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Algebraic Kripke-style semantics for substructural fuzzy logics*

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【Abstract】 This paper deals with Kripke-style semantics, which will be called *algebraic* Kripke-style semantics, for fuzzy logics based on uninorms (so called uninorm-based logics). First, we recall algebraic semantics for uninorm-based logics. In the general framework of uninorm-based logics, we next introduce various types of general algebraic Kripke-style semantics, and connect them with algebraic semantics. Finally, we analogously consider particular algebraic Kripke-style semantics, and also connect them with algebraic semantics.

【Key Words】 (Algebraic) Kripke-style semantics, algebraic semantics, substructural logic, many-valued logic, fuzzy logic.

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1. Introduction

This paper is a contribution to the study of Kripke-style semantics, which is said to be *algebraic* Kripke-style semantics, for *substructural fuzzy* logics: *substructural* logics lacking structural rules like weakening or contraction, and *fuzzy* logics dealing with vagueness. (Logics complete with respect to (w.r.t.) linearly ordered algebras are fuzzy in Cintula's sense (Cintula (2006)).) After Kripke first introduced the so-called *Kripke semantics* for modal and intuitionistic logics in Kripke (1963; 1965a; 1965b) using binary accessibility relations, many semantics generalizing them, the so-called *Kripke-style semantics*, have been provided for many-valued logics. As Yang (2014a) mentioned, there are at least two trends in generalization for many-valued logics.

One is to provide model structures with binary relations, but without operations. Various types of these semantics have been provided for three- and four-valued logics (see Bimbo & Dunn (2002), Dunn (1976; 2000), Thomason (1969), Yang (2009; 2012a). For instance, Thomason (1969) gave three-valued Kripke-style semantics for the Nelson's system **N** of constructable falsity by allowing partial evaluations ("gaps" (*N*)), and Dunn (1976; 2000) provided three-valued Kripke-style semantics for the relevance logic **RM** (the logic **R** of relevant implication with mingle) by allowing non-functional evaluations ("gluts" (*B*)). These logics all have *non-operational and binary relational* semantics. Thus, semantics in this trend are said to be non-operational and binary relational Kripke-style semantics.

The other trend is to provide model structures with both operations and binary relations. Various types of these semantics have been provided for infinite-valued or fuzzy logics (see Diaconescu & Georgescu (2007), Montagna & Ono (2002), Montagna & Sacchetti (2003), Yang (2012b)). In particular, after semantics for the infinite-valued Łukasiewicz logic **Ł** was introduced by Urquhart (1986), many Kripke-style semantics were recently provided for fuzzy logics based on t-norms (so called t-norm-based logics) by Montagna-Ono (2002), Montagna-Sacchetti (2003; 2004), and Diaconescu-Georgescu (2007). These logics all have *both operational and binary relational* semantics. Thus, semantics with this trend are said to be *operational and binary relational* Kripke-style semantics.

This paper is a contribution to the study of operational and binary relational Kripke-style semantics for fuzzy logics based on uninorms (so called uninorm-based logics). (Uninorms are functions introduced by Yager and Rybalov (1996) as a generalization of t-norms where the identity can lie anywhere in [0, 1].) We will investigate one particular kind of semantics, which will be called *algebraic* Kripke-style semantics, for uninorm-based logics. The particular systems investigated in this paper are all already known *substructural* logics lacking structural rules such as weakening and contraction. Thus, this is also an investigation of such semantics for substructural logics.

The aim of this paper is to introduce *algebraic* Kripke-style semantics, Kripke-style semantics being equivalent to algebraic semantics, for uninorm-based logics. Note that, as mentioned in Yang (2014a), to show that Kripke-style semantics can also be established for logics having algebraic semantics is one way to verify that Kripke-style semantics are very powerful as semantics for logics. Actually, as the results in Montagna & Sacchetti (2003) show, algebraic Kripke-style semantics is *very powerful* in the sense that most prominent t-norm-based logics, which are algebraically complete, also have algebraic Kripke-style semantics. But, while algebraic semantics for weakening-free uninorm-based logics have been introduced (see Gabbay & Metcalfe (2007), Metcalfe & Montagna (2007), Metcalfe, Olivetti, & Gabbay (2009)), Kripke-style semantics for such logics have not yet been studied enough yet. To our knowledge, Yang (2012b; 2014b) only provided such semantics for the uninorm logic **UL** and some of its extensions. This gives rise to the question:

Do algebraically complete uninorm-based logics also have algebraic Kripke-style semantics?

W.r.t. the propositional case, the answer is yes because we can introduce algebraic Kripke-style semantics for the system **UL** and its axiomatic extensions. However, w.r.t. the predicate case, the answer is no. It is, indeed, already known that in the predicate case there are important t-norm-based logics such as **BL** (Basic fuzzy logic), **Ł** and **Π** (Product logic), which do not have algebraic Kripke-style semantics (see Montagna & Sacchetti (2003)). Thus, in a sense, algebraic Kripke-style semantics for uninorm-based logics are not interesting in the predicate case.

Therefore, in this paper, we consider only propositional uninorm-based logics. (Henceforth, by uninorm-based logics, we denote propositional uninorm-based logics.)

This paper verifies the (positive) answer by introducing algebraic Kripke-style semantics for uninorm-based logics. For this, first, in Section 2 we introduce not merely the general framework of uninorm-based logics, but also particular infinite-valued logics, more precisely, **UL** and its most prominent axiomatic extensions, and the corresponding algebraic semantics as the necessary definitions for treating the question. In Section 3, we introduce various types of algebraic Kripke-style semantics for the system **UL** and its axiomatic extensions *in general*, and connect them with algebraic semantics. In Section 4, we analogously consider *particular* algebraic Kripke-style semantics for the uninorm-based logics, i.e., algebraic Kripke-style semantics for the prominent infinite-valued logics as the axiomatic extensions of **UL**, and also connect them with algebraic semantics.

For convenience, we shall adopt the notations and terminology similar to those in Dunn (2000), Metcalfe & Montagna (2007), Montagna & Sacchetti (2003; 2004) Yang (2012b; 2014b), and assume familiarity with them (together with results found therein).

2. The logic **UL** and its axiomatic extensions

The term *uninorm-based logics* refers to substructural fuzzy logic systems with uninorm-based semantics, where the (strong) conjunction and implication connectives & and → are interpreted by a (left-continuous) conjunctive uninorm and its residuum. In this framework the weakest logic is **UL**. This logic (and its axiomatic extensions, henceforth extensions for short) can be based on a countable propositional language with formulas *Fm* built inductively as usual from a set of propositional variables *VAR*, binary connectives $\to$, &, $\wedge$, $\vee$, and constants **T**, **F**, **f**, **t**, with defined connectives:

df1. $\neg\phi := \phi \to \mathbf{f}$, and
df2. $\phi \leftrightarrow \psi := (\phi \to \psi) \wedge (\psi \to \phi)$.

We further define $\phi_{\mathbf{t}}$ as $\phi \wedge \mathbf{t}$. We use the axiom systems to provide a consequence relation.

We start with the following axiomatization of **UL**, the most basic fuzzy logic introduced here.

Definition 2.1 (Metcalfe & Montagna (2007)) **UL** consists of the following axiom schemes and rules:

A1. $\phi \to \phi$
A2. $(\phi \wedge \psi) \to \phi$, $(\phi \wedge \psi) \to \psi$
A3. $((\phi \to \psi) \wedge (\phi \to \chi)) \to (\phi \to (\psi \wedge \chi))$
A4. $\phi \to (\phi \vee \psi)$, $\psi \to (\phi \vee \psi)$
A5. $((\phi \to \chi) \wedge (\psi \to \chi)) \to ((\phi \vee \psi) \to \chi)$
A6. $(\phi \,\&\, \psi) \to (\psi \,\&\, \phi)$
A7. $(\phi \,\&\, \mathbf{t}) \leftrightarrow \phi$
A8. $\mathbf{F} \to \phi$
A9. $\phi \to \mathbf{T}$
A10. $(\phi \to (\psi \to \chi)) \leftrightarrow ((\phi \,\&\, \psi) \to \chi)$

A11. $(\phi \rightarrow \psi) \rightarrow ((\psi \rightarrow \chi) \rightarrow (\phi \rightarrow \chi))$
A12. $(\phi \rightarrow \psi)_t \vee (\psi \rightarrow \phi)_t$
$\phi \rightarrow \psi, \phi \vdash \psi$ (modus ponens, mp)
$\phi, \psi \vdash \phi \wedge \psi$ (adjunction, adj).

We moreover define ϕ^n_t as ϕ_t & ⋯ & ϕ_t, n factors. For the remainder we shall follow the customary notation and terminology. A *theory* over **UL** is a set T of formulas. A *proof* is a sequence of formulas whose each member is either an axiom of **UL** or a member of T or follows from some preceding members of the sequence using the two rules in Definition 2.1. T $\vdash \phi$, more exactly T $\vdash_{\mathbf{UL}} \phi$, means that ϕ is *provable* in T w.r.t. **UL**, i.e., there is a **UL**-proof of ϕ in T. The system **UL** enjoys the local deduction theorem (LDT) and substitution rule.

Proposition 2.2 (Novak (1990)) Let T be a theory over L, and ϕ, ψ formulas. (LDT) T $\cup \{\phi\} \vdash_{\mathbf{UL}} \psi$ if and only if (iff) there is n such that T $\vdash_{\mathbf{UL}} \phi^n_t \rightarrow \psi$.

Proposition 2.3 (Cintula (2006), Cintula & Noguera (2011)) For all formulas ϕ, ψ, and χ, we have

(Cong) $\phi \leftrightarrow \psi \vdash_{\mathbf{UL}} \chi(\phi) \leftrightarrow \chi(\psi)$.

For convenience, '¬,' '→,' '∧,' and '∨' are used ambiguously as propositional connectives and as algebraic operators, but context should clarify their meanings.

The algebraic counterpart of **UL** is the class of the so-called *UL-algebras*. Let $x_t := x \wedge t$. They are defined as follows.

Definition 2.4 (Metcalfe & Montagna (2007)) A *UL-algebra* is a structure **A** = (A, ⊤, ⊥, t, f, ∧, ∨, *, →) such that:

(I) (A, ⊤, ⊥, ∧, ∨) is a bounded lattice with top element ⊤ and bottom element ⊥.

(II) (A, *, t) is a commutative monoid.

(III) f is an arbitrary element of A.

(IV) $y \leq x \rightarrow z$ iff $x * y \leq z$, for all x, y, z ∈ A (residuation).

(V) $t \leq (x \rightarrow y)_t \vee (y \rightarrow x)_t$ (prelinearity).

If the lattice order is linear or total, we will say that **A** is a UL-chain.

Additional (unary) negation and (binary) equivalence operations are defined as follows: $\neg x := x \rightarrow f$ and $x \leftrightarrow y := (x \rightarrow y) \wedge (y \rightarrow x)$.

The class of all UL-algebras is a variety which will be denoted by UL.[1]

Definition 2.5 Let K be a class of UL-algebras. We define consequence relation $\models_K$ in the following way: $T \models_K \phi$ iff for each **A** ∈ K and **A**-evaluation v, $v(\phi) \geq t$ whenever $v(\psi) \geq t$ for each $\psi \in T$.

We write $\models_K \phi$ instead of $\varnothing \models_K \phi$, and $T \models_{\mathbf{A}} \phi$ instead of $T \models_{\{\mathbf{A}\}} \phi$.

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1) It is the variety of pointed bounded commutative residuated lattices with prelinearity. For the definition of variety, see Dunn (2000).

That UL is the proper algebraic semantics for **UL** is witnessed by the following completeness result.

Theorem 2.6 (Metcalfe & Montagna (2007)) Let T be a theory over **UL**, and φ a formula. T $\vdash_{\mathbf{UL}}$ φ iff T $\models_{\mathrm{UL}}$ φ.

This completeness result can be refined by taking into account the following representation of UL-algebras related to the prelinearity property of UL-algebras.

Proposition 2.7 (Tsinakis & Blount (2003)) Each UL-algebra is a subdirect product of UL-chains.

This leads to the completeness of **UL** w.r.t. the class of chains of UL.

Corollary 2.8 (Metcalfe & Montagna (2007)) Let T be a theory over **UL** and φ a formula. T $\vdash_{\mathbf{UL}}$ φ iff T $\models_{\{\text{UL-chains}\}}$ φ.

We next consider general algebraic completeness results for extensions of **UL**. Suitable classes of algebras for extensions of **UL** are defined as follows.

Definition 2.9 (Metcalfe & Montagna (2007)) A UL-algebra **A** is an **L**-algebra iff all axioms of the corresponding logic **L** are valid in **A**, i.e., v(φ) $\geq$ t for all evaluations v for **A**.

We define below the notions of strong completeness, finite strong completeness, and weak completeness.

Definition 2.10 Let **L** be an extension of **UL** and $\mathcal{L}$ be a class of **L**-algebras. We define:

- **L** has the property of *strong completeness*, SC for short, if for T a theory over **L**, T $\vdash_{\mathbf{L}} \phi$ iff T $\models_{\mathcal{L}} \phi$.
- **L** has the property of *finite strong completeness*, FSC for short, if for T a finite theory over **L**, T $\vdash_{\mathbf{L}} \phi$ iff T $\models_{\mathcal{L}} \phi$.
- **L** has the property of *weak completeness*, WC for short, if $\vdash_{\mathbf{L}} \phi$ iff $\models_{\mathcal{L}} \phi$.

By V($\mathcal{K}$), we denote the variety generated by the class $\mathcal{K}$; Q($\mathcal{K}$), the quasivariety generated by the class $\mathcal{K}$; $ISP_{\sigma\text{-}f}(\mathcal{K})$, the smallest class of all isomorphic copies of isomorphisms(I), subalgebras(S), and σ-filtered products($P_{\sigma\text{-}f}$) of families of members of $\mathcal{K}$. Then, we can obtain the following algebraic properties for each type of completeness.

Theorem 2.11 (Cf. Cintula et al. (2009)) Let **L** be an extension of **UL**.

1. **L** has the WC iff $\mathcal{L}$ = V($\mathcal{L}$).
2. **L** has the FSC iff $\mathcal{L}$ = Q($\mathcal{L}$).
3. **L** has the SC iff $\mathcal{L}$ = $ISP_{\sigma\text{-}f}(\mathcal{L})$.

Theorem 2.12 (Cf. Cintula et al. (2009)) Let **L** be an extension of **UL** and let $\mathcal{L}$ be a class of **L**-chains. Then the following are

equivalent.
(i) **L** has the SC.
(ii) Every countable **L**-chain belongs to IS(Ł).
(iii) Every countable subdirectly irreducible **L**-chain belongs to IS(Ł).

Proof: By almost the same proof as Theorem 3.5 in Cintula et al. (2009). □

Theorem 2.13 (Cf. Cintula et al. (2009)) Let **L** be an extension of **UL** and let Ł be a class of **L**-chains. Then the following are equivalent.
(i) **L** has the FSC.
(ii) Every **L**-chain is partially embeddable into Ł.
(iii) Every countable **L**-chain is partially embeddable into Ł.
(iv) Every subdirectly irreducible **L**-chain is partially embeddable into Ł.
(v) Every countable subdirectly irreducible **L**-chain is partially embeddable into Ł.

Proof: By almost the same proof as Theorem 3.8 in Cintula et al. (2009). □

A lot of the well-known fuzzy logics can be presented as extensions of **UL**. We define unary connective - and constant u as follows: (df3) $-\phi := \phi \rightarrow \mathbf{t}$ and (df4) u := -**T**. Tables 1 and 2 collect some axiom schemes and the extensions of **UL** that they define (see Cintula et al. (2009), Gabbay & Metcalfe (2007), Metcalfe & Montagna (2007)}).[2] The system **UL** is actually an algebraizable logic in the sense of Blok and Pigozzi (see Blok & Pigozzi (1989)) and UL is its equivalent algebraic semantics. This implies that all the extensions of **UL** are also algebraizable and their equivalent algebraic semantics are the subvarieties of UL defined by the translations of the axioms into equations. In particular, there is an order-reversing isomorphism between axiomatic extension of **UL** and subvarieties of UL (cf. see Cintula et al. (2009)).

Table 1: Some usual axiom schemes in fuzzy logics

Axiom schema	Abbreviation of Name
$\neg\neg\phi \to \phi$	(Inv)
$\phi \to \phi \,\&\, \phi$	(Ctr)
$\phi \,\&\, \phi \to \phi$	(Exp)
$(\mathbf{T}\to\phi) \vee (\phi\to(\psi\wedge u)) \vee (\psi\to(\phi\&(\phi\to\psi)))$	(RDiv)
$\mathbf{t} \leftrightarrow \mathbf{f}$	(FP)
$u \leftrightarrow u \,\&\, u$	(U)
$(\phi\to\mathbf{F}) \vee (\mathbf{T}\to\phi) \vee ((\phi\to(\phi\&\psi)) \to \psi)$	(RCan)
$\phi \to (\psi \to \phi)$	(W)
$(\phi \to \mathbf{F}) \vee ((\phi \to (\phi \,\&\, \psi)) \to \psi)$	(Can)
$((\phi \,\&\, \psi) \to \mathbf{F}) \vee ((\phi \to (\phi \,\&\, \psi)) \to \psi)$	(WCan)
$(\phi \wedge \psi) \to (\phi \,\&\, (\phi \to \psi))$	(Div)
$(\phi \wedge \neg\phi) \to \mathbf{F}$	(PC)
$\neg\neg\phi \to (((\psi \,\&\, \phi) \to (\chi \,\&\, \phi)) \to (\psi \to \chi))$	(Prod)
$((\phi \,\&\, \psi) \to \mathbf{F}) \vee ((\phi \wedge \psi) \to (\phi \,\&\, \psi))$	(WNM)

2) These tables collect only some of the most prominent extensions of **UL**, even though many other ones have been studied in the literature (see Ciabattoni, Esteva, & Godo (2002), Cintula et al. (2009), Montagna, Noguera, & Horčík (2006), Wang S, Wang B, & Pei (2005), Wang S, Wang B, & Ren (2005)}).

Table 2: Some extensions of **UL** obtained by adding the corresponding additional axiom schemes

Logic	Additional axiom schemes
IUL	(Inv)
UML	(Ctr) and (Exp)
IUML	(Ctr), (Exp) and (FP)
BUL	(RDiv) and (U)
IBUL	(RDiv), (U) and (Inv)
CBUL	(RDiv), (U) and (RCan)
CRL	(RDiv), (U), (Inv), (RCan) and (FP)
MTL	(W)
IMTL	(W) and (Inv)
SMTL	(W) and (PC)
ΠMTL	(W), (PC) and (Prod)
BL	(W) and (Div)
Ł	(W), (Div), and (Inv)
G	(W), (Div), (Ctr) and (Exp)
SBL	(W), (Div) and (PC)
Π	(W), (Div) and (Can)

Let L be a logic listed in Table 2. Table 3 collects the properties of a UL-algebra corresponding to the axiom schemes in Table 1. An L-algebra is a UL-algebra satisfying algebraic properties corresponding to the additional axiom schemes.

