi Lindley distribution is a strong competitor to other distributions used here for fitting data set 1 and data set 2.

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