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DIFFERENTIAL GEOMETRIC PROPERTIES ON THE HEISENBERG GROUP

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Abstract. In this paper, we show that there exists no left invariant Riemannian metric h on the Heisenberg group H such that (H, h) is a symmetric Riemannian manifold, and there does not exist any H-invariant metric $\bar{h}$ on the Heisenberg manifold H/Γ such that the Riemannian connection on $(H/\Gamma, \bar{h})$ is a Yang-Mills connection. Moreover, we get necessary and sufficient conditions for a group homomorphism of $(SU(2), g)$ with an arbitrarily given left invariant metric g into (H, h) with an arbitrarily given left invariant metric h to be a harmonic and an affine map, and get the totality of harmonic maps of $(SU(2), g)$ into H with a left invariant metric, and then show the fact that any affine map of $(SU(2), g)$ into H, equipped with a properly given left invariant metric on H, does not exist.

1. Introduction

In this paper, we study various differential geometric properties on the Heisenberg group H with an arbitrarily given left invariant Riemannian metric h. And then, we show that there does not exist any H-invariant metric such that the Riemannian connection for a H-invariant metric on the Heisenberg manifold H/Γ, which is locally isomorphic to H, is a Yang-Mills connection, where Γ is the discrete subgroup (of the Heisenberg group H) with integer entries. Moreover, we get necessary and sufficient conditions for a group homomorphism of $SU(2)$ into H to be a harmonic and an affine map.

In §2, we show that there exists no left invariant metric h on the Heisenberg group H such that (H, h) is a symmetric Riemannian manifold (cf. Theorem 2.4). And, a complete estimation of the Ricci curvature on (H, h) with an arbitrarily given left invariant Riemannian metric h is given (cf. Proposition 2.5).

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In §3, we get the fact (cf. Theorem 3.1) that there does not exist any H-invariant metric $\bar{h}$ on the Heisenberg manifold H/Γ such that the Riemannian connection on $(H/\Gamma, \bar{h})$ becomes a Yang-Mills connection, where Π is the natural projection of H into H/Γ and h is a left invariant metric on H.

In §4, we obtain a necessary and sufficient condition for a group homomorphism of $(SU(2), g)$, with an arbitrarily given left invariant Riemannian metric g on $SU(2)$, into the Heisenberg group (H, h), with an arbitrarily given left invariant Riemannian metric h on H, to be a harmonic map (cf. Theorem 4.4). Using this result, we get a necessary and sufficient condition for a group homomorphism of $(SU(2), g)$ into H with a properly given left invariant metric to be harmonic (cf. Corollary 4.6). And, as a by-product, we obtain the totality of harmonic maps of $(SU(2), g)$ into the Heisenberg group H, with the given left invariant metric (cf. Theorem 4.11). Moreover, we get a necessary and sufficient condition for a group homomorphism of $(SU(2), g)$ into the Heisenberg group (H, h) to be an affine map, (cf. Theorem 4.7). And then, using Theorem 4.7, we show the fact that any affine map from $(SU(2), g)$ into H, equipped with the given left invariant metric on H, does not exist (cf. Proposition 4.12).

2. Heisenberg group (H, h) with a left invariant metric h

Let H be the Heisenberg group (cf. [7, 12]), that is,

$$H = \left\{ \begin{pmatrix} 1 & a_{12} & a_{13} \\ 0 & 1 & a_{23} \\ 0 & 0 & 1 \end{pmatrix} \mid a_{12}, a_{23}, a_{13} \in \mathbb{R} \right\}. \tag{2.1}$$

Denote by $x,\ y,\ z$ coordinates on H, say for $A \in H$, $x(A) = a_{12}$, $y(A) = a_{23}$, $z(A) = a_{13}$. If L_B is the left translation by an element $B \in H$, we have

$$L_B^*\, dx = dx, \quad L_B^*\, dy = dy, \quad L_B^*\, (dz - xdy) = dz - xdy. \tag{2.2}$$

On H, the vector fields

$$\mathbf{v}_1 := \frac{\partial}{\partial x}, \quad \mathbf{v}_2 := \frac{\partial}{\partial y} + x\frac{\partial}{\partial z}, \quad \mathbf{v}_3 := \frac{\partial}{\partial z} \tag{2.3}$$

are dual to $dx,\ dy,\ dz - xdy$, and are left invariant. We denote by $\mathfrak{h}$ the Lie algebra of all left invariant vector fields on H. Now we take an inner product $<\ ,\ >_0$ on $\mathfrak{h}$ such that $\{\mathbf{v}_a\}_{a=1}^3$ is an orthonormal basis on $\mathfrak{h}$. Let the left invariant metric on H associated with the inner product $<\ ,\ >_0$ be h_0. Then, the Riemannian manifold (H, h_0) is referred to as the *Heisenberg Riemannian Lie group.*

In general, the Riemannian connection ∇ on a Riemannian manifold (M, g) is given by

$$\begin{aligned} 2g(\nabla_X Y, Z) &= Xg(Y, Z) + Yg(X, Z) - Zg(X, Y) \\ &\quad + g([X, Y], Z) + g([Z, X], Y) \\ &\quad - g([Y, Z], X) \qquad (X, Y, Z \in \mathfrak{X}(M)). \end{aligned} \tag{2.4}$$

Let $<\,,\,>$ be an arbitrarily given inner product on the Lie algebra $\mathfrak{h}$. Then, we can put

$$
\begin{aligned}
&< \mathbf{v}_a, \mathbf{v}_a >=: {k_a}^2 \ (a = 1, 2, 3), \quad && < \mathbf{v}_1, \mathbf{v}_2 >=: k_1 k_2 \cos\varphi_1, \\
&< \mathbf{v}_2, \mathbf{v}_3 >=: k_2 k_3 \cos\varphi_2, && < \mathbf{v}_3, \mathbf{v}_1 >=: k_3 k_1 \cos\varphi_3,
\end{aligned} \tag{2.5}
$$

where each k_a is positive constant and $0 < \varphi_1, \varphi_2, \varphi_3 < \pi$. So, the space of all left invariant Riemannian metrics on H is given by

$$
\begin{aligned}
&\{(k_1, k_2, k_3, \varphi_1, \varphi_2, \varphi_3) \mid \text{all } k_a > 0,\ 0 < \varphi_1, \varphi_2, \varphi_3 < \pi, \\
&\qquad (1 - \cos^2\varphi_1 - \cos^2\varphi_2 - \cos^2\varphi_3 + 2\cos\varphi_1\cos\varphi_2\cos\varphi_3) > 0\}.
\end{aligned}
$$

Now we normalize left invariant Riemannian metrics by putting $k_1 = 1$, and put

$$
\begin{aligned}
\mathcal{M} := \{&(1, k_2, k_3, \varphi_1, \varphi_2, \varphi_3) \mid k_2 > 0,\ k_3 > 0,\ 0 < \varphi_1, \varphi_2, \varphi_3 < \pi, \\
&(1 - \cos^2\varphi_1 - \cos^2\varphi_2 - \cos^2\varphi_3 + 2\cos\varphi_1\cos\varphi_2\cos\varphi_3) > 0\},
\end{aligned} \tag{2.6}
$$

and

$$
\mathcal{A} := \{(1, k_2, k_3, \varphi_1 = \pi/2, \varphi_2 = \pi/2, \varphi_3 = \pi/2) \in \mathcal{M} \mid k_2 > 0,\ k_3 > 0\}. \tag{2.7}
$$

Here, $\mathcal{A}$ is the family of all orthogonal bases with same orientations in the usual Euclidean space $\mathbb{R}^3$.

