J. Korean Math. Soc. **53** (2016), No. 5, pp. 1101–1114
http://dx.doi.org/10.4134/JKMS.j150416
pISSN: 0304-9914 / eISSN: 2234-3008

GRADIENT RICCI ALMOST SOLITONS ON TWO CLASSES OF ALMOST KENMOTSU MANIFOLDS

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Abstract. Let $(M^{2n+1},\phi,\xi,\eta,g)$ be a $(k,\mu)'$-almost Kenmotsu manifold with $k<-1$ which admits a gradient Ricci almost soliton (g,f,λ), where λ is the soliton function and f is the potential function. In this paper, it is proved that λ is a constant and this implies that M^{2n+1} is locally isometric to a rigid gradient Ricci soliton $\mathbb{H}^{n+1}(-4)\times\mathbb{R}^n$, and the soliton is expanding with $\lambda=-4n$. Moreover, if a three dimensional Kenmotsu manifold admits a gradient Ricci almost soliton, then either it is of constant sectional curvature -1 or the potential vector field is pointwise colinear with the Reeb vector field.

1. Introduction

It is well-known that a *Ricci soliton* on a Riemannian manifold (M,g) (see Hamilton [13]) is defined by

$$\frac{1}{2}\mathcal{L}_V g+S=\lambda g \tag{1.1}$$

for a vector field V and a certain constant λ on M and it is denoted by (g,V,λ), where S denotes the Ricci tensor. In general, V and λ are called the *potential vector field* and *soliton constant*, respectively. Obviously, if the potential vector field is either a Killing vector field or vanishing, then a Ricci soliton reduces to an Einstein metric (that is, the Ricci tensor is a constant multiple of the Riemannian metric if the dimension of the manifold is greater than 2). In 1982, Hamilton [12] introduced the notion of Ricci flow to find a canonical metric on a smooth manifold. The Ricci flow is an evolution equation for metrics on a Riemannian manifold defined by

$$\frac{\partial}{\partial_t}g_{ij}(t)=-2S_{ij}.$$

It is worth mentioning that Ricci solitons with complete potential vector field correspond to self-similar solutions to the Ricci flow. Moreover, if the

Received July 9, 2015.
2010 *Mathematics Subject Classification.* Primary 53D15; Secondary 53C25, 53C35.
Key words and phrases. gradient Ricci almost soliton, $(k,\mu)'$-almost Kenmotsu manifold, 3-dimensional Kenmotsu manifold, Einstein metric.

potential vector field V is the gradient of some function f on M, then (1.1) becomes

$$\nabla\nabla f + S = \lambda g \tag{1.2}$$

and is called a *gradient Ricci soliton.* According to Perelman [16], we know that a Ricci soliton on a compact manifold is always a gradient Ricci soliton.

Recently, the notion of Ricci soliton was generalized to Ricci almost soliton on a Riemannian manifold (M, g) (see Pigola et al. [19]). Generally, (1.1) is called a *Ricci almost soliton* if λ is a variable function on M. In addition, a Ricci almost soliton is said to be a *gradient Ricci almost soliton* if the potential vector field V is the gradient of some function f on M and it is denoted by (g, f, λ), where λ is called the *soliton function.* A Ricci almost soliton is said to be *shrinking*, *steady* or *expanding* according as $\lambda > 0$, $\lambda = 0$ or $\lambda < 0$, respectively. Otherwise, it will be called *indefinite.* It is worth pointing out that Barros et al. [1] proved that a Ricci almost soliton on a compact manifold is a gradient Ricci almost soliton provided that the scalar curvature is a constant.

The studies of Ricci solitons on almost contact metric manifolds were initiated by R. Sharma [20]. In the paper, the author improved some results proved by Boyer and Galicki [3]. More precisely, he obtained an odd-dimensional analogue of the well-known Goldberg's conjecture in the framework of contact geometry. Recently, Ricci solitons on three dimensional Kenmotsu manifolds and η-Einstein Kenmotsu manifolds of dimension > 3 were studied by Ghosh [9], Cho [5] and Ghosh [10], respectively. Generalizing some results shown in [5, 9], the present author jointly with Liu in [23] studied Ricci solitons on three dimensional η-Einstein almost Kenmotsu manifolds. Moreover, the existences of gradient Ricci solitons on $(k, \mu)'$-almost Kenmotsu manifolds were also investigated by the present author et al. [22].

We recall that Ghosh [11] and Sharma [21] recently obtained some results concerning Ricci almost solitons on some types of contact metric manifolds, which extended some earlier results proved by Sharma [20] and Cho and Sharma [6]. Motivated by these results, we shall investigate gradient Ricci almost solitons on three dimensional Kenmotsu manifolds and $(k, \mu)'$-almost Kenmotsu manifolds with $k < -1$, respectively. We mainly generalize some results proved by Wang et al. [22] and obtain some new examples of non-trivial gradient Ricci almost solitons on three dimensional Kenmotsu manifolds.

This paper is organized as follows. In Section 2, we recall some fundamental formulas and properties of almost Kenmotsu manifolds. In Section 3, we prove that a gradient Ricci almost soliton on a $(k, \mu)'$-almost Kenmotsu manifold with $k < -1$ is, in fact, a rigid gradient Ricci soliton and this extend some corresponding results shown in [22]. In Section 4, it is proved that a gradient Ricci almost soliton on a three dimensional Kenmotsu manifold either is trivial, (i.e., an Einstein metric, and hence the Kenmotsu manifold is of constant sectional curvature -1), or the potential vector field is pointwise colinear with the Reeb vector field. Finally, we show that $(g, \beta\xi, \lambda)$ (where β is a variable function) on a three dimensional Kenmotsu manifold is a Ricci soