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# A CONVERSE THEOREM ON $h$-STABILITY VIA IMPULSIVE VARIATIONAL SYSTEMS

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Abstract. In this paper we develop useful relations which estimate the difference between the solutions of nonlinear impulsive differential systems with different initial values. Then we obtain the converse $h$-stability theorem of Massera's type for the nonlinear impulsive systems by employing the $t_\infty$-similarity of the associated impulsive variational systems and relations.

## 1. Introduction

The qualitative properties of solutions of differential equations with impulse effects have been developed by a large number of mathematicians due to the wide application of these systems to the control theory, biology, electronics, etc (see e.g. [2, 3, 13]).

Simeonov and Bainov [20] studied the exponential stability of the solutions for impulsive differential equations by using the comparison method and piecewise continuous auxiliary functions which are analogues to Lyapunov's functions. Also, Kulev and Bainov [12] introduced the notions of various types of uniform Lipschitz stability for impulsive differential systems and obtained sufficient conditions for these notions and their relations.

Pinto [17] introduced the notion of $h$-stability for differential systems without impulse effect with the intention of obtaining results about stability for weakly stable differential systems under some perturbations. The various notions of $h$-stability given in [17, 18] include several types of known stability properties as uniform stability, exponential asymptotic stability [15] and uniform Lipschitz stability [9].

Choi et al. [5, 6] studied $h$-stability for the linear impulsive differential equations by means of the notions of similarity, $t_\infty$-similarity, and impulsive integral

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inequality. In [4], we proved that two concepts of $h$-stability and $h$-stability in variation for nonlinear impulsive differential systems are equivalent via $t_\infty$-similarity of the associated variational impulsive systems and impulsive integral inequalities. Also, we characterized $h$-stability for nonlinear impulsive differential systems by using the notions of piecewise continuous auxiliary functions and impulsive differential inequalities. Many authors [2, 3, 5, 12, 13, 16, 20] have studied the various types of stability of solutions for impulsive differential equations. However, to the best of our knowledge, there are no papers published on the converse $h$-stability theorem for nonlinear impulsive differential systems.

Motivated by the above discussion, we develop useful relations which estimate the difference between the solutions of nonlinear impulsive differential systems with different initial values. Then we obtain the converse $h$-stability theorem of Massera's type for the nonlinear impulsive systems by employing the $t_\infty$-similarity of the associated impulsive variational systems and relations.

## 2. Preliminary notes and definitions

Let $\mathbb{R}_+ = [0, \infty)$ and $\mathbb{R}^n$ be the $n$-dimensional Euclidean space with a convenient vector norm $|\cdot|$. We consider the impulsive differential system with impulses at fixed times

$$\begin{cases} x' &= f(t,x),\ t \neq \tau_k, \\ \Delta x &= I_k(x),\ t = \tau_k,\ k = 1, 2, \dots, \end{cases} \tag{2.1}$$

where $\Delta x(t) = x(t{+}0) - x(t{-}0)$. Assume that the following basic conditions hold:

(A1) $\{\tau_k\}$ is an unbounded increasing sequence satisfying $0 \leq t_0 = \tau_0 < \tau_1 < \tau_2 < \cdots$ and $\lim_{k\to\infty} \tau_k = \infty$.

(A2) The function $f : \mathbb{R}_+ \times \mathbb{R}^n \to \mathbb{R}^n$ is continuous and has a continuous partial derivative $f_x = \frac{\partial f}{\partial x}$ in $(\tau_{k-1}, \tau_k] \times \mathbb{R}^n,\ k = 1, 2, \dots$, and $f(t, 0) = 0$ for each $t \in \mathbb{R}_+$.

(A3) For any $x \in \mathbb{R}^n$ and any $k = 1, 2, \dots$, the functions $f$ and $f_x$ have finite limits as $(t, y) \to (\tau_k, x), t > \tau_k$.

(A4) Each function $I_k : \mathbb{R}^n \to \mathbb{R}^n$ is continuously differentiable in $\mathbb{R}^n$ and there exist nonnegative constants $l_k$ such that

$$|I_k(x) - I_k(y)| \leq l_k |x - y|,\ k \in \mathbb{N}, x, y \in \mathbb{R}^n,$$

and $I_k(0) = 0, k = 1, 2, \dots$.

(A5) The solution $x(t, t_0, x_0)$ of system (2.1) which satisfies the initial condition $x(t_0{+}0, t_0, x_0) = x_0$ is defined in the interval $(t_0, \infty)$, and is left continuous. And at the moments $\tau_k$ the following relations hold

$$x(\tau_k{-}0) = x(\tau_k),\ \Delta x(\tau_k) = x(\tau_k{+}0) - x(\tau_k{-}0).$$

Then it follows from the condition (A5) that the solution $x(t,t_0,x_0)$ of system (2.1) through the initial value $(t_0,x_0)$ is described as follows:

$$x(t,t_0,x_0)=\begin{cases} x(t,t_0,x_0),\ t_0\le t\le\tau_1,\\ x(t,\tau_1,x_1^+),\ \tau_1<t\le\tau_2,\\ \cdots\\ x(t,\tau_k,x_k^+),\ \tau_k<t\le\tau_{k+1},\\ \cdots,\end{cases}$$

where $x_k^+=x_k+I_k(x_k)$ and $x_k=x(\tau_k)$. For global existence of solutions for system (2.1), see [21]. The basic notions and important theory for system (2.1) were described in detail in [2, 3, 13].

Together with system (2.1), we consider the associated impulsive variational systems

$$\begin{cases} v' &= f_x(t,0)v,\ t\neq\tau_k,\\ \Delta v &= \frac{\partial I_k(0)}{\partial x}v,\ t=\tau_k,\ k\in\mathbb{N},\\ v(t_0+0) &= v_0\end{cases} \tag{2.2}$$

and

$$\begin{cases} z' &= f_x(t,x(t,t_0,x_0))z,\ t\neq\tau_k,\\ \Delta z &= \frac{\partial I_k(x(\tau_k))}{\partial x}z,\ t=\tau_k,\ k\in\mathbb{N},\\ z(t_0+0) &= z_0.\end{cases} \tag{2.3}$$

It follows from [13, Theorem 2.4.1] that the fundamental matrix solution $\Phi(t,t_0,0)$ of system (2.2) is given by

$$\Phi(t,t_0,0)=\frac{\partial x(t,t_0,0)}{\partial x_0},\ t\neq\tau_k$$

and the fundamental matrix solution $\Phi(t,t_0,x_0)$ of system (2.3) is given by

$$\Phi(t,t_0,x_0)=\frac{\partial x(t,t_0,x_0)}{\partial x_0},\ t\neq\tau_k. \tag{2.4}$$

Let $M_n(\mathbb{R})$ be the set of all $n\times n$ matrices over $\mathbb{R}$ and let $PC(\mathbb{R}_+,M_n(\mathbb{R}))$ denote the class of piecewise continuous functions from $\mathbb{R}_+$ to $M_n(\mathbb{R})$ with discontinuities of the first kind only at $t=\tau_k,k\in\mathbb{N}$.