Table 3: Some properties of a UL-algebra

Property of a UL-algebra	Name Ab. of Axiom scheme
$\neg\neg x \leq x$	(Inv)
$x \leq x * x$	(Ctr)
$x * x \leq x$	(Exp)
$t \leq (\top \rightarrow x) \vee (x \rightarrow (y \wedge u)) \vee (y \rightarrow (x*(x \rightarrow y)))$	(RDiv)
$t = f$	(FP)
$u = u * u$	(U)

$t \leq (x \rightarrow \bot) \vee (\top \rightarrow x) \vee ((x \rightarrow (x * y)) \rightarrow y)$	(RCan)
$x \leq y \rightarrow x$	(W)
$t \leq (x \rightarrow \bot) \vee ((x \rightarrow (x * y)) \rightarrow y)$	(Can)
$t \leq ((x * y) \rightarrow \bot) \vee ((x \rightarrow (x * y)) \rightarrow y)$	(WCan)
$t \leq (x \wedge y) \rightarrow (x * (x \rightarrow y))$	(Div)
$x \wedge \neg x \leq \bot$	(PC)
$\neg\neg x \leq ((y * x) \rightarrow (z * x)) \rightarrow (y \rightarrow z)$	(Prod)
$t \leq ((x * y) \rightarrow \bot) \vee ((x \wedge y) \rightarrow (x * y))$	(WNM)

Definition 2.14 An L-algebra is *standard* iff its lattice reduct is [0, 1].

Since the Lindenbaum-Tarski algebra of L can be provided, the completeness results of L are obtained as follows.

Theorem 2.15 (i) For L a logic listed in Table 2 and T a (finite w.r.t. **BUL**, **IBUL**, **CBUL**, **CRL**, **BL**, **SBL**, **Ł** and **Π**) theory over L, the following are equivalent:

(1) $T \vdash_L \phi$.

(2) $T \models_L \phi$.

(3) For every linearly ordered L-algebra and evaluation v, $v(\phi) \geq t$ whenever $v(\psi) \geq t$ for each $\psi \in T$, denoted by $T \models^1_L \phi$.

(ii) For L (except **IUL**) in (i), the following are also equivalent:

(1) $T \vdash_L \phi$.

(2) For every standard L-algebra and evaluation v, $v(\phi) \geq t$ whenever $v(\psi) \geq t$ for each $\psi \in T$.

Proof: See Cintula et al. (2009), Gabbay & Metcalfe (2007), Metcalfe & Montagna (2007). (Note that, though the standard weak completeness (but not the standard finite strong completeness) of **BUL**, **IBUL**, **CBUL**, and **CRL** is proved in Gabbay & Metcalfe (2007), we can easily prove the standard FSC of **BUL**, **IBUL**, **CBUL**, and **CRL**.) □

3. General algebraic Kripke-style semantics

Here, we consider *general* algebraic Kripke-style semantics for **UL** and its extensions.

Definition 3.1 (Algebraic Kripke frame) An *algebraic Kripke frame* is a structure **X** = (X, ⊤, ⊥, t, f, ≤, ∗, →) such that (X, ⊤, ⊥, t, f, ≤, ∗, →) is a linearly ordered residuated pointed bounded commutative monoid. The elements of **X** are called *nodes*.

Definition 3.2 (UL frame) A *UL frame* is an algebraic Kripke frame, where ∗ is conjunctive (i.e., ⊥ ∗ ⊤ = ⊥) and left-continuous (i.e., whenever $\sup\{x_i : i \in I\}$ exists, $x * \sup\{x_i : i \in I\} = \sup\{x * x_i : i \in I\}$), and so its residuum → is defined as $x \to y := \sup\{z : x * z \leq y\}$ for all x, y ∈ X.

Definition 3.2 ensures that a UL frame has a supremum w.r.t. ∗, i.e., for every x, y ∈ X, the set $\{z : x * z \leq y\}$ has the supremum. **X** is said to be *complete* if ≤ is a complete order.

An *evaluation* or *forcing* on an algebraic Kripke frame is a relation ⊩ between nodes and propositional variables, and arbitrary formulas subject to the conditions below: for every propositional variable p,

(AHC) if x $\Vdash$ p and y $\leq$ x, then y $\Vdash$ p;
(min) $\bot$ $\Vdash$ p; and

for arbitrary formulas,

(t) x $\Vdash$ **t** iff x $\leq$ t;
(f) x $\Vdash$ **f** iff x $\leq$ f;
($\bot$) x $\Vdash$ **F** iff x = $\bot$;
($\wedge$) x $\Vdash$ $\phi \wedge \psi$ = iff x $\Vdash$ ϕ and x $\Vdash$ ψ;
($\vee$) x $\Vdash$ $\phi \vee \psi$ iff x $\Vdash$ ϕ or x $\Vdash$ ψ;
(&) x $\Vdash$ ϕ & ψ iff there are y, z $\in$ X such that y $\Vdash$ ϕ, z $\Vdash$ ψ, and x $\leq$ y $*$ z;
($\rightarrow$) x $\Vdash$ $\phi \rightarrow \psi$ iff for all y $\in$ X, if y $\Vdash$ ϕ, then x $*$ y $\Vdash$ ψ.

An evaluation or forcing on a UL frame is an evaluation or forcing further satisfying that (max) for every atomic sentence p, {x : x $\Vdash$ p} has a maximum.