Let $<\,,\,>$ be an arbitrarily given inner product on the Lie algebra $\mathfrak{h}$ which is determined by $(k_1 = 1, k_2, k_3, \varphi_1, \varphi_2, \varphi_3)$ which belongs to $\mathcal{M}$. Let h be the left invariant Riemannian metric on H which is induced by the inner product $<\,,\,>$ on $\mathfrak{h}$. For simplicity, here and from now on we use the following notations:

$$
\begin{aligned}
&h_{ab} := h(\mathbf{v}_a, \mathbf{v}_b), \\
&\lambda := (1 - \cos^2\varphi_1 - \cos^2\varphi_2 - \cos^2\varphi_3 + 2\cos\varphi_1\cos\varphi_2\cos\varphi_3)^{\frac{1}{2}} \ (> 0).
\end{aligned} \tag{2.8}
$$

Then

$$
|(h_{ab})_{a,b}|^{\frac{1}{2}} = k_2 k_3 \lambda. \tag{2.9}
$$

Putting

$$
\begin{aligned}
&\mathbf{d}_1 := \mathbf{v}_1, \quad \mathbf{d}'_2 := \mathbf{v}_2 - < \mathbf{v}_1, \mathbf{v}_2 > \mathbf{v}_1, \quad \mathbf{d}_2 := < \mathbf{d}'_2, \mathbf{d}'_2 >^{-1/2} \mathbf{d}'_2, \\
&\mathbf{d}'_3 := \mathbf{v}_3 - < \mathbf{d}_1, \mathbf{v}_3 > \mathbf{d}_1 - < \mathbf{d}_2, \mathbf{v}_3 > \mathbf{d}_2, \quad \mathbf{d}_3 := < \mathbf{d}'_3, \mathbf{d}'_3 >^{-1/2} \mathbf{d}'_3,
\end{aligned} \tag{2.10}
$$

we have an orthonormal frame

$$
\{\mathbf{d}_1,\ \mathbf{d}_2,\ \mathbf{d}_3\} \tag{2.11}
$$

on (H, h). We have from (2.5) and (2.10)

$$
\|\mathbf{d}'_2\|_h = k_2 \sin\varphi_1, \quad \|\mathbf{d}'_3\|_h = k_3 \lambda (\sin\varphi_1)^{-1}. \tag{2.12}
$$

By the help of (2.5), (2.10) and (2.12), we obtain

$$
\begin{aligned}
&\mathbf{v}_1 = \mathbf{d}_1, \quad \mathbf{v}_2 = k_2 \cos\varphi_1 \mathbf{d}_1 + k_2 \sin\varphi_1 \mathbf{d}_2, \\
&\mathbf{v}_3 = k_3 \cos\varphi_3 \mathbf{d}_1 + k_3(\cos\varphi_2 - \cos\varphi_3\cos\varphi_1)(\sin\varphi_1)^{-1}\mathbf{d}_2 \\
&\qquad + k_3 \lambda (\sin\varphi_1)^{-1} \mathbf{d}_3.
\end{aligned} \tag{2.13}
$$

By virtue of (2.3), (2.5), (2.10), (2.12) and (2.13), we get

$$
\begin{aligned}
[\mathbf{d}_1, \mathbf{d}_2] &= k_2^{-1} k_3 \{(\sin\varphi_1)^{-1} \cos\varphi_3 \mathbf{d}_1 \\
&\qquad + (\cos\varphi_2 - \cos\varphi_3 \cos\varphi_1)(\sin\varphi_1)^{-2} \mathbf{d}_2 + \lambda(\sin\varphi_1)^{-2} \mathbf{d}_3\}, \\
[\mathbf{d}_2, \mathbf{d}_3] &= k_2^{-1} k_3 \cos\varphi_3 \{\lambda^{-1} \cos\varphi_3 \mathbf{d}_1 \\
&\qquad + (\lambda \sin\varphi_1)^{-1} (\cos\varphi_2 - \cos\varphi_3 \cos\varphi_1) \mathbf{d}_2 + (\sin\varphi_1)^{-1} \mathbf{d}_3\}, \\
[\mathbf{d}_3, \mathbf{d}_1] &= k_3 (k_2 \lambda \sin\varphi_1)^{-1} (\cos\varphi_2 - \cos\varphi_3 \cos\varphi_1) \\
&\qquad \{\cos\varphi_3 \mathbf{d}_1 + (\sin\varphi_1)^{-1} (\cos\varphi_2 - \cos\varphi_3 \cos\varphi_1) \mathbf{d}_2 \\
&\qquad\qquad + \lambda(\sin\varphi_1)^{-1} \mathbf{d}_3\}.
\end{aligned}
\tag{2.14}
$$

Let ∇ be the Riemannian connection on (H, h). And, let R be the curvature tensor field on (H, h), that is,

$$
R(X, Y)Z = ([\nabla_X, \nabla_Y] - \nabla_{[X,Y]})(Z) \quad (X, Y, Z \in \mathfrak{X}(H)).
$$

From (2.4) and (2.14), we get

$$
\begin{aligned}
\nabla_{\mathbf{d}_1} \mathbf{d}_1 &= k_3 \cos\varphi_3 \ (k_2 \sin\varphi_1)^{-1} \\
&\qquad \{-\mathbf{d}_2 + \lambda^{-1} (\cos\varphi_2 - \cos\varphi_3 \cos\varphi_1) \mathbf{d}_3\}, \\
\nabla_{\mathbf{d}_1} \mathbf{d}_2 &= k_2^{-1} k_3 \{(\sin\varphi_1)^{-1} \cos\varphi_3 \mathbf{d}_1 + (2\lambda)^{-1} (\sin^2\varphi_3 - \cos^2\varphi_3) \mathbf{d}_3\}, \\
\nabla_{\mathbf{d}_1} \mathbf{d}_3 &= (2k_2\lambda)^{-1} \{2k_3 (\sin\varphi_1)^{-1} \cos\varphi_3 (\cos\varphi_3 \cos\varphi_1 - \cos\varphi_2) \mathbf{d}_1 \\
&\qquad + k_3 (\cos^2\varphi_3 - \sin^2\varphi_3) \mathbf{d}_2\}, \\
\nabla_{\mathbf{d}_2} \mathbf{d}_2 &= k_3 (k_2 \sin\varphi_1)^{-1} (\cos\varphi_2 - \cos\varphi_3 \cos\varphi_1) \\
&\qquad \{(\sin\varphi_1)^{-1} \mathbf{d}_1 - \lambda^{-1} \cos\varphi_3 \mathbf{d}_3\}, \\
\nabla_{\mathbf{d}_2} \mathbf{d}_3 &= k_3 (k_2 \lambda \sin\varphi_1)^{-1} \\
&\qquad \{(2\sin\varphi_1)^{-1} (2\lambda^2 + \sin^2\varphi_1 \cos^2\varphi_3 - \sin^2\varphi_1 \sin^2\varphi_3) \mathbf{d}_1 \\
&\qquad\qquad + \cos\varphi_3 (\cos\varphi_2 - \cos\varphi_3 \cos\varphi_1) \mathbf{d}_2\}, \\
\nabla_{\mathbf{d}_3} \mathbf{d}_3 &= k_3 (k_2 \sin\varphi_1)^{-1} \\
&\qquad \{(\sin\varphi_1)^{-1} (\cos\varphi_3 \cos\varphi_1 - \cos\varphi_2) \ \mathbf{d}_1 + \cos\varphi_3 \mathbf{d}_2\}.
\end{aligned}
\tag{2.15}
$$