We consider two linear homogeneous impulsive systems

$$\begin{cases} x'=A(t)x,\ t\neq\tau_k,\\ \Delta x=A_kx,\ t=\tau_k,k\in\mathbb{N},\end{cases} \tag{2.5}$$

and

$$\begin{cases} y'=B(t)y,\ t\neq\tau_k,\\ \Delta y=B_ky,\ t=\tau_k,k\in\mathbb{N},\end{cases} \tag{2.6}$$

under the assumption that the following conditions hold:

(B1) $A, B \in PC(\mathbb{R}_+, M_n(\mathbb{R}))$.

(B2) $A_k, B_k \in M_n(\mathbb{R}), \det(E + A_k) \neq 0$, and $\det(E + B_k) \neq 0, k \in \mathbb{N}$.

Then it follows from [2, p. 7] that the solutions of system (2.5) can be written in the form

$$x(t, t_0, x_0) = X(t, t_0)x_0, \tag{2.7}$$

where $X(t)$ is a fundamental matrix of system (2.5) and the Cauchy matrix $X(t, t_0)$ [2, 13] is given by

$$X(t, t_0) = \begin{cases} U_k(t, t_0),\ 0 \leq t_0 \leq t \leq \tau_k, \\ U_{k+1}(t, \tau_k^+)(E + A_k)U_k(\tau_k, t_0),\ t_0 \leq \tau_k < t \leq \tau_{k+1}, \\ U_{k+1}(t, \tau_k^+) \prod_{j=1}^{k}(E + A_j)U_j(\tau_j, \tau_{j-1}^+),\ t_0 = \tau_0^+ \leq \tau_1 < \tau_k < t \leq \tau_{k+1}. \end{cases}$$

Here $U_k(t, s)(k \in \mathbb{N}, t, s \in (\tau_{k-1}, \tau_k])$ is the Cauchy matrix for the linear differential system

$$x' = A(t)x,\ t \in (\tau_{k-1}, \tau_k], k = 1, 2, \ldots. \tag{2.8}$$

Denote by $\mathfrak{S}$ the set of all matrix functions $S : \mathbb{R}_+ \to M_n(\mathbb{R})$ which belong to $PC(\mathbb{R}_+, M_n(\mathbb{R}))$ and are bounded in $\mathbb{R}_+$ together with their inverse $S^{-1}(t)$. Let $\mathcal{M}$ be the set of the linear homogeneous impulsive systems as follows

$$\mathcal{M} := \{(A, A_k) \mid A \in PC(\mathbb{R}_+, M_n(\mathbb{R})),\ A_k \in M_n(\mathbb{R}),\ \det(E + A_k) \neq 0,\ k \in \mathbb{N}\}.$$

We recall the notion of $t_\infty$-similarity with impulse effect in $\mathcal{M}$ which is analogue to the concept of $t_\infty$-similarity introduced by Conti [8].

**Definition 2.1** ([3])**.** We say that $(A, A_k) \in \mathcal{M}$ is $t_\infty$*-similar* to $(B, B_k) \in \mathcal{M}$ if there exists a matrix-valued function $S \in \mathfrak{S}$ such that

$$S'(t) - A(t)S(t) + S(t)B(t) \equiv F_0 \in L_1,\ t \neq \tau_k, \tag{2.9}$$

$$\Delta S(\tau_k) - A_k S(\tau_k) + S(\tau_k^+)B_k \equiv F_k \in l_1,\ t = \tau_k, k \in \mathbb{N}, \tag{2.10}$$

where $\Delta S(\tau_k) = S(\tau_k^+) - S(\tau_k)$. We say that system (2.5) is $t_\infty$*-similar* to system (2.6) if $(A, A_k) \in \mathcal{M}$ is $t_\infty$-similar to $(B, B_k) \in \mathcal{M}$ for each $k \in \mathbb{N}$. Note that the relation $(A, A_k) \sim (B, B_k) : S$ is an equivalence relation.

*Remark* 2.2. The notion of $t_\infty$-similarity preserves various concepts of stability: stability, uniform stability, uniform asymptotic stability and strict stability (see [3, Theorem 10.3]). In the case when $F_0 = 0$ in (2.9) and $F_k = 0$ in (2.10) for each $k \in \mathbb{N}$, then $t_\infty$-similarity reduces to kinematical similarity for impulsive linear equations [1].

**Definition 2.3** ([20])**.** We say that the function $V : \mathbb{R}_+ \times \mathbb{R}^n \to \mathbb{R}^n$ belongs to the *class* $\upsilon_0$ if

(i) $V$ is continuous in $G_k = \{(t, x) \in \mathbb{R}_+ \times \mathbb{R}^n :\ \tau_{k-1} < t < \tau_k\}$ for each $k \in \mathbb{N}$ and $V(t, 0) = 0$ for each $t \in \mathbb{R}_+$;

(ii) for any $k \in \mathbb{N}$ and $x \in \mathbb{R}^n$ there exist the finite limits

$$V(\tau_k - 0, x) = \lim_{\substack{(t,y)\to(\tau_k,x) \\ t<\tau_k}} V(t,y), \; V(\tau_k + 0, x) \quad = \quad \lim_{\substack{(t,y)\to(\tau_k,x) \\ t>\tau_k}} V(t,y)$$

and the equality $V(\tau_k - 0, x) = V(\tau_k, x)$ holds.

We note that if $t \neq \tau_k$, then $V(t+0,x)$ denotes $V(t,x)$.

**Definition 2.4** ([20])**.** The function $V \in v_0$ belongs to the *class* $v_1$ if it is continuously differentiable in $G = \cup_{k=1}^{\infty} G_k$.

*Remark* 2.5 ([22]). For $(t,x) \in G$ we define the *derivative* of the function $V \in v_1$ with respect to system (2.1) by

$$\dot{V}_{(2.1)}(t,x) = \frac{\partial V(t,x)}{\partial t} + \frac{\partial V(t,x)}{\partial x} f(t,x)$$

and the *upper right derivative* of the function $V \in v_0$ with respect to the solutions of system (2.1) by

$$D^+_{(2.1)} V(t,x) = \lim_{s\to 0+} \sup \frac{1}{s}[V(t+s, x(t+s,t,x)) - V(t,x)].$$

We note that if the function $V \in v_0$ is locally Lipschitz continuous with respect to $x \in \mathbb{R}^n$, then for $(t,x) \in G_k$ we have

$$D^+_{(2.1)} V(t,x) = \lim_{s\to 0+} \sup \frac{1}{s}[V(t+s, x+sf(t,x)) - V(t,x)].$$

We recall some notions of various types of $h$-stability for the zero solution of system (2.1). For nonlinear differential systems without impulse effect, this notion was introduced by Pinto [17, 18].