Definition 3.3 (i) (Algebraic Kripke model) An *algebraic Kripke model* is a pair (**X**, $\Vdash$), where **X** is an algebraic Kripke frame and $\Vdash$ is a forcing on **X**.

(ii) (UL model) A *UL model* is a pair (**X**, $\Vdash$), where **X** is a UL frame and $\Vdash$ is a forcing on **X**. A UL model (**X**, $\Vdash$) is said to be *complete* if **X** is a complete frame and $\Vdash$ is a forcing on **X**.

Definition 3.4 (Cf. Montagna & Sacchetti (2004)) Given an algebraic Kripke model (**X**, $\Vdash$), a node x of **X** and a formula φ, we say that *x forces φ* to express x $\Vdash$ φ. We say that φ is *true* in (**X**, $\Vdash$) if t $\Vdash$ φ, and that φ is *valid* in the frame **X** (expressed by **X** $\models$ φ) if φ is true in (**X**, $\Vdash$) for every forcing $\Vdash$ on **X**.

For soundness and completeness for **UL**, let $\vdash_{\mathbf{UL}}$ φ be the theoremhood of φ in **UL**. First we note the following lemma.

Lemma 3.5 (i) (Hereditary Lemma, HL) Let **X** be an algebraic Kripke frame. For any sentence φ and for all nodes x, y $\in$ **X**, if x $\Vdash$ φ and y $\leq$ x, then y $\Vdash$ φ.

(ii) Let $\Vdash$ be a forcing on a UL frame, and φ a sentence. Then the set {x $\in$ X : x $\Vdash$ φ} has a maximum.

Proof: (i) Easy. (ii) See Lemma 2.11 in Montagna & Sacchetti (2003). □

Proposition 3.6 (Soundness, Yang (2012b)) If $\vdash_{\mathbf{UL}}$ φ, then φ is valid in every UL frame.

By a *chain*, we mean a linearly ordered algebra. The next proposition connects algebraic Kripke semantics and algebraic semantics for **UL** (cf. see Montagna & Sacchetti (2004)).

Proposition 3.7 (Yang (2012b))

(i) The $\{\top, \bot, t, f, \leq, *, \rightarrow\}$ reduct of a UL-chain **A** is a UL frame, which is complete iff **A** is complete.

(ii) Let **X** = $(X, \top, \bot, t, f, \leq, *, \rightarrow)$ be a UL frame. Then the structure **A** = $(X, \top, \bot, t, f, max, min, *, \rightarrow)$ is a UL-algebra (where *max* and *min* are meant w.r.t. $\leq$).

(iii) Let **X** be the $\{\top, \bot, t, f, \leq, *, \rightarrow\}$ reduct of a UL-chain **A**, and let v be an evaluation in **A**. Let for every atomic formula p and for every $x \in$ **A**, $x \Vdash p$ iff $x \leq v(p)$. Then (**X**, $\Vdash$) is a UL model, and for every formula ϕ and for every $x \in$ **A**, we obtain that: $x \Vdash \phi$ iff $x \leq v(\phi)$.

(iv) Let (**X**, $\Vdash$) be a UL model, and let **A** be the UL-algebra defined as in (ii). Define for every atomic formula p, $v(p) = max\{x \in X : x \Vdash p\}$. Then for every formula ϕ, $v(\phi) = max\{x \in X : x \Vdash \phi\}$.

Theorem 3.8 (Strong completeness, Yang (2012b))

(i) **UL** is strongly complete w.r.t. the class of all UL-frames.

(ii) **UL** is strongly complete w.r.t. the class of complete UL-frames.

We further consider frames of extensions of **UL**.

Definition 3.9 Let X be a class of algebraic Kripke frames and ϕ a formula.

(i) (Validity) We say that ϕ is valid in $\mathcal{X}$ if it is valid in every frame in $\mathcal{X}$.

(ii) (Semantic consequence) Given a set T of formulas, we say that ϕ is a *semantic consequence* of T w.r.t. $\mathcal{X}$ (in symbols, T $\vDash_{\mathcal{X}}$ ϕ) if for every **X** $\in$ $\mathcal{X}$ and for every forcing $\Vdash$ on **X**, if t $\Vdash$ ψ for all $\psi \in$ T, then t $\Vdash$ ϕ.

Definition 3.10 A UL frame **X** is an **L** frame iff all axioms of **L** are valid in **X**. We say that a UL model (**X**, $\Vdash$) is an ***L*** *model* if **X** is an **L** frame.

Analogously algebraic Kripke semantics and algebraic semantics for **L**, an extension of **UL**, can be connected as follows.

Proposition 3.11 (i) The $\{\top, \bot, t, f, \leq, *, \to\}$ reduct of an **L**-chain **A** is an **L** frame, which is complete iff **A** is complete.

(ii) Let **X** = $(X, \top, \bot, t, f, \leq, *, \to)$ be an **L** frame. Then the structure **A** = $(X, \top, \bot, t, f, max, min, *, \to)$ is an **L**-algebra (where *max* and *min* are meant w.r.t. $\leq$).