Putting $h(R(\mathbf{d}_a, \mathbf{d}_b)\mathbf{d}_c, \mathbf{d}_e) =: R^e_{cab}$, we have from (2.15)

$$
\begin{aligned}
R^1_{212} &= {k_3}^2 (4\ {k_2}^2 \lambda^2 \sin^2\varphi_1)^{-1} \\
&\qquad (-3 + 3\cos^2\varphi_1 + 4\cos^2\varphi_2 + 4\cos^2\varphi_3 - 8\cos\varphi_1 \cos\varphi_2 \cos\varphi_3), \\
R^1_{313} &= {k_3}^2 (4\ {k_2}^2 \lambda^2 \sin^2\varphi_1)^{-1} \\
&\qquad \{\sin^2\varphi_1 - 4(\cos\varphi_2 - \cos\varphi_3 \cos\varphi_1)^2\},
\end{aligned}
\tag{2.16}
$$

$$R^2_{323} = {k_3}^2(4\ {k_2}^2\lambda^2 \sin^2\varphi_1)^{-1}$$
$$(4\cos^2\varphi_3\cos^2\varphi_1 - 4\cos^2\varphi_3 - \cos^2\varphi_1 + 1),$$
$$R^1_{223} = -{k_3}^2\cos\varphi_3\ ({k_2}^2\lambda\sin\varphi_1)^{-1},$$
$$R^1_{231} = {k_3}^2(\cos\varphi_3\cos\varphi_1 - \cos\varphi_2)({k_2}^2\ \lambda\sin^2\varphi_1)^{-1},$$
$$R^3_{123} = {k_3}^2\cos\varphi_3(\cos\varphi_3\cos\varphi_1 - \cos\varphi_2)({k_2}^2\lambda^2\sin\varphi_1)^{-1}.$$

Let $\rho(:= \rho_h)$ be the Ricci operator on (H, h), that is,

$$\rho(X,Y) = \sum_{a=1}^{3} h(R(X,\mathbf{d}_a)\mathbf{d}_a, Y) \quad (X, Y \in \mathfrak{X}(H)). \tag{2.17}$$

Putting $\rho(\mathbf{d}_a, \mathbf{d}_b) =: \rho_{ab}$ $(a, b = 1, 2, 3)$, we obtain from (2.16) and (2.17)

$$\begin{aligned}
\rho_{11} &= {k_3}^2(\cos^2\varphi_3 - \sin^2\varphi_3)(2\ {k_2}^2\lambda^2)^{-1},\\
\rho_{22} &= {k_3}^2\{\sin^2\varphi_1(2\sin^2\varphi_3 - 1) - 2\lambda^2\}(2{k_2}^2\ \lambda^2\sin^2\varphi_1)^{-1},\\
\rho_{33} &= {k_3}^2(2\lambda^2 - \sin^2\varphi_1)(2{k_2}^2\lambda^2\sin^2\varphi_1)^{-1},\\
\rho_{12} &= {k_3}^2\cos\varphi_3(\cos\varphi_2 - \cos\varphi_3\cos\varphi_1)({k_2}^2\lambda^2\sin\varphi_1)^{-1},\\
\rho_{23} &= {k_3}^2(\cos\varphi_2 - \cos\varphi_3\cos\varphi_1)({k_2}^2\lambda\sin^2\varphi_1)^{-1},\\
\rho_{31} &= {k_3}^2\cos\varphi_3({k_2}^2\lambda\sin\varphi_1)^{-1}.
\end{aligned} \tag{2.18}$$

In general, the Riemannian metric g on a Riemannian manifold (M, g) is called *Einstein* if $\rho_g = c \cdot g$ for some constant c (cf. [14]), where ρ_g is the Ricci operator on (M, g).

In our situation, we get:

Proposition 2.1. *Let H be the Heisenberg Lie group. Then there exists no left invariant Einstein metric on H.*

Proof. Assume that (H, h) is an Einstein manifold, that is, $\rho_{ab} := \rho_h(\mathbf{d}_a, \mathbf{d}_b) = c \cdot h(\mathbf{d}_a, \mathbf{d}_b) = c \cdot \delta_{ab}$ $(a, b = 1, 2, 3)$ for some constant c. Then, from this assumption, (2.6) and $\rho_{31} = {k_3}^2\cos\varphi_3({k_2}^2\lambda\sin\varphi_1)^{-1}$ appeared in (2.18), we obtain

$$\varphi_3 = \frac{\pi}{2}. \tag{2.19}$$

By the help of (2.8), (2.18) and (2.19), we have

$$\rho_{22} \cdot \rho_{33} < 0, \tag{2.20}$$

which contradicts the fact that $\rho_{22} = \rho_{33} = c$. So, we obtain this proposition. □

Remark 2.2. It is well known that *three dimensional Einstein manifold is a space of constant curvature* (cf. [4, p. 293]). Wolf (cf. [15]) showed the fact that *three dimensional nilpotent Lie group is not a constant curvature space.*

Proposition 2.1 follows from these facts. More generally, the following theorem (cf. [6, Theorem 2.4]) is well known;

Let G be a nilpotent Lie group. Then, there does not exist any left invariant Einstein metric on G.

The above Proposition 2.1 also follows from this theorem.

In general, Riemannian manifold (M, g) is said to be *symmetric* (resp. *locally symmetric*) at $x \in M$, if there exists an involutive isometry of M (resp. an open neighborhood U of x) which has x as an isolated fixed point. And (M, g) is said to be *symmetric* (resp. *locally symmetric*), if it is symmetric (resp. locally symmetric) at every point of M. Moreover the following theorem (cf. [4, Theorem 3, p. 303]) is well known.

Theorem 2.3. *Let ∇ and R be the Riemannian connection and the curvature tensor field on a Riemannian manifold (M, g), respectively. Then (M, g) is locally symmetric if and only if $\nabla R = 0$.*

In our situation, we obtain:

Theorem 2.4. *Let h be an arbitrarily given left invariant Riemannian metric on the Heisenberg group H. Then the Riemannian manifold (H, h) is not (locally) symmetric. That is, there does not exist any left invariant Riemannian metric h on H such that the Riemannian manifold (H, h) is symmetric.*

Proof. From (2.15) and (2.16), we get

$$\begin{aligned} & h((\nabla_{\mathbf{d}_1} R)(\mathbf{d}_1, \mathbf{d}_3)\mathbf{d}_2, \mathbf{d}_1) \\ = & - {k_3}^3 (2\, {k_2}^3 \lambda^3 \sin^6 \varphi_1)^{-1} [\lambda^2 (\lambda^2 + \sin^2 \varphi_1 \cos^2 \varphi_3)^2 \\ & + \mu^2 \{\lambda^2(\lambda^2 - \mu^2) - (\mu^2 + \sin^2 \varphi_1 \cos^2 \varphi_3)^2\}], \\ & h((\nabla_{\mathbf{d}_1} R)(\mathbf{d}_1, \mathbf{d}_2)\mathbf{d}_2, \mathbf{d}_1) \\ = & - {k_3}^3 \mu ({k_2}^3 \lambda^2 \sin^2 \varphi_1)^{-1}. \end{aligned} \tag{2.21}$$

Here and from now on, we put $\mu := (\cos \varphi_2 - \cos \varphi_3 \cos \varphi_1)$.

We assume that (H, h) is symmetric, that is, $\nabla R = 0$. Then from the second formula of (2.21), we have

$$\mu = 0. \tag{2.22}$$

By virtue of (2.22) and the first formula of (2.21), we obtain

$$h((\nabla_{\mathbf{d}_1} R)(\mathbf{d}_1, \mathbf{d}_3)\mathbf{d}_2, \mathbf{d}_1) < 0, \tag{2.23}$$

which is absurd since $\nabla R = 0$ by the assumption. So, we get this theorem. $\square$

In general, for the Ricci curvature tensor field *Ric* of (0,2)-type in a Riemannian manifold (M, g) and a nonzero vector $\mathbf{x}_p \in T_p(M)$,

$$r(\mathbf{x}_p) := \frac{Ric(\mathbf{x}_p, \mathbf{x}_p)}{\|\mathbf{x}_p\|_g^2}.$$

is said to be the *Ricci curvature* of (M, g) with respect to $\mathbf{x}_p$.