**Definition 2.6.** The zero solution of system (2.1) is said to be

(hS) *$h$-stable* if there exist a positive bounded left-continuous function $h : \mathbb{R}_+ \to \mathbb{R}$, $\delta > 0$, and a constant $c \geq 1$ such that

$$|x(t,t_0,x_0)| \leq c|x_0|h(t)h(t_0)^{-1}, \; t \geq t_0$$

for $|x_0| \leq \delta$(here $h(t)^{-1} = \frac{1}{h(t)}$);

(GhS) *globally $h$-stable* if system (2.1) is $h$-stable for every $x_0 \in D$, where $D \subset \mathbb{R}^n$ is a region which includes the origin;

(hSV) *$h$-stable in variation* if system (2.3) is $h$-stable;

(GhSV) *globally $h$-stable in variation* if system (2.3) is globally $h$-stable.

The various notions of $h$-stability given by Definition 2.6 include several notions of well-known stability such as uniform stability, uniform asymptotic stability, uniform Lipschitz stability, exponential stability, and non-uniform stability. For linear impulsive differential systems, we note that the notions of various types of the above $h$-stability are equivalent.

## 3. Main results

In this section we develop a useful inequality which estimates the difference between the solutions of nonlinear impulsive systems with different initial values by using fundamental solutions of the associated impulsive variational systems. Then we obtain the converse $h$-stability theorem of Massera's type for the nonlinear impulsive systems by employing the $t_\infty$-similarity of the associated impulsive variational systems and an inequality.

To do this, we need the following lemma for differential systems.

**Lemma 3.1** ([14, Theorem 1.2.9]). *Let $g \in C(\mathbb{R}_+ \times \mathbb{R}^n, \mathbb{R}^n)$, and $g_y$ exist and be continuous on $\mathbb{R}_+ \times \mathbb{R}^n$. Assume that $y(t, t_0, x_0)$ and $\hat{y}(t, t_0, y_0)$ are the solutions of differential system of the following form*

$$y' = g(t, y) \tag{3.1}$$

*through $(t_0, x_0)$ and $(t_0, y_0)$, respectively, existing for $t \geq t_0$, such that $x_0$ and $y_0$ belong to a convex subset of $\mathbb{R}^n$. Then*

$$y(t, t_0, x_0) - \hat{y}(t, t_0, y_0) = \int_0^1 \Psi(t, t_0, y_0 + s(x_0 - y_0))ds \cdot (x_0 - y_0)$$

*holds for $t \geq t_0$, where $\Psi(t, t_0, x_0) = \frac{\partial y(t,t_0,x_0)}{\partial x_0}$.*

We need some lemmas to obtain the converse $h$-stability theorem for system (2.1). We obtain the following result which estimates the difference between the solutions of nonlinear impulsive system (2.1) with different initial values in terms of the fundamental solutions of the associated impulsive variational systems.

**Lemma 3.2.** *Suppose that $D \subset \mathbb{R}^n$ is a convex subset and each map $I_k$ in condition (A4) is Lipschitzian in $x$ with nonnegative constants $l_k$, i.e.,*

$$|I_k(x) - I_k(y)| \leq l_k|x - y|, \ k \in \mathbb{N}. \tag{3.2}$$

*Then, for $t \geq t_0$,*

$$\begin{aligned}
&|x(t, t_0, x_0) - \hat{x}(t, t_0, y_0)| \\
\leq \ &\sup_{\eta \in D} |\Phi(t, \tau_k^+, \eta)| \prod_{i=1}^{k} (1 + l_i)|\Phi(\tau_i, \tau_{i-1}^+, \eta)||x_0 - y_0|, \ \tau_k < t \leq \tau_{k+1}, k \in \mathbb{N},
\end{aligned}$$

*where $x(t, t_0, x_0)$ and $\hat{x}(t, t_0, y_0)$ are the solutions of system (2.1) through $(t_0, x_0)$ and $(t_0, y_0)$, respectively, existing for $t \geq t_0$, such that $x_0$ and $y_0$ belong to a convex subset of $\mathbb{R}^n$ and $\tau_0^+ = t_0$. Moreover, $\Phi(t, t_0, x_0)$ is the fundamental matrix solution of system (2.3) given by (2.4).*

*Proof.* If $t_0 \leq t \leq \tau_1$, then it follows from Lemma 3.1 that

$$x(t, t_0, x_0) - \hat{x}(t, t_0, y_0) = \int_0^1 \Phi(t, t_0, y_0 + s(x_0 - y_0))ds \cdot (x_0 - y_0). \tag{3.3}$$

Thus, we have

$$
\begin{aligned}
(3.4)\qquad |x(t,t_0,x_0)-\hat{x}(t,t_0,y_0)| &\le \int_0^1 |\Phi(t,t_0,y_0+s(x_0-y_0))|ds|x_0-y_0| \\
&\le \sup_{\eta\in D} |\Phi(t,t_0,\eta)||x_0-y_0|,\ t_0\le t\le \tau_1,
\end{aligned}
$$

where $D$ is a convex subset of $\mathbb{R}^n$ containing $x_0$ and $y_0$.

If $\tau_1 < t \le \tau_2$, then it follows from Lemma 3.1 and (3.3) that

$$
\begin{aligned}
&x(t,\tau_1,x(\tau_1^+)) - \hat{x}(t,\tau_1,\hat{x}(\tau_1^+)) \\
&= \int_0^1 \Phi(t,\tau_1^+,p_1(s))ds[x(\tau_1^+)-\hat{x}(\tau_1^+)] \\
&= \int_0^1 \Phi(t,\tau_1^+,p_1(s))ds[\int_0^1 \Phi(\tau_1,t_0,p_0(s))ds(x_0-y_0) + I_1(x(\tau_1)) - I_1(\hat{x}(\tau_1))],
\end{aligned}
$$

where $p_0(s) = y_0 + s(x_0-y_0)$ and $p_1(s) = \hat{x}(\tau_1^+) + s(x(\tau_1^+)-\hat{x}(\tau_1^+))$ and $0\le s\le 1$. In view of (3.2) and (3.4), we obtain

(3.5)

$$
\begin{aligned}
&|x(t,\tau_1,x(\tau_1^+)) - \hat{x}(t,\tau_1,\hat{x}(\tau_1^+))| \\
&\le \int_0^1 |\Phi(t,\tau_1^+,p_1(s))|ds[\int_0^1 |\Phi(\tau_1,t_0,p_0(s))|ds|x_0-y_0| + |I_1(x(\tau_1)) \\
&\quad - I_1(\hat{x}(\tau_1))|] \\
&\le \int_0^1 |\Phi(t,\tau_1^+,p_1(s))|ds[\int_0^1 |\Phi(\tau_1,t_0,p_0(s))|ds|x_0-y_0| + l_1|x(\tau_1)-\hat{x}(\tau_1)|] \\
&\le \int_0^1 |\Phi(t,\tau_1^+,p_1(s))|ds\int_0^1 |\Phi(\tau_1,t_0,p_0(s))|ds|x_0-y_0|(1+l_1) \\
&\le \sup_{\eta\in D} |\Phi(t,\tau_1^+,\eta)|(1+l_1)|\Phi(\tau_1,t_0,\eta)||x_0-y_0|,\ \tau_1<t\le\tau_2,
\end{aligned}
$$

where $p_0(s) = y_0 + s(x_0-y_0)$ and $p_1(s) = \hat{x}(\tau_1^+) + s(x(\tau_1^+)-\hat{x}(\tau_1^+))$ and $0\le s\le 1$.