(iii) Let **X** be the $\{\top, \bot, t, f, \leq, *, \to\}$ reduct of an **L**-chain **A**, and let v be an evaluation in **A**. Let for every atomic formula p and for every x $\in$ **A**, x $\Vdash$ p iff x $\leq$ v(p). Then (**X**, $\Vdash$) is an **L** model, and for every formula ϕ and for every x $\in$ **A**, we obtain that: x $\Vdash$ ϕ iff x $\leq$ v(ϕ).

(iv) Let (**X**, $\Vdash$) be an **L** model, and let **A** be the **L**-algebra defined as in (ii). Define for every atomic formula p, v(p) = max{x $\in$ X : x $\Vdash$ p}. Then for every formula ϕ, v(ϕ) =

$\max\{x \in X : x \Vdash \phi\}$.

Proof: Its proof is almost the same as Proposition 3.7. (We just note that Definitions 2.9 and 3.10 ensure that **L** frames and (corresponding) **L**-algebras satisfy additional frame and (corresponding) algebraic properties for **L**.) □

Theorem 3.12 Let **L** be an extension of **UL** and let X a class of **L** frames. Theorems 2.11 to 2.13 hold true for X.

Proof: It directly follows from Proposition 3.11, jute noting that **L** frames form **L**-algebras (see Proposition 3.7 (ii)). □

4. Particular algebraic Kripke-style semantics

An extension **L** of **UL** in Section 3 is not necessarily an infinite-valued logic. (Note that Classical logic (CL) is an extension of **UL**.) Here, we consider algebraic Kripke-style semantics for the prominent infinite-valued logics as the extensions of **UL** introduced in Table 2.

Let L be a logic listed in Table 2. For soundness and completeness of L, we connect properties of frames to properties of logics. Table 4 collects the properties of a UL frame corresponding to the axiom schemes in Table 1.

Table 4: Some properties of a UL frame

Property of a UL frame	Name Ab. of Axiom scheme
$\neg\neg x \leq x$	(Inv)
$x \leq x * x$	(Ctr)
$x * x \leq x$	(Exp)
If $y \leq u$, $t \leq \top \rightarrow x$ or $t \leq x \rightarrow y$ or $t \leq y \rightarrow (x * (x \rightarrow y))$; otherwise, $t \leq \top \rightarrow x$ or $t \leq x \rightarrow u$ or $t \leq y \rightarrow (x * (x \rightarrow y))$	(RDiv)
$t = f$	(FP)
$u = u * u$	(U)
$t \leq x \rightarrow \bot$ or $t \leq \top \rightarrow x$ or $t \leq (x \rightarrow (x * y)) \rightarrow y$	(RCan)
$x \leq y \rightarrow x$	(W)
$t \leq x \rightarrow \bot$ or $t \leq (x \rightarrow (x * y)) \rightarrow y$	(Can)
$t \leq (x * y) \rightarrow \bot$ or $t \leq (x \rightarrow (x * y)) \rightarrow y$	(WCan)
If $x \leq y$, there is a z such that $z * y = x$	(Div)
If $x \leq \neg x$, $x \leq \bot$; otherwise, $\neg x \leq \bot$	(PC)
$\neg\neg x \leq ((y * x) \rightarrow (z * x)) \rightarrow (y \rightarrow z)$	(Prod)
If $x \leq y$, $t \leq (x * y) \rightarrow \bot$ or $t \leq x \rightarrow (x * y)$; otherwise, $t \leq (x * y) \rightarrow \bot$ or $t \leq y \rightarrow (x * y)$	(WNM)

Notice that in UL frames *min* and *max*, each of which corresponds to the meet and join operators $\wedge$ and $\vee$, respectively, can be defined using the inequation $\leq$. Thus, the algebraic properties corresponding to the frame properties of (RDiv), (RCan), (Can), (WCan), (Div), (PC), and (WNM) can be expressed as in Table 3, using meet $\wedge$ and join $\vee$. (The other properties of a UL-algebra are the same as those of a UL frame in Table 4.)

Let an *L frame* be a UL frame satisfying frame properties corresponding to axiom schemes for L. Since UL frames are linearly ordered (and so form UL-algebras), we can obtain the completeness for L, using Theorem 2.10.

Before showing Proposition 4.2 below, we prove the following:

Lemma 4.1 t $\Vdash \phi \rightarrow \psi$ iff for all x $\in$ **X**, if x $\Vdash \phi$, then x $\Vdash \psi$.

Proof: Since t * x = x for all x $\in$ **X**, it directly follows from ($\rightarrow$). □

Proposition 4.2 Let **X** = (X, $\top$, $\bot$, t, f, $\leq$, $*$, $\rightarrow$) be a UL frame, and let (L) be a name listed in Table 4 and $(L)_F$ be the corresponding property of a UL frame.

(i) Let (L) be a name among (INV), (Ctr), (Exp), (FP), (RDiv), (U), and (RCan). **X** $\vDash$ (L) iff **X** satisfies $(L)_F$.

(ii) Let **X** be a UL frame satisfying the property corresponding to (W), i.e., an MTL frame. **X** $\vDash$ (L) iff **X** satisfies $(L)_F$.