In our situation, we get:

Proposition 2.5. *Assume that h is the left invariant Riemannian metric on H which is defined by $(k_1 = 1, k_2, k_3, \varphi_1 = \frac{\pi}{2}, \varphi_1 = \frac{\pi}{2}, \varphi_2 = \frac{\pi}{2}, \varphi_3 = \frac{\pi}{2}) \in \mathcal{A}$. Let X be an arbitrarily given nonzero vector field which is left invariant on H. Then the Ricci curvature $r(X)$ is completely estimated as follows*:

$$\frac{-{k_3}^2}{2{k_2}^2} \ \leq \ r(X) \ \leq \ \frac{{k_3}^2}{2{k_2}^2}.$$

And the scalar curvature S_h on the manifold (H, h) is $(-{k_3}^2)/(2{k_2}^2)$.

Proof. By the help of the assumption, we have from (2.5) and (2.8)

$$\lambda = 1, \quad h(\mathbf{v}_a, \mathbf{v}_b) = \delta_{ab}{k_a}^2 \quad (a, b = 1, 2, 3). \tag{2.24}$$

From (2.18) and (2.24), we obtain

$$\rho_{11} = \rho_{22} = \frac{-{k_3}^2}{2{k_2}^2}, \ \rho_{33} = \frac{{k_3}^2}{2{k_2}^2}, \ \rho_{12} = \rho_{23} = \rho_{31} = 0. \tag{2.25}$$

Using (2.25), we get the scalar curvature

$$S_h := \rho_{11} + \rho_{22} + \rho_{33} = \frac{-{k_3}^2}{2{k_2}^2}. \tag{2.26}$$

Moreover, by virtue of (2.25) we obtain the fact that

$$\frac{-{k_3}^2}{2{k_2}^2} \ \leq \ r(X) \ \leq \ \frac{{k_3}^2}{2{k_2}^2} \tag{2.27}$$

for an arbitrarily given left invariant vector field $X(\neq 0)$ on H. □

3. Yang-Mills Riemannian connections on the Heisenberg Riemannian manifold H/Γ

We retain the notations as in Section 2.

Let Γ be the discrete subgroup (of the Heisenberg group H) with integer entries. We put $M =: H/\Gamma$. We refer to M as the *Heisenberg manifold.* For $B \in H$, $L_B^* dx = dx$, $L_B^* dy = dy$, $L_B^*(dz - xdy) = dz - xdy$. In particular, these forms are invariant under the action of Γ; let $\Pi : H \to M = H/\Gamma$ be the natural projection, then there exist 1-forms μ_1, μ_2 and μ_3 on M such that $\Pi^\star(\mu_1) = dx$, $\Pi^\star(\mu_2) = dy$, $\Pi^\star(\mu_3) = dz - x\,dy$. For each left invariant metric $h(\in \mathcal{M})$ on H, there exists $\bar{h}$ on M such that $\Pi^\star \bar{h} = h$.

From now on, all the calculations on $(M := H/\Gamma, \bar{h})$ will be done on H and its Lie algebra $\mathfrak{h}$. In fact, because M is a homogeneous space, the curvature is the same in all its points, and M is locally isomorphic to H (cf. [7, 11]).

In general, a Yang-Mills connection is a critical point of the Yang-Mills functional

$$\mathcal{YM}(D) = \frac{1}{2}\int_M \|R^D\|^2 \; v_g \tag{3.1}$$

on the space $\mathcal{C}_E$ of all connections in a smooth vector bundle E over a closed (compact and connected) Riemannian manifold (M, g), where R^D is the curvature of $D \in \mathcal{C}_E$. Equivalently, D is a Yang-Mills connection if it satisfies the Yang-Mills equation (cf. [8, 9, 10])

$$\delta_D R^D = 0, \tag{3.2}$$

(the Euler-Lagrange equations of the variational principle associated with (3.1)) where δ_D is the formal adjoint of the covariant exterior differential operator d_D which is defined by $D \in \mathcal{C}_E$. Let ∇ be the Riemannian connection on an n-dimensional Riemannian manifold (M, g). We find from (3.1) and (3.2) that the Riemannian connection ∇ in the tangent bundle TM over the manifold (M, g) is a Yang-Mills connection if and only if (cf. [8, 9, 10])

$$(\delta_\nabla R^\nabla)(X)Y = -\sum_{i=1}^{n} (\nabla_{e_i} R^\nabla)(e_i, X)Y = 0 \tag{3.3}$$

for arbitrarily given $X, Y \in \mathfrak{X}(M)$ and a (locally defined) orthonormal frame on (M, g).

In our situation, we obtain:

Theorem 3.1. *Let $h(\in \mathcal{M})$ be an arbitrarily given left invariant Riemannian metric on H. Then, the Riemannian connection ∇ in the tangent bundle TM over $(M := H/\Gamma, \bar{h})$ is not a Yang-Mills connection. In other words, there exists no H-invariant Riemannian connection which is a Yang-Mills connection in the tangent bundle over the Heisenberg manifold $M(= H/\Gamma)$.*

Proof. From (2.15) and (2.16), we get

$$\begin{aligned} &h((\nabla_{\mathbf{d}_2} R)(\mathbf{d}_2, \mathbf{d}_1)\mathbf{d}_2, \mathbf{d}_1) = -{k_3}^3 \cos\varphi_3 ({k_2}^3 \lambda^2 \sin\varphi_1)^{-1}, \\ &h((\nabla_{\mathbf{d}_3} R)(\mathbf{d}_3, \mathbf{d}_1)\mathbf{d}_2, \mathbf{d}_1) = -{k_3}^3 \cos\varphi_3 (2{k_2}^3 \lambda^2 \sin\varphi_1)^{-1}. \end{aligned} \tag{3.4}$$

By the help of (3.4), we obtain

$$\sum_{a=1}^{3} h((\nabla_{\mathbf{d}_a} R)(\mathbf{d}_a, \mathbf{d}_1)\mathbf{d}_2, \mathbf{d}_1) = -3{k_3}^3 \cos\varphi_3 (2{k_2}^3 \lambda^2 \sin\varphi_1)^{-1}. \tag{3.5}$$

And, from (2.15) and (2.16), we have

$$\begin{aligned} &h((\nabla_{\mathbf{d}_1} R)(\mathbf{d}_1, \mathbf{d}_2)\mathbf{d}_2, \mathbf{d}_1) \\ = &-{k_3}^3 (\cos\varphi_2 - \cos\varphi_3 \cos\varphi_1)({k_2}^3 \lambda^2 \sin^2\varphi_1)^{-1}, \\ &h((\nabla_{\mathbf{d}_3} R)(\mathbf{d}_3, \mathbf{d}_2)\mathbf{d}_2, \mathbf{d}_1) \\ = &-{k_3}^3 (\cos\varphi_2 - \cos\varphi_3 \cos\varphi_1)(2{k_2}^3 \lambda^2 \sin^2\varphi_1)^{-1}. \end{aligned} \tag{3.6}$$

We have from (3.6)

$$\begin{aligned}\sum_{a=1}^{3} &h((\nabla_{\mathbf{d}_a} R)(\mathbf{d}_a, \mathbf{d}_2)\mathbf{d}_2, \mathbf{d}_1) \\ &= -3{k_3}^3(\cos\varphi_2 - \cos\varphi_3\cos\varphi_1)(2{k_2}^3\lambda^2\sin^2\varphi_1)^{-1}.\end{aligned} \tag{3.7}$$

Using $\lambda^2 + \mu^2 = \sin^2\varphi_1 \sin^2\varphi_3$, we have from (2.15) and (2.16)

$$\begin{aligned}&h((\nabla_{\mathbf{d}_1} R)(\mathbf{d}_1, \mathbf{d}_3)\mathbf{d}_2, \mathbf{d}_1) \\ &= {k_3}^3(2{k_2}^3\lambda^3\sin^2\varphi_1)^{-1}\{\mu^2 - \lambda^2\cos^2\varphi_3(\sin^2\varphi_3 + 1) - \lambda^2\sin^4\varphi_3\}, \\ &h((\nabla_{\mathbf{d}_2} R)(\mathbf{d}_2, \mathbf{d}_3)\mathbf{d}_2, \mathbf{d}_1) \\ &= {k_3}^3(2{k_2}^3\lambda^3\sin^2\varphi_1)^{-1}(\sin^2\varphi_1\cos^2\varphi_3 - \lambda^2).\end{aligned} \tag{3.8}$$

From (3.8), we get

$$\begin{aligned}\sum_{a=1}^{3} &h((\nabla_{\mathbf{d}_a} R)(\mathbf{d}_a, \mathbf{d}_3)\mathbf{d}_2, \mathbf{d}_1) \\ &= {k_3}^3(2{k_2}^3\lambda^3\sin^2\varphi_1)^{-1}(\mu^2 + \sin^2\varphi_1\cos^2\varphi_3 - 2\lambda^2).\end{aligned} \tag{3.9}$$

Now, we assume that the Riemannian connection ∇ on $(M := H/\Gamma, \bar{h})$ is a Yang-Mills connection, that is,

$$\delta_\nabla R (:= \delta_\nabla R^\nabla) = 0. \tag{3.10}$$

Then, from (2.6), (3.5) and (3.7), we have

$$\varphi_2 = \varphi_3 = \frac{\pi}{2}, \quad \mu(:= \cos\varphi_2 - \cos\varphi_3\cos\varphi_1) = 0. \tag{3.11}$$

By virtue of (3.9) and (3.11), we get

$$\sum_{a=1}^{3} h((\nabla_{\mathbf{d}_a} R)(\mathbf{d}_a, \mathbf{d}_3)\mathbf{d}_2, \mathbf{d}_1) = h(-(\delta_\nabla R)(\mathbf{d}_3)\mathbf{d}_2, \mathbf{d}_1) < 0, \tag{3.12}$$

which contradicts the assumption that the connection ∇ is a Yang-Mills connection. Thus, we obtain this theorem. □

4. Harmonic and affine maps of $SU(2)$ into the Heisenberg group H

4.1. Harmonic and affine maps of a closed (compact and closed) Riemannian manifold into another Riemannian manifold

Harmonic maps of a compact Riemannian manifold (M, g) into another Riemannian manifold (N, h) are the extrema of the energy functional (cf. [1, 13])

$$E(\phi) = \frac{1}{2}\int_M \|d\phi\|^2 v_g, \tag{4.1}$$

where $\|d\phi\|$ is the norm of the differential $d\phi$ of a map $\phi \in C^\infty(M, N)$ with respect to the metrics g, h. Moreover, a necessary and sufficient condition (cf.

[1, 13]) for a map $\phi(\in C^\infty(M,N))$ to be a critical point of the energy functional E, (that is, to be a harmonic map), is

$$\tau(\phi) := \sum_{i=1}^{n} \left(\tilde{\nabla}_{\mathbf{e}_i}\phi_*\mathbf{e}_i - \phi_*{}^g\nabla_{\mathbf{e}_i}\mathbf{e}_i\right) = 0 \ \text{ everywhere on } M. \tag{4.2}$$

Here, ${}^g\nabla$ is the Levi-Civita connection on (M,g), and $\tilde{\nabla}$ is the induced connection in the bundle $\phi^{-1}TN = \{(x,Y) \mid x \in M,\ Y \in T_{\phi(x)}N\}$ over the n-dimensional manifold M which is induced by the map ϕ and the Riemannian connection ${}^h\nabla$ on (N,h), $\tau(\phi)$ is an element of $\Gamma(M;\phi^{-1}TN)$ called the *tension field* of ϕ, and $\{\mathbf{e}_i\}_{i=1}^n$ is a (locally defined) orthonormal frame field on (M,g).

On the other hand, we call a smooth map ϕ of (M,g) into (N,h) an *affine map* (cf. [4, p. 225]) if the map ϕ maps each parallel vector field along each curve c of M into a parallel vector field along the curve $\phi\circ c$ of N. So, a necessary and sufficient condition for a map ϕ between (M,g) and (N,h) to be an affine map (cf. [3], [4, Proposition 1.2, p. 225]) is

$$\phi_\star({}^g\nabla_X Y) = \tilde{\nabla}_X\phi_\star Y \qquad (X, Y \in \mathfrak{X}(M)). \tag{4.3}$$

4.2. Harmonic and affine group homomorphisms

We retain the notations as in the subsection 4.1.

Let G be an n-dimensional closed (compact and connected) Lie group with an arbitrarily given left invariant metric g, and H an m-dimensional Lie group with an arbitrarily given left invariant metric h. Let $\mathfrak{g}$ (resp. $\mathfrak{h}$) be the Lie algebra of all left invariant vector fields on G (resp. H). Let $\phi: G \to H$ be a group homomorphism, $\{\mathbf{e}_i\}_{i=1}^n$ (resp. $\{\mathbf{d}_a\}_{a=1}^m$) an orthonormal basis of $(\mathfrak{g},g)$ (resp. $(\mathfrak{h},h)$). We use the following notations:

$$\begin{aligned} &(d\phi)(\mathbf{e}_i) =: \sum_{a=1}^m {\phi_i}^a\ \mathbf{d}_a, \\ &{}^g\nabla_{\mathbf{e}_i}\mathbf{e}_j =: \sum_{k=1}^n {\alpha_{ij}}^k\ \mathbf{e}_k, \quad {}^h\nabla_{\mathbf{d}_a}\mathbf{d}_b =: \sum_{c=1}^m {\beta_{ab}}^c\ \mathbf{d}_c. \end{aligned} \tag{4.4}$$

Here ${}^g\nabla$ (resp. ${}^h\nabla$) is the Levi-Civita connection on (G,g) (resp. (H,h)), and $d\phi$ $(=\phi_*)$ is the differential of the group homomorphism ϕ. From (4.4) we obtain

$$\begin{aligned} \tilde{\nabla}_{\mathbf{e}_i}\phi_*\mathbf{e}_j &= \sum_{a,b,c=1}^m {\phi_i}^a\ {\phi_j}^b\ {\beta_{ab}}^c\ \mathbf{d}_c, \\ \phi_*({}^g\nabla_{\mathbf{e}_i}\mathbf{e}_j) &= \sum_{k=1}^n\sum_{a=1}^m {\alpha_{ij}}^k\ {\phi_k}^a\ \mathbf{d}_a, \end{aligned} \tag{4.5}$$

since each ${\phi_i}^a$ is constant. By the help of (4.2) and (4.5), we obtain:

Proposition 4.1. *Let (G, g) be an n-dimensional closed Lie group with an arbitrarily given left invariant metric g on G, (H, h) an m-dimensional Lie group with an arbitrarily given left invariant metric h on H. Then a group homomorphism $\phi : (G, g) \to (H, h)$ is a harmonic map if and only if the tension field $\tau(\phi)$ vanishes, namely,*

$$\sum_{i=1}^{n} \left(\sum_{a,b=1}^{m} {\phi_i}^a \ {\phi_i}^b \ {\beta_{ab}}^c - \sum_{j=1}^{n} {\alpha_{ii}}^j \ {\phi_j}^c \right) = 0 \tag{4.6}$$

for all $c = 1, 2, \ldots, m$.