If $\tau_2 < t \le \tau_3$, then it follows from Lemma 3.1 that

$$
\begin{aligned}
x(t,\tau_2,x(\tau_2^+)) - \hat{x}(t,\tau_2,\hat{x}(\tau_2^+)) &= \int_0^1 \Phi(t,\tau_2^+,p_2(s))ds[x(\tau_2^+)-\hat{x}(\tau_2^+)] \\
= \int_0^1 \Phi(t,\tau_2^+,p_2(s))ds[x(\tau_2)-\hat{x}(\tau_2) &+ I_2(x(\tau_2)) - I_2(\hat{x}(\tau_2))],
\end{aligned}
$$

where $p_2(s) = \hat{x}(\tau_2^+) + s(x(\tau_2^+)-\hat{x}(\tau_2^+))$ and $0\le s\le 1$. In view of (3.2) and (3.5), we have

$$
\begin{aligned}
&|x(t,\tau_2,x(\tau_2^+)) - \hat{x}(t,\tau_2,\hat{x}(\tau_2^+))| \\
&\le \int_0^1 |\Phi(t,\tau_2^+,p_2(s))|ds[|x(\tau_2)-\hat{x}(\tau_2)| + |I_2(x(\tau_2)) - I_2(\hat{x}(\tau_2))|]
\end{aligned}
$$

$$
\begin{aligned}
&\le \int_0^1 |\Phi(t,\tau_2^+,p_2(s))|ds[|x(\tau_2)-\hat{x}(\tau_2)|+l_2|x(\tau_2)-\hat{x}(\tau_2)|]\\
&\le \int_0^1 |\Phi(t,\tau_2^+,p_2(s))|ds(1+l_2)|x(\tau_2)-\hat{x}(\tau_2)|\\
&\le \int_0^1 |\Phi(t,\tau_2^+,p_2(s))|ds(1+l_1)(1+l_2)\int_0^1 |\Phi(\tau_2,\tau_1^+,p_1(s))|ds\\
&\quad \int_0^1 |\Phi(\tau_1,t_0,p_0(s))|ds|x_0-y_0|\\
&\le \sup_{\eta\in D}|\Phi(t,\tau_2^+,\eta)|ds(1+l_1)(1+l_2)|\Phi(\tau_2,\tau_1^+,\eta)||\Phi(\tau_1,t_0,\eta)||x_0-y_0|,
\end{aligned}
$$

where $\tau_2 < t \le \tau_3$, $p_0(s) = y_0+s(x_0-y_0)$ and $p_i(s) = \hat{x}(\tau_i^+)+s(x(\tau_i^+)-\hat{x}(\tau_i^+))$, $i = 1,2$ and $0 \le s \le 1$.

It follows from mathematical induction that

$$
\begin{aligned}
&x(t,\tau_k,x(\tau_k^+))-\hat{x}(t,\tau_k,\hat{x}(\tau_k^+)) = \int_0^1 \Phi(t,\tau_k^+,p_k(s))ds[x(\tau_k^+)-\hat{x}(\tau_k^+)]\\
&= \int_0^1 \Phi(t,\tau_k^+,p_k(s))ds[x(\tau_k)-\hat{x}(\tau_k)+I_k(x(\tau_k))-I_k(\hat{x}(\tau_k))],
\end{aligned}
$$

where $\tau_k < t \le \tau_{k+1}$, $k\in\mathbb{N}$, $p_k(s) = \hat{x}(\tau_k^+)+s(x(\tau_k^+)-\hat{x}(\tau_k^+))$, $0\le s\le 1$. Thus it follows from (3.2) that

$$
\begin{aligned}
&|x(t,\tau_k,x(\tau_k^+))-\hat{x}(t,\tau_k,\hat{x}(\tau_k^+))|\\
&\le \int_0^1 |\Phi(t,\tau_k^+,p_k(s))|ds[|x(\tau_k)-\hat{x}(\tau_k)|+|I_k(x(\tau_k))-I_k(\hat{x}(\tau_k))|]\\
&\le \int_0^1 |\Phi(t,\tau_k^+,p_k(s))|ds(1+l_k)|x(\tau_k)-\hat{x}(\tau_k)|\\
&\le \sup_{\eta\in D}|\Phi(t,\tau_k^+,\eta)|\prod_{i=1}^k(1+l_i)|\Phi(\tau_i,\tau_{i-1}^+,\eta)||x_0-y_0|,\ \tau_k<t\le\tau_{k+1}, k\in\mathbb{N},
\end{aligned}
$$

where $p_k(s) = \hat{x}(\tau_k^+)+s(x(\tau_k^+)-\hat{x}(\tau_k^+))$, $0\le s\le 1$, and $\tau_0^+ = t_0$. This completes the proof. □

As a consequence of Lemma 3.2, we easily obtain the following result.

**Corollary 3.3** ([4])**.** *Let $I_k(x) = d_k x$ in condition* (A4)*, and let $d_k$ be constants. Assume that $x(t,t_0,x_0)$ and $\hat{x}(t,t_0,y_0)$ are the solutions of system* (2.1) *through $(t_0,x_0)$ and $(t_0,y_0)$, respectively, existing for $t\ge t_0$, such that $x_0$ and $y_0$ belong to a convex subset of $\mathbb{R}^n$. Then,*

$$
\begin{aligned}
&x(t,t_0,x_0)-\hat{x}(t,t_0,y_0)\\
&= \int_0^1 \Phi(t,\tau_k^+,p_k(s))ds\prod_{i=1}^k(1+d_i)\int_0^1 \Phi(\tau_i,\tau_{i-1}^+,p_{i-1}(s))ds\cdot(x_0-y_0),
\end{aligned}
$$

*holds for* $\tau_k < t \le \tau_{k+1}$, $t \ge t_0$, *where* $p_0(s) = y_0 + s(x_0 - y_0)$, $p_i(s) = \hat{x}(\tau_i^+) + s(x(\tau_i^+) - \hat{x}(\tau_i^+))$, $i = 1, \ldots, k$, $0 \le s \le 1$, *and* $\tau_0^+ = t_0$.

We can obtain the following relation between the solution $x(t, t_0, x_0)$ of system (2.1) and $\Phi(t, t_0, x_0)$ given by (2.4).

**Corollary 3.4.** *Assume that* $\hat{x}(t, t_0, y_0) = 0$ *and* $y_0 = 0$ *in Corollary 3.3. Then,*

$$x(t, t_0, x_0) = \int_0^1 \Phi(t, \tau_k^+, sx(\tau_k^+))ds \prod_{i=1}^{k}(1 + d_i) \int_0^1 \Phi(\tau_i, \tau_{i-1}^+, sx(\tau_{i-1}^+))ds \cdot x_0,$$

*holds for* $\tau_k < t \le \tau_{k+1}$, $k \in \mathbb{N}$, $t \ge t_0$, *where* $\tau_0 = t_0$.

*Remark* 3.5. Assume that $I_k(x) = 0$, $k \in \mathbb{N}$, in condition (A4). From Corollary 3.3 and the uniqueness of solutions, it follows that

$$x(t, t_0, x_0) = \int_0^1 \Phi(t, t_0, sx_0)ds \cdot x_0, \ t \ge t_0.$$

From Theorems 3.9 and 3.11 in [4], we can show that two notions of hS and hSV for system (2.1) are equivalent.

**Lemma 3.6** ([4, Theorem 3.11])**.** *Assume that impulsive variational system* (2.2) *is* $t_\infty$*-similar to impulsive variational system* (2.3) *and* $\sup_{k\in\mathbb{N}} \frac{h(\tau_k)}{h(\tau_k^+)}$ *is bounded. Then, the zero solution of system* (2.3) *is* $h$*-stable provided the zero solution of system* (2.1) *is* $h$*-stable.*

To prove our main result, we need the following stability criterion for linear impulsive systems which is adapted from Lemma 1 in [18]. We state the result of [4, Lemma 3.7] without its proof.

**Lemma 3.7.** *Suppose that the zero solution of system* (2.3) *is* $h$*-stable. Then, there exist a constant* $c \ge 1$ *and a positive bounded left-continuous function* $h$ *defined on* $\mathbb{R}_+$ *such that for every* $x_0 \in \mathbb{R}^n$,

$$|\Phi(t, t_0, x_0)| \le ch(t)h(t_0)^{-1}, \ t \ge t_0 \ge 0, \tag{3.6}$$

*where* $\Phi(t, t_0, x_0)$ *is the fundamental matrix solution of system* (2.3).

Choi and Koo [4] provided some sufficient conditions for $h$-stability of solutions of system (2.1) by using the comparison method and piecewise continuous auxiliary functions which are analogues to Lyapunov functions. In order to obtain the main result (Theorem 3.8), Lemma 3.2 and the notion of $t_\infty$-similarity of the associated impulsive variational systems are used. Now, we can obtain the converse $h$-stability theorem of Massera's type for nonlinear impulsive differential systems which guarantee the existence of piecewise continuous Lyapunov's functions with certain properties. It is adapted from Lemma 3 in [20] and Theorem 15.3 in [2].

**Theorem 3.8.** *Assume that* $(f_x(t,0), \frac{\partial I_k(0)}{\partial x}) \in \mathcal{M}$ *is* $t_\infty$*-similar to*

$$(f_x(t, x(t,t_0,x_0)), \frac{\partial I_k(x(\tau_k))}{\partial x}) \in \mathcal{M}$$

*and* $\sum_{k=1}^{\infty} l_k < \infty$ *for nonnegative constants* $l_k$ *of each map* $I_k$ *in the condition* (A4)*. Suppose further that the zero solution of system* (2.1) *is globally* $h$*-stable such that* $h'(t)$ *exists and is continuous on* $\mathbb{R}_+$*. Then, there exist a positive constant* $c$ *and a function* $V : \mathbb{R}_+ \times \mathbb{R}^n \to \mathbb{R}_+$ *such that*

(i) $V \in v_0$ *and* $V(t,x)$ *is Lipschitzian in* $x$ *with Lipschitz constant* $L$ *for each* $t \in \mathbb{R}_+$;

(ii) $|x| \le V(t,x) \le c|x|, \ (t,x) \in \mathbb{R}_+ \times \mathbb{R}^n$;

(iii) $D^+_{(2.1)} V(t,x) \le \frac{h'(t)}{h(t)} V(t,x), \ t \ne \tau_k$;

(iv) $V(\tau_k^+, x + I_k(x)) \le V(\tau_k, x), \ x \in \mathbb{R}^n, \ k \in \mathbb{N}$,

*where* $L = c\exp(\sum_{i=1}^{\infty} l_i)$*.*

*Proof.* Since the zero solution of system (2.1) is globally $h$-stable, there exist a constant $c \ge 1$ and a positive bounded continuous function $h : \mathbb{R}_+ \to \mathbb{R}$ such that

$$|x(t,t_0,x_0)| \le c|x_0|h(t)h(t_0)^{-1}, \ t \ge t_0, \ x_0 \in \mathbb{R}^n.$$

Define the function $V : \mathbb{R}_+ \times \mathbb{R}^n \to \mathbb{R}_+$ by

$$V(t,x) = \sup_{\tau \ge 0} |x(t+\tau,t,x)|h(t+\tau)^{-1}h(t), \ t \ne \tau_k,$$
$$V(\tau_k, x) = V(\tau_k^-, x), \ k = 1,2,\dots,$$

where $x(t+\tau,t,x)$ is a solution of system (2.1) through $(t,x) \in \mathbb{R}_+ \times \mathbb{R}^n$. Then, we obtain

$$|x| = |x(t,t,x)| \le \sup_{\tau \ge 0} |x(t+\tau,t,x)|h(t+\tau)^{-1}h(t)$$

and

$$V(t,x) \le c|x|h(t+\tau)h(t)^{-1}h(t+\tau)^{-1}h(t) = c|x|.$$

Thus the property (ii) is proved for $t \ne \tau_k$ for each $k \in \mathbb{N}$.

From the definition of $hS$ and uniqueness of solutions of system (2.1), it follows that $V(t,x)$ is defined on $\mathbb{R}_+ \times \mathbb{R}^n$.

We show that $V(t,x)$ is Lipschitzian in $x$ for each $t \in \mathbb{R}_+$. Let $(t,x),(t,y) \in \mathbb{R}_+ \times \mathbb{R}^n$. From Lemmas 3.2, 3.6 and 3.7, we have

$$\begin{aligned}
(3.7) \quad & |V(t,x) - V(t,y)| \\
&\le |\sup_{\tau \ge 0} |x(t+\tau,t,x)|h(t+\tau)^{-1}h(t) - \sup_{\tau \ge 0} |x(t+\tau,t,y)|h(t+\tau)^{-1}h(t)| \\
&\le \sup_{\tau \ge 0} |x(t+\tau,t,x) - x(t+\tau,t,y)|h(t+\tau)^{-1}h(t) \\
&\le |x-y| \sup_{\eta \in D} |\Phi(t+\tau,\tau_k^+,\eta)| \prod_{i=1}^{k} (1+l_i)|\Phi(\tau_i,\tau_{i-1}^+,\eta)|h(t+\tau)^{-1}h(t)
\end{aligned}$$

$$
\begin{aligned}
&\le |x-y|c\prod_{i=1}^{\infty}(1+l_i)h(t+\tau)h(t)^{-1}h(t+\tau)^{-1}h(t)\\
&\le |x-y|c\exp(\sum_{i=1}^{\infty} l_i)\\
&\le L|x-y|,\ \tau_k < t+\tau \le \tau_{k+1},
\end{aligned}
$$

where $D$ is a convex subset of $\mathbb{R}^n$ containing $x$ and $y$, and $L = c\exp(\sum_{i=1}^{\infty} l_i)$. This implies that $V(t,x)$ is Lipschitzian in $x$ for each $t \in \mathbb{R}_+$. Let $x, y \in \mathbb{R}^n, \tau_{k-1} < t < \tau_k$ and $\delta > 0$ be such that $t+\delta < \tau_k$. Then we have

$$
\begin{aligned}
|V(t+\delta,x)-V(t,y)| \le{}& |V(t+\delta,x)-V(t+\delta,y)|\\
&+|V(t+\delta,y)-V(t+\delta,x(t+\delta,t,y))|\\
&+|V(t+\delta,x(t+\delta,t,y))-V(t,y)|.
\end{aligned}
\tag{3.8}
$$

In view of (3.7) and $\lim_{\delta\to 0}|y-x(t+\delta,t,y)| = 0$ for $t \ne \tau_k$, then the first two terms in the right-hand side of estimate (3.8) are small if $|x-y|$ and $\delta$ are small.

Denote $a(\delta) = \sup_{\tau>\delta}|x(t+\tau,t,y)|h(t+\tau)^{-1}h(t)$. The function $a(\delta)$ is non-increasing for $\delta \ge 0$ and $\lim_{\delta\to 0_+} a(\delta) = a(0)$ since $|x(t+\tau,t,x)|h(t+\tau)h(t)^{-1}$ is a bounded and piecewise continuous function for $\tau \ge 0$ and is continuous in some neighborhood of $\tau = 0$.

Then, for the third term in (3.8), we obtain

(3.9)

$$
\begin{aligned}
&|V(t+\delta,x(t+\delta,t,y))-V(t,y)|\\
&= |\sup_{s>0}|x(t+\delta+s,t+\delta,x(t+\delta,t,y))|h(t+\delta+s)^{-1}h(t+\delta)\\
&\quad -\sup_{\tau>0}|x(t+\tau,t,y)|h(t+\tau)^{-1}h(t)|\\
&= |\sup_{\tau>\delta}|x(t+\tau,t,y)|h(t+\tau)^{-1}h(t+\delta) - \sup_{\tau>0}|x(t+\tau,t,y)|h(t+\tau)^{-1}h(t)|\\
&= |a(\delta)h(t)^{-1}h(t+\delta)-a(0)| \to 0 \text{ as } \delta \to 0_+,
\end{aligned}
$$

since $h(t)$ is continuous. Hence $V(t,x)$ is continuous for $x \in \mathbb{R}^n$ and $t \ne \tau_k$.

Let $x \in \mathbb{R}^n$, $t \in \mathbb{R}_+$, $t \ne \tau_k$ and $\delta > 0$. Then

$$
\begin{aligned}
&D^+_{(2,1)}V(t,x)\\
&= \lim_{\delta\to 0_+}\sup\frac{1}{\delta}[V(t+\delta,x(t+\delta,t,x))-V(t,x)]\\
&= \lim_{\delta\to 0_+}\sup\frac{1}{\delta}[\sup_{\tau>0}|x(t+\delta+\tau,t+\delta,x(t+\delta,t,x))|h(t+\delta+\tau)^{-1}h(t+\delta)\\
&\quad -\sup_{\tau>0}|x(t+\tau,t,x)|h(t+\tau)^{-1}h(t)]\\
&\le \lim_{\delta\to 0_+}\sup\frac{1}{\delta}[\sup_{\tau>\delta}|x(t+\tau,t,x)|h(t+\tau)^{-1}h(t+\delta)
\end{aligned}
$$

$$
\begin{aligned}
&\quad - \sup_{\tau>0} |x(t+\tau,t,x)| h(t+\tau)^{-1} h(t)] \\
&\le \lim_{\delta\to 0_+} \sup \frac{1}{\delta}[\sup_{\tau>0} |x(t+\tau,t,x)| h(t+\tau)^{-1} h(t)(h(t+\delta)h(t)^{-1}-1)] \\
&\le \lim_{\delta\to 0_+} [\sup \frac{1}{\delta}[h(t+\delta)h(t)^{-1}-1]V(t,x) \\
&\le \frac{h'(t)}{h(t)} V(t,x),\ t\ne\tau_k.
\end{aligned}
$$

Since, for small $\delta>0$ and $t\ne\tau_k$,

$$
\begin{aligned}
&|V(t+\delta,x+\delta f(t,x)) - V(t,x)| \\
&\le |V(t+\delta,x+\delta f(t,x)) - V(t+\delta,x(t+\delta,t,x))| \\
&\quad + |V(t+\delta,x(t+\delta,t,x)) - V(t,x)| \\
&\le L|x+\delta f(t,x) - x(t+\delta,t,x)| + |V(t+\delta,x(t+\delta,t,x)) - V(t,x)|,
\end{aligned}
$$

it follows that

$$
D^+_{(2.1)} V(t,x) \le \frac{h'(t)}{h(t)} V(t,x),\ t\ne\tau_k.
$$

Thus, the property (iii) is proved for $t\ne\tau_k$ for each $k\in\mathbb{N}$.

Let $\tau_k\in\mathbb{R}_+$ and $x\in\mathbb{R}^n$ be fixed and $t_i\in(\tau_k,\tau_{k+1})$, $x_i\in\mathbb{R}^n$ for $i=1,2$. Then

$$
\begin{aligned}
|V(t_1,x_1) - V(t_2,x_2)| \le\ & |V(t_1,x_1) - V(t_1,x(t_1,\tau_k,x))| \\
& + |V(t_2,x_2) - V(t_2,x(t_2,\tau_k,x))| \\
& + |V(t_1,x(t_1,\tau_k,x)) - V(t_2,x(t_2,\tau_k,x))|.
\end{aligned}
\tag{3.10}
$$

Since $V(t,x)$ and $f(t,x)$ are locally Lipschitz continuous, we obtain

$$
\begin{aligned}
|V(t_i,x_i) - V(t_i,x(t_i,\tau_k,x))| &\le c|x_i - x(t_i,\tau_k,x)| \\
&\le c|x_i - x| + c|x - x(t_i,\tau_k,x)| \\
&\to 0 \text{ as } x_i\to x,\ t_i\to\tau_k,\ i=1,2,
\end{aligned}
$$

where $\lim_{t_i\to\tau_k} x(t_i,\tau_k^+,x) = x$ for $i=1,2$. Also, we obtain

$$
\begin{aligned}
&|V(t_1,x(t_1,\tau_k^+,x)) - V(t_2,x(t_2,\tau_k^+,x))| \\
&= \sup_{s>0} |x(t_1+s,t_1,x(t_1,\tau_k^+,x))| h(t_1+s)^{-1} h(t_1) \\
&\quad - \sup_{s>0} |x(t_2+s,t_2,x(t_2,\tau_k^+,x))| h(t_2+s)^{-1} h(t_2)| \\
&= |a(t_1-\tau_k)h(\tau_k^+)h(t_1)^{-1} - |a(t_2-\tau_k)h(\tau_k^+)h(t_2)^{-1}| \to 0 \text{ as } t_i\to\tau_k,\ i=1,2,
\end{aligned}
$$

since $a(\delta)=\sup_{\tau>\delta}|x(t+\tau,t,y)|h(t+\tau)^{-1}h(t)$ is non-increasing for $\delta\ge 0$ and $\lim_{\delta\to 0_-} a(\delta) = a(0)$. This implies that the limit $V(\tau_k^+,x)$ exists. From the similar manner, we have the existence of limit $V(\tau_k^-,x)$. From the equality of $V(\tau^-,x) = V(\tau_k,x)$, we have $V\in v_0$.

Let $\eta(t, t_0, x_0)$ be the solution of the initial value problem

$$\frac{d\eta}{dt} = f(t, \eta),\ \eta(t_0) = x_0.$$

In fact that the relation $x(s, \mu, \eta(\mu, \tau_k, x + I_k(x))) = x(s, \lambda, \eta(\lambda, \tau_k, x))$ holds for $\tau_{k-1} < \lambda < \tau_k < \mu < \tau_{k+1}$ and $s > \mu$, we obtain

$$V(\mu, \eta(\mu, \tau_k, x + I_k(x))) \leq V(\lambda, \eta(\lambda, \tau_k, x)).$$

Passing to the limit as $\mu \to \tau_k^+$ and $\lambda \to \tau_k^-$ yields

$$V(\tau_k^+, x + I_k(x)) \leq V(\tau_k^-, x) = V(\tau_k, x).$$

This completes the proof of theorem. □

*Remark* 3.9. (i) In the case where system (2.1) has no impulses, i.e., all $I_k$ are identically zero, then Theorem 3.8 reduces to Theorem 2.4 in [7].

(ii) Note that the convergence of the series $\sum_{k=1}^{\infty} l_k$ is equivalent to the convergence of the infinite product $\prod_{k=1}^{\infty} l_k$.

(iii) The questions about known stability properties of the solutions of various classes of differential systems with impulse effect have been studied in [2, 10, 11, 12, 16, 19, 20]. We improved well-known results on the converse Lyapunov theorems on various types of stability of solutions for system (2.1).

We can obtain the following Massera type converse theorem for the uniformly exponential asymptotic stability of impulsive differential equations as a special case of Theorem 3.8.

**Corollary 3.10.** *Assume that* $(f_x(t, 0), \frac{\partial I_k(0)}{\partial x}) \in \mathcal{M}$ *is* $t_\infty$*-similar to*

$$(f_x(t, x(t, t_0, x_0)), \frac{\partial I_k(x(\tau_k))}{\partial x}) \in \mathcal{M}.$$

*Suppose further that the zero solution of system* (2.1) *is globally* $h$*-stable with* $h(t) = e^{-\lambda t}$ *for a nonnegative constant* $\lambda$*. Then, there exist a positive constant* $c$ *and a function* $V : \mathbb{R}_+ \times \mathbb{R}^n \to \mathbb{R}_+$ *such that*

(i) $V \in v_0$ *and* $V(t, x)$ *is Lipschitzian in* $x$ *with Lipschitz constant* $L$ *for each* $t \in \mathbb{R}_+$;

(ii) $|x| \leq V(t, x) \leq c|x|,\ (t, x) \in \mathbb{R}_+ \times \mathbb{R}^n$;

(iii) $D^+_{(2.1)} V(t, x) \leq -\lambda V(t, x),\ t \neq \tau_k$;

(iv) $V(\tau_k^+, x + I_k(x)) \leq V(\tau_k, x),\ k \in \mathbb{N},\ x \in \mathbb{R}^n$.

## 4. Examples

In this section we give two examples which illustrate some results from the previous section.

**Example 4.1** ([5, Example 2.10])**.** To illustrate Lemma 3.7, we consider the linear impulsive differential equation

$$\begin{cases} x' &= a(t)x,\ t \neq \tau_k, \\ \Delta x &= a_k x, \quad t = \tau_k,\ k \in \mathbb{N}, \end{cases} \tag{4.1}$$

where $a \in PC(\mathbb{R}_+, \mathbb{R}), a_k \in \mathbb{R}$, and $\det(1 + a_k) \neq 0, k \in \mathbb{N}$. Suppose that $\int_{t_0}^{\infty} |a(s)|ds < \infty$ and $\sum_{t_0 \le \tau_k \le \infty} |a_k| < \infty$ for each $t_0 \in \mathbb{R}_+$. Then the solution $x(t, t_0, x_0)$ of equation (4.1) given by

$$x(t, t_0, x_0) = \prod_{t_0 \le \tau_k < t} (a_k + 1) \exp(\int_{t_0}^{t} a(s)ds)x_0$$

satisfies

$$\begin{aligned} |x(t, t_0, x_0)| &= |\prod_{t_0 \le \tau_k < t} (a_k + 1) \exp(\int_{t_0}^{t} a(s)ds)||x_0| \\ &\le |x_0| \exp(\int_{t_0}^{t} |a(s)|ds + \sum_{t_0 \le \tau_k < t} |a_k|) \\ &\le c|x_0|h(t)h(t_0)^{-1},\ t \ge t_0, \end{aligned}$$

where $c = 1$ and $h : \mathbb{R}_+ \to \mathbb{R}$ is a positive bounded left-continuous function given by

$$h(t) = \begin{cases} \exp(\int_0^t |a(s)|ds),\ t_0 \le t \le \tau_1, \\ \cdots, \\ \exp(\int_0^t |a(s)|ds + \sum_{t_0 \le \tau_k < t} |a_k|),\ \tau_k < t \le \tau_{k+1},\ k \in \mathbb{N}. \end{cases}$$

Hence the zero solution of equation (4.1) is $h$-stable.