Proof: First note that **X** is the {$\top$, $\bot$, t, f, $\leq$, *, $\rightarrow$} reduct of a unique UL-algebra **A**, and evaluations on **A** are related to forcing on **X** by conditions (iii) and (iv) of Proposition 3.7. Thus a formula is valid in every model on **X** iff it is valid on **A** under any evaluation. Therefore, by Proposition 3.7, in order to prove (i), it suffices to show that a linearly ordered UL-algebra is an L-algebra iff it satisfies the corresponding frame properties, and similarly for (ii).

As an example, we prove that **X** $\vDash$ (RDiv) iff **X** satisfies $(RDiv)_F$. For this, we consider a linearly ordered UL-algebra **A** defined as in Proposition 3.11 (ii). For the right-to-left direction, let $y \leq u$, and $t \leq \top \to x$ or $t \leq x \to y$ or $t \leq y \to (x * (x \to y))$. Then, $t \leq \top \to x$ or $t \leq x \to (y \wedge u)$ or $t \leq y \to (x * (x \to y))$. If $y > u$, analogously we have $t \leq \top \to x$ or $t \leq x \to (y \wedge u)$ or $t \leq y \to (x * (x \to y))$. Hence, it holds that $t \leq \top \to x$ or $t \leq x \to (y \wedge u)$ or $t \leq y \to (x * (x \to y))$. Then, by Lemma 3.9 and ($\vee$), it holds true that **A** $\models$ (RDiv) and so **X** $\models$ (RDiv), as wished. For the left-to-right direction, let $y \leq u$, and $t \nleq \top \to x$ and $t \nleq x \to y$ and $t \nleq y \to (x * (x \to y))$. Consider $t \nleq \top \to x$. Then, $t > \top \to x$. Define $y \Vdash p$ by $y \leq z = t$. Let p be the propositional variable whose evaluation is x. Then, $z \Vdash \mathbf{t}$ and $z \nVdash \mathbf{T} \to p$ and so $t \nVdash \mathbf{T} \to p$. Therefore, it holds that $t \nVdash \mathbf{T} \to \phi$. Similarly, it holds that $t \nVdash \phi \to (\psi \wedge u))$ and $t \nVdash \psi \to (\phi \,\&\, (\phi \to \psi))$. Hence, it holds that $t \nVdash (\mathbf{T} \to \phi) \vee (\phi \to (\psi \wedge u)) \vee (\psi \to (\phi \,\&\, (\phi \to \psi)))$. Otherwise, analogously it also holds that $t \nVdash (\mathbf{T} \to \phi) \vee (\phi \to (\psi \wedge u)) \vee (\psi \to (\phi \,\&\, (\phi \to \psi)))$. Thus, **A** $\not\models$ (RDiv) and so **X** $\not\models$ (RDiv), as required.

The proofs for the other cases are left to the interested reader. □

Corollary 4.3 (Soundness) If $\vdash_L \phi$, then ϕ is valid in every L frame.

Analogously algebraic Kripke semantics and algebraic semantics for L can be connected as in Proposition 3.11.

Theorem 4.4 (Completeness)

(i) L is (finitely) strong complete w.r.t. the class of all L frames.

(ii) L (except **IUL**) is weakly complete w.r.t. the class of all complete L frames.

(iii) L (except **IUL**) is (finitely) strong complete w.r.t. the class of all complete L frames.

Proof: (i) to (iii) follow from Proposition 3.11 and Theorem 2.15. □

5. Concluding remark

We investigated algebraic Kripke-style semantics for substructural fuzzy logics based on uninorms, so called uninorm-based logics. We first introduced the general framework of uninorm-based logics and particular infinite-valued logics as **UL** and its extensions, and the corresponding algebraic semantics. Next, we introduced various types of general and particular algebraic Kripke-style semantics for the uninorm-based logics and connect them with algebraic semantics.

We investigated here just algebraic Kripke-style semantics for uninorm-based logics. Note that, while Yang provided algebraic semantics for **MIAL** and its extensions in Yang (2015). we did not consider algebraic Kripke-style semantics for these systems. To provide such semantics for these logics remains a problem to be solved.

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준구조 퍼지 논리를 위한 대수적 크립키형 의미론

양 은 석

이 글에서 우리는 유니놈에 기반한 퍼지 논리를 위한 대수적 크립키형 의미론을 다룬다. 이를 위하여 먼저 유니놈에 기반한 논리 체계들을 위한 대수적 의미론을 재고한다. 다음으로 유니놈에 기반한 체계들의 일반적 구조에서 다양한 종류의 일반적 대수적 크립키형 의미론을 소개하고 그것들을 대수적 의미론과 연관 짓는다. 마지막으로 우리는 유사하게 특수한 대수적 의미론을 소개하고 이를 또한 대수적 의미론과 연관 짓는다.

주요어: (대수적) 크립키형 의미론, 대수적 의미론, 준구조 논리, 다치 논리, 퍼지 논리