Similarly, by virtue of (4.3) and (4.5), we obtain:

Proposition 4.2. *Let (G, g) be an n-dimensional closed Lie group with an arbitrarily given left invariant metric g on G, (H, h) an m-dimensional Lie group with an arbitrarily given left invariant metric h on H. Then, a necessary and sufficient condition for a group homomorphism $\phi : (G, g) \to (H, h)$ to be an affine map is*

$$\phi_*({}^g\nabla_{\mathbf{e}_i}\mathbf{e}_j) = \tilde{\nabla}_{\mathbf{e}_i}\phi_*\mathbf{e}_j \quad (i, j = 1, 2, \ldots, n), \quad i.e.,$$

for each $c \in \{1, 2, \ldots, m\}$

$$\sum_{k=1}^{n} {\alpha_{ij}}^k {\phi_k}^c - \sum_{a,b=1}^{m} {\phi_i}^a {\phi_j}^b {\beta_{ab}}^c = 0 \quad (i, j = 1, 2, \ldots, n). \tag{4.7}$$

4.3. Left invariant Riemannian metrics on $SU(2)$

Let $\mathfrak{su}(2)$ be the Lie algebra of $SU(2)$. The Killing form B of $\mathfrak{su}(2)$ satisfies

$$B(X, Y) = 4\, Trace(XY) \quad (X, Y \in \mathfrak{su}(2)). \tag{4.8}$$

We define an inner product $< \, , \, >_0$ on $\mathfrak{su}(2)$ by

$$< X, Y >_0 := -B(X, Y) \quad (X, Y \in \mathfrak{su}(2)). \tag{4.9}$$

Here and from now on, let g be an arbitrarily given left invariant Riemannian metric on $SU(2)$. The following lemma is well known (cf. [3]).

Lemma 4.3. *Let g be a left invariant Riemannian metric on $SU(2)$. Let $< \, , \, >$ be an inner product on $\mathfrak{su}(2)$ defined by $< X, Y > := g_e(X_e, Y_e)$, where $X, Y \in \mathfrak{su}(2)$ and e is the identity matrix of $SU(2)$. Then there exists an orthonormal basis $\{X_1, X_2, X_3\}$ of $\mathfrak{su}(2)$ with respect to $< \, , \, >_0$ $(= -B)$ such that*

$$\begin{aligned} &[X_1, X_2] = (1/\sqrt{2})X_3, \qquad [X_2, X_3] = (1/\sqrt{2})X_1, \\ &[X_3, X_1] = (1/\sqrt{2})X_2, \qquad < X_i, X_j > = \delta_{ij}{a_i}^2, \end{aligned} \tag{4.10}$$

where a_i $(i = 1, 2, 3)$ are positive constants determined by the given left invariant Riemannian metric g on $SU(2)$.

4.4. Harmonic group homomorphisms of $SU(2)$ into the Heisenberg group H

We retain the notations as in §2 and the subsection 4.3. We fix an orthonormal basis $\{X_1, X_2, X_3\}$ of $\mathfrak{su}(2)$ with respect to $<\ ,\ >_0$ satisfying (4.10) in Lemma 4.3, and denote by $g_{(a_1,a_2,a_3)}$ the left invariant Riemannian metric on $SU(2)$ which is determined by positive real numbers a_1, a_2, a_3 in Lemma 4.3. Moreover, we normalize left invariant Riemannian metrics on $SU(2)$ by putting $a_1 = 1$. We denote by $g_{(1,a_2,a_3)}$, or simply by $g_{(a_2,a_3)}(=: g)$, the left invariant Riemannian metric which is determined by positive real numbers $a_1 = 1, a_2, a_3$.

For the orthonormal basis $\{X_1, X_2, X_3\}$ of $\mathfrak{su}(2)$ with respect to $<\ ,\ >_0 := -B$ in Lemma 4.3, if we put

$$\mathbf{e}_1 := X_1, \quad \mathbf{e}_2 := {a_2}^{-1} X_2, \quad \mathbf{e}_3 := {a_3}^{-1} X_3,$$

then $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ is an orthonormal frame on $(SU(2), g_{(a_2,a_3)})$. From (4.10), we have

$$\begin{aligned} &[\mathbf{e}_1, \mathbf{e}_2] = a_3(\sqrt{2}\ a_2)^{-1}\mathbf{e}_3, \quad [\mathbf{e}_2, \mathbf{e}_3] = (\sqrt{2}\ a_2\ a_3)^{-1}\mathbf{e}_1, \\ &[\mathbf{e}_3, \mathbf{e}_1] = a_2(\sqrt{2}\ a_3)^{-1}\mathbf{e}_2. \end{aligned} \tag{4.11}$$

By virtue of (2.4) and (4.11), we get

$$\begin{aligned} &{}^g\nabla_{\mathbf{e}_1}\mathbf{e}_2 = \frac{(a_2)^2 + (a_3)^2 - 1}{2\sqrt{2}\ a_2 a_3}\mathbf{e}_3, \quad {}^g\nabla_{\mathbf{e}_2}\mathbf{e}_3 = \frac{1 - (a_2)^2 + (a_3)^2}{2\sqrt{2}\ a_2 a_3}\mathbf{e}_1, \\ &{}^g\nabla_{\mathbf{e}_3}\mathbf{e}_1 = \frac{1 + (a_2)^2 - (a_3)^2}{2\sqrt{2}\ a_2 a_3}\mathbf{e}_2, \quad {}^g\nabla_{\mathbf{e}_i}\mathbf{e}_i = 0 \ \ (i = 1, 2, 3). \end{aligned} \tag{4.12}$$

Putting $g({}^g\nabla_{\mathbf{e}_i}\mathbf{e}_j, \mathbf{e}_k) =: {\alpha_{ij}}^k$, we have from (4.11) and (4.12)

$$\begin{aligned} &{\alpha_{12}}^3 = -{\alpha_{13}}^2 = \frac{(a_2)^2 + (a_3)^2 - 1}{2\sqrt{2}\ a_2 a_3}, \\ &{\alpha_{23}}^1 = -{\alpha_{21}}^3 = \frac{1 - (a_2)^2 + (a_3)^2}{2\sqrt{2}\ a_2 a_3}, \\ &{\alpha_{31}}^2 = -{\alpha_{32}}^1 = \frac{1 + (a_2)^2 - (a_3)^2}{2\sqrt{2}\ a_2 a_3}, \quad {\alpha_{ij}}^k = 0 \ \ \text{otherwise}. \end{aligned} \tag{4.13}$$

Similarly, putting $h({}^h\nabla_{\mathbf{d}_a}\mathbf{d}_b, \mathbf{d}_c) =: {\beta_{ab}}^c$ $(a, b, c = 1, 2, 3)$, from (2.15) we have the non-zero terms of ${\beta_{ab}}^c$

$$\begin{aligned} &{\beta_{11}}^2 = -{\beta_{12}}^1 = -k_3 \cos\varphi_3\ (k_2 \sin\varphi_1)^{-1}, \\ &{\beta_{11}}^3 = -{\beta_{13}}^1 = k_3 \cos\varphi_3 (\cos\varphi_2 - \cos\varphi_3 \cos\varphi_1)(k_2 \lambda \sin\varphi_1)^{-1}, \\ &{\beta_{12}}^3 = -{\beta_{13}}^2 = k_3(\sin^2\varphi_3 - \cos^2\varphi_3)(2k_2\lambda)^{-1}, \\ &{\beta_{21}}^2 = -{\beta_{22}}^1 = k_3(cos\varphi_3 \cos\varphi_1 - \cos\varphi_2)(k_2 \sin^2\varphi_1)^{-1}, \end{aligned} \tag{4.14}$$

$$\begin{aligned}
{\beta_{21}}^3 &= -{\beta_{23}}^1 = k_3(\sin^2\varphi_1 \sin^2\varphi_3 - \sin^2\varphi_1\cos^2\varphi_3 - 2\lambda^2)\\
&\qquad\qquad (2k_2\lambda\sin^2\varphi_1)^{-1},\\
{\beta_{22}}^3 &= -{\beta_{23}}^2 = k_3\cos\varphi_3(\cos\varphi_3\cos\varphi_1 - \cos\varphi_2)(k_2\lambda\sin\varphi_1)^{-1},\\
{\beta_{31}}^2 &= -{\beta_{32}}^1 = k_3(\sin^2\varphi_1\cos^2\varphi_3 - \sin^2\varphi_1\sin^2\varphi_3 + 2\cos^2\varphi_2\\
&\qquad + 2\cos^2\varphi_3\cos^2\varphi_1 - 4\cos\varphi_1\cos\varphi_2\cos\varphi_3)\\
&\qquad\qquad (2k_2\lambda\sin^2\varphi_1)^{-1},\\
{\beta_{31}}^3 &= -{\beta_{33}}^1 = k_3(\cos\varphi_2 - \cos\varphi_3\cos\varphi_1)(k_2\sin^2\varphi_1)^{-1},\\
{\beta_{32}}^3 &= -{\beta_{33}}^2 = -k_3\cos\varphi_3(k_2\sin\varphi_1)^{-1}.
\end{aligned}$$