**Example 4.2** ([7, 11])**.** To illustrate Theorem 3.8, we consider the Ricatti scalar equation with impulse effect

$$\begin{cases} x' = f(t, x) &= \lambda(t)(-x + x^2),\ t \neq \tau_k, \\ \Delta x = I_k(x) &= d_k x,\ t = \tau_k,\ k = 1, 2, \ldots, \\ x(t_0^+) &= x_0, \end{cases} \tag{4.2}$$

where $\lambda : \mathbb{R}_+ \to \mathbb{R}_+$ is a continuous function on $[\tau_{k-1}, \tau_k],\ k = 1, 2, \ldots,$ with points of discontinuity of first kind at $t = \tau_k$, and $\det(d_k + 1) \neq 0,\ k \in \mathbb{N}$. Assume $\int_{t_0}^{\infty} \lambda(s)ds < \infty$ for each $t_0 \in \mathbb{R}_+$ and $\sum_{k=1}^{\infty} |d_k| < \infty$. Then the solution $x(t, t_0, x_0)$ of equation (4.2) with $x_0 \in D \subset \mathbb{R}$ is given by

$$x(t) = \begin{cases} \frac{x_0}{x_0 + (1 - x_0)\exp(\int_{t_0}^{t} \lambda(s)ds)},\ t_0 \le t \le \tau_1, \\ \frac{(d_1+1)x_0}{\exp(\int_{t_0}^{t} \lambda(s)ds) + x_0[(d_1+1) - \exp(\int_{t_0}^{t} \lambda(s)ds - d_1 \exp(\int_{\tau_1}^{t} \lambda(s)ds]},\ \tau_1 < t \le \tau_2, \\ \cdots, \\ \frac{\prod_{i=1}^{k}(d_i+1)x_0}{e^{\int_{t_0}^{t} \lambda(s)ds} + x_0 g(t,k)},\ \tau_k < t \le \tau_{k+1},\ k = 2, 3, \ldots, \end{cases}$$

where

$$g(t,k)=\prod_{i=1}^{k}(d_i+1)-e^{\int_{t_0}^{t}\lambda(s)ds}-d_1e^{\int_{\tau_1}^{t}\lambda(s)ds}-\cdots-d_k\prod_{i=1}^{k-1}(d_i+1)e^{\int_{\tau_k}^{t}\lambda(s)ds}.$$

Then, we obtain the associated impulsive variational systems for equation (4.2) along the solution $x(t,t_0,x_0)$ of equation (4.2) as the following:

$$\begin{cases} v'=f_x(t,0)v & =-\lambda(t)v,\ t\neq\tau_k,\\ \Delta v=\frac{\partial I_k(0)}{\partial x}v & =d_kv,\ t=\tau_k,\ k\in\mathbb{N},\\ v(t_0+0) & =v_0 \end{cases}\tag{4.3}$$

and

$$\begin{cases} z'=f_x(t,x(t,t_0,x_0))z & =\lambda(t)[-1+2x(t,t_0,x_0)]z,\ t\neq\tau_k,\\ \Delta z=\frac{\partial I_k(x(\tau_k))}{\partial x}z & =d_kz,\ t=\tau_k,\ k\in\mathbb{N},\\ z(t_0+0) & =z_0. \end{cases}\tag{4.4}$$

Then we have

$$\begin{aligned}|\Phi(t,t_0,0)=\frac{\partial x(t,t_0,x_0)}{\partial x_0}| &=|\prod_{t_0\le\tau_k<t}(d_k+1)\exp(-\int_{t_0}^{t}\lambda(s)ds)|\\ &\le\exp(\int_{t_0}^{t}\lambda(s)ds+\sum_{t_0\le\tau_k<t}|d_k|)\\ &\le ch(t)h(t_0)^{-1},\ t\ge t_0,\end{aligned}$$

where $h(t)=\exp(-\int_0^t\lambda(s)ds)$ and $c=\exp(\sum_{t_0\le\tau_k\le\infty}|d_k|)$. Thus the zero solution of equation (4.3) is $h$-stable. Also, there exist $F_0\in L_1$ with $t\neq\tau_k$ and $F_k\equiv 0\in l_1, k\in\mathbb{N}$, such that

$$\begin{aligned}&S'(t)-f_x(t,0)S(t)+S(t)f_x(t,x(t,t_0,x_0))\equiv F_0\in L_1,\ t\neq\tau_k,\\ &\Delta S(\tau_k)-d_kS(\tau_k)+S(\tau_k^+)d_k\equiv F_k\in l_1,\ t=\tau_k,\ k\in\mathbb{N},\end{aligned}$$

where $\Delta S(\tau_k)=S(\tau_k^+)-S(\tau_k)$ and $S(t)=1$, since

$$\begin{aligned}\int_0^\infty|F_0(s)|ds &=\int_0^\infty|f_x(s,x(s,t_0,x_0))-f_x(s,0)|ds\\ &=\int_0^\infty|2\lambda(s)x(s,t_0,x_0)|ds\\ &\le 2|\prod_{i=1}^{k}(d_i+1)||x_0|\int_0^\infty|\frac{\lambda(s)}{e^{\int_{t_0}^{s}\lambda(\tau)d\tau}+x_0g(s,k)}|ds\\ &\le|x_0|2\exp(\sum_{k=1}^{\infty}|d_k|)\int_0^\infty\lambda(s)\exp(-\int_{t_0}^{s}\lambda(\tau)d\tau)ds<\infty,\end{aligned}$$

where the limit $\lim_{k\to\infty} x_0 g(t,k)$ exists and $0 < \lim_{k\to\infty} x_0 g(t,k) \equiv C(x_0)$. Furthermore, we have

$$\Delta S(\tau_k) - d_k S(\tau_k) + S(\tau_k^+) d_k = 0 \equiv F_k \in l_1,\ t = \tau_k,\ k \in \mathbb{N}.$$

Thus $(f_x(t,0), d_k) \in \mathcal{M}$ is $t_\infty$-similar to $(f_x(t, x(t,t_0,x_0)), d_k) \in \mathcal{M}$ for each $k \in \mathbb{N}$. Therefore, the zero solution of equation (4.4) is also $h$-stable by Lemma 3.6.

Finally, all assumptions of Theorem 3.8 are satisfied, then there exist a positive constant $c$ and a function $V : \mathbb{R}_+ \times I \to \mathbb{R}$ with $I \subset \mathbb{R}$ which satisfies conditions (i)-(iv) of Theorem 3.8.

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