From (4.4), (4.5) and (4.14), we obtain the fact that

$$\sum_{i=1}^{3} h(\tilde{\nabla}_{\mathbf{e}_i}\phi_\star\mathbf{e}_i, \mathbf{d}_1) = \sum_{i=1}^{3}\sum_{a,b=1}^{3} {\phi_i}^a{\phi_i}^b{\beta_{ab}}^1 = 0 \text{ if and only if}$$

$$\begin{aligned}
\sum_{i=1}^{3}[&\sin\varphi_1\cos\varphi_3\{\lambda{\phi_i}^1{\phi_i}^2 + (\cos\varphi_3\cos\varphi_1 - \cos\varphi_2){\phi_i}^3{\phi_i}^1\}\\
&+ (\sin^2\varphi_1 - 2\cos^2\varphi_2 - \cos^2\varphi_3 - \cos^2\varphi_3\cos^2\varphi_1\\
&\qquad + 4\cos\varphi_1\cos\varphi_2\cos\varphi_3){\phi_i}^2{\phi_i}^3\\
&+ \lambda(\cos\varphi_2 - \cos\varphi_3\cos\varphi_1)\{({\phi_i}^2)^2 - ({\phi_i}^3)^2\}] = 0,
\end{aligned} \tag{4.15}$$

where each $h(\phi_\star\mathbf{e}_i, \mathbf{d}_a) =: {\phi_i}^a$ is a real constant on H, and $(h(\phi_\star\mathbf{e}_i, \mathbf{d}_a))^2 =: ({\phi_i}^a)^2$. Similarly, from (4.4), (4.5) and (4.14), we get the fact that

$$\sum_{i=1}^{3} h(\tilde{\nabla}_{\mathbf{e}_i}\phi_\star\mathbf{e}_i, \mathbf{d}_2) = \sum_{i=1}^{3}\sum_{a,b=1}^{3} {\phi_i}^a{\phi_i}^b{\beta_{ab}}^2 = 0 \text{ if and only if}$$

$$\begin{aligned}
\sum_{i=1}^{3}[&\lambda\sin\varphi_1\cos\varphi_3\{({\phi_i}^3)^2 - ({\phi_i}^1)^2\}\\
&+ (\sin^2\varphi_1\cos^2\varphi_3 - \sin^2\varphi_1\sin^2\varphi_3 + \cos^2\varphi_2 + \cos^2\varphi_3\cos^2\varphi_1\\
&\qquad - 2\cos\varphi_1\cos\varphi_2\cos\varphi_3){\phi_i}^3{\phi_i}^1\\
&+ (\cos\varphi_3\cos\varphi_1 - \cos\varphi_2)\{\lambda{\phi_i}^1{\phi_i}^2 - \sin\varphi_1\cos\varphi_3{\phi_i}^2{\phi_i}^3\}] = 0.
\end{aligned} \tag{4.16}$$

Moreover, from (4.4), (4.5) and (4.14), we have the fact that

$$\sum_{i=1}^{3} h(\tilde{\nabla}_{\mathbf{e}_i}\phi_\star\mathbf{e}_i, \mathbf{d}_3) = \sum_{i=1}^{3}\sum_{a,b=1}^{3} {\phi_i}^a{\phi_i}^b{\beta_{ab}}^3 = 0 \quad \text{if and only if} \tag{4.17}$$

$$\sum_{i=1}^{3}[\sin\varphi_1\cos\varphi_3(\cos\varphi_2 - cos\varphi_3\cos\varphi_1)\{(\phi_i{}^1)^2-(\phi_i{}^2)^2\}$$
$$+(\sin^2\varphi_1\sin^2\varphi_3-\sin^2\varphi_1\cos^2\varphi_3-\lambda^2)\phi_i{}^1\phi_i{}^2$$
$$+\lambda(\cos\varphi_2-\cos\varphi_3\cos\varphi_1)\phi_i{}^3\phi_i{}^1-\lambda\sin\varphi_1\cos\varphi_3\phi_i{}^2\phi_i{}^3]=0.$$

By virtue of Proposition 4.1, (4.13), (4.15), (4.16) and (4.17), we obtain:

Theorem 4.4. *Let g (resp. h) be an arbitrarily given left invariant metric on $SU(2)$ (resp. the Heisenberg group H). Let $(1, k_2, k_3, \varphi_1, \varphi_2, \varphi_3)$ be an element of $\mathcal{M}$ which corresponds to the metric h on H. Then, a necessary and sufficient condition for a group homomorphism $\phi : (S(2), g) \to (H, h)$ to be a harmonic map is that*

$$\sum_{i=1}^{3}[\sin\varphi_1\cos\varphi_3\{\lambda\phi_i{}^1\phi_i{}^2+(\cos\varphi_3\cos\varphi_1-\cos\varphi_2)\phi_i{}^3\phi_i{}^1\}$$
$$+(\sin^2\varphi_1-2\cos^2\varphi_2-\cos^2\varphi_3-\cos^2\varphi_3\cos^2\varphi_1$$
$$+4\cos\varphi_1\cos\varphi_2\cos\varphi_3)\phi_i{}^2\phi_i{}^3$$
$$+\lambda(\cos\varphi_2-\cos\varphi_3\cos\varphi_1)\{(\phi_i{}^2)^2-(\phi_i{}^3)^2\}]=0,$$

$$\sum_{i=1}^{3}[\lambda\sin\varphi_1\cos\varphi_3\{(\phi_i{}^3)^2-(\phi_i{}^1)^2\}$$
$$+(\sin^2\varphi_1\cos^2\varphi_3-\sin^2\varphi_1\sin^2\varphi_3+\cos^2\varphi_2+\cos^2\varphi_3\cos^2\varphi_1$$
$$-2\cos\varphi_1\cos\varphi_2\cos\varphi_3)\phi_i{}^3\phi_i{}^1$$
$$+(\cos\varphi_3\cos\varphi_1-\cos\varphi_2)\{\lambda\phi_i{}^1\phi_i{}^2-\sin\varphi_1\cos\varphi_3\phi_i{}^2\phi_i{}^3\}]=0, \text{ and}$$

$$\sum_{i=1}^{3}[\sin\varphi_1\cos\varphi_3(\cos\varphi_2-\cos\varphi_3\cos\varphi_1)\{(\phi_i{}^1)^2-(\phi_i{}^2)^2\}$$
$$+(\sin^2\varphi_1\sin^2\varphi_3-\sin^2\varphi_1\cos^2\varphi_3-\lambda^2)\phi_i{}^1\phi_i{}^2$$
$$+\lambda(\cos\varphi_2-\cos\varphi_3\cos\varphi_1)\phi_i{}^3\phi_i{}^1-\lambda\sin\varphi_1\cos\varphi_3\phi_i{}^2\phi_i{}^3]=0.$$

Remark 4.5. We can see that the necessary and sufficient condition (in Theorem 4.4) for a group homomorphism $\phi : (SU(2), g) \to (H, h)$ to be harmonic is independent of the lengths of the left invariant frame vector fields $\{\mathbf{v}_a\}_{a=1}^3$, and only relies on the angle between $\mathbf{v}_a$ and $\mathbf{v}_b$ $(a, b = 1, 2, 3)$ and the components of the matrix $(\phi_i{}^a)_{ai}$ of the Lie algebra homomorphism $d\phi(= \phi_\star) : \mathfrak{su}(2) \to \mathfrak{h}$ with respect to the orthonormal bases $\{\mathbf{e}_i\}_{i=1}^3$ and $\{\mathbf{d}_a\}_{a=1}^3$.

And, by the help of (2.8) and Theorem 4.4, we get:

Corollary 4.6. *Let g be an arbitrarily given left invariant metric on $SU(2)$. And, let h be the left invariant metric on H which corresponds to $(1, k_2, k_3, \varphi_1 = \pi/2, \varphi_2 = \pi/2, \varphi_3 = \pi/2) \in \mathcal{A}$. Then, a necessary and sufficient condition for* *a group homomorphism* $\phi : (SU(2), g) \to (H, h)$ *to be a harmonic map is that*

$$\sum_{i=1}^{3} {\phi_i}^2 {\phi_i}^3 = 0, \quad \sum_{i=1}^{3} {\phi_i}^3 {\phi_i}^1 = 0, \quad i.e.,$$
$$\sum_{i=1}^{3} h(\phi_\star \mathbf{e}_i, \mathbf{d}_2) h(\phi_\star \mathbf{e}_i, \mathbf{d}_3) = 0, \quad \sum_{i=1}^{3} h(\phi_\star \mathbf{e}_i, \mathbf{d}_3) h(\phi_\star \mathbf{e}_i, \mathbf{d}_1) = 0.$$

In order to obtain the totality (cf. Theorem 4.11) of group homomorphisms from $(SU(2), g)$ with an arbitrarily given left invariant metric g into the Heisenberg group H with a properly given left invariant metric which are also harmonic, we introduce the following:

Lemma 4.7 ([5, Theorem 1, p. 181])**.** *Let* K *be a compact connected topological group, and let* U *be a neighborhood of the identity* e *of* K *satisfying* $U = U^{-1}$*. Then, there is a positive integer* n *such that every* $k \in K$ *can be expressed as a product* $k = k_1 \cdots k_n$ $(k_i \in U)$ *of* n *elements of* U*.*

Lemma 4.8 ([2, Lemma 1.12, p. 110])**.** *Let* K *and* L *be Lie groups with Lie algebras* $\mathfrak{k}$ *and* $\mathfrak{l}$*, respectively. Let* ψ *be an analytic homomorphism of* K *into* L*. Then,* $d\psi_e$ *is a Lie algebra homomorphism of* $\mathfrak{k}$ *into* $\mathfrak{l}$ *and*

$$\psi(\exp X) = \exp d\psi_e(X) \quad (X \in \mathfrak{k}).$$

Lemma 4.9 ([2, Proposition 1.6, p. 104])**.** *Let* K *be a Lie group,* $\mathfrak{k}$ *the Lie algebra of* K*. Then, there exists an open neighborhood* N_0 *of* 0 *in* $\mathfrak{k}$ *and an open neighborhood* N_e *of* e *in* K *such that* exp *is an analytic diffeomorphism of* N_0 *onto* N_e*.*

Lemma 4.10 ([2, Corollary 6.7, p. 133; Proposition 6.10, p. 135])**.** *Let* K *be a compact connected Lie group with the Lie algebra* $\mathfrak{k}$*. Then the map* $\exp : \mathfrak{k} \to \mathrm{K}$ *is surjective.*

Here and from now on, exp is the exponential map from the Lie algebra of a given Lie group into the Lie group (cf. [2, p. 104]).

In our situation, by virtue of Corollary 4.6 and Lemmas 4.7 $\sim$ 4.10, we obtain:

Theorem 4.11. *Let* g *be an arbitrarily given left invariant metric on* $SU(2)$*. And, let* h *be the left invariant metric on* H *which corresponds to* $(1, k_2, k_3, \varphi_1 = \pi/2, \varphi_2 = \pi/2, \varphi_3 = \pi/2) \in \mathcal{A}$*.*

Assume that $\phi : (SU(2), g) \to (H, h)$ *is an arbitrarily given group homomorphism which is also a harmonic map. Then the map* ϕ *is completely determined by*

$$\phi(\exp\left(\sum_{i=1}^{3} t^i \mathbf{e}_i)\right) := \exp\left(\sum_{i=1}^{3}\sum_{a=1}^{3} t^i {C_i}^a \mathbf{d}_a\right) \tag{4.18}$$

for each $(t^1, t^2, t^3) \in \{(t^1, t^2, t^3) \in \mathbb{R}^3 \mid (t^i)^2 < \delta$ *for a sufficiently small positive real number* $\delta\}$*, where* $C = ({C_i}^a)_{ai}, ({C_i}^a = h(\phi_\star \mathbf{e}_i, \mathbf{d}_a))$*, is one, and only one real matrix of type* $(3,3)$ *whose components satisfy*

$$\sum_{i=1}^{3} {C_i}^2 {C_i}^3 = \sum_{i=1}^{3} {C_i}^3 {C_i}^1 = 0. \tag{4.19}$$

Conversely, if there is a real matrix $C = ({C_i}^a)_{ai}$ *of type* $(3,3)$ *such that the components of the matrix* C *satisfy the condition* (4.19)*, then the group homomorphism* $\phi : (SU(2), g) \to (H, h)$ *defined by* (4.18) *is a harmonic map.*

4.5. Affine group homomorphisms of $SU(2)$ into the Heisenberg group H

We retain the notations as in §2 and the subsections 4.3, 4.4. In this subsection, we obtain:

Proposition 4.12. *Let* g *be an arbitrarily given left invariant metric on* $SU(2)$ *and, let* h *be the left invariant metric on* H *which corresponds to* $(1, k_2, k_3, \varphi_1 = \pi/2, \varphi_2 = \pi/2, \varphi_3 = \pi/2) \in \mathcal{A}$*. Then, there does not exist any affine group homomorphism* $\phi : (SU(2), g) \to (H, h)$ *which is not a trivial group homomorphism* (*i.e., which satisfies the condition* $\phi(SU(2)) \neq e$)*.*

Proof. From (2.8), (4.14) and the assumption $\varphi_1 = \varphi_2 = \varphi_3 = \pi/2$, we get

$$\begin{aligned} &{\beta_{12}}^3 (= -{\beta_{13}}^2) = k_3 (2k_2)^{-1}, \qquad {\beta_{23}}^1 (= -{\beta_{21}}^3) = k_3 (2k_2)^{-1}, \\ &{\beta_{31}}^2 (= -{\beta_{32}}^1) = -k_3 (2k_2)^{-1}, \qquad {\beta_{ab}}^c = 0 \ \text{ otherwise.} \end{aligned} \tag{4.20}$$

By the help of (4.13), (4.20) and Proposition 4.2, we obtain the fact that a necessary and sufficient condition for a group homomorphism $\phi : (SU(2), g) \to (H, h)$ to be an affine map is

$$\begin{aligned} &{\phi_i}^1 = {\phi_i}^2 = 0 \ (i = 1, 2, 3), \\ &\{1 - (a_2)^2 + (a_3)^2\} {\phi_1}^3 = \{1 + (a_2)^2 - (a_3)^2\} {\phi_1}^3 = 0, \\ &\{1 + (a_2)^2 - (a_3)^2\} {\phi_2}^3 = \{(a_2)^2 + (a_3)^2 - 1\} {\phi_2}^3 = 0, \\ &\{(a_2)^2 + (a_3)^2 - 1\} {\phi_3}^3 = \{1 - (a_2)^2 + (a_3)^2\} {\phi_3}^3 = 0. \end{aligned} \tag{4.21}$$

Moreover, the condition (4.21) is equivalent to

$${\phi_i}^a = 0 \quad (i = 1, 2, 3; \ a = 1.2.3), \tag{4.22}$$

since a_2 and a_3 are positive real numbers which correspond to the left invariant Riemannian metric g on $SU(2)$. So, by virtue of (4.22) and Lemmas 4.7 $\sim$ 4.10, we get this proposition. □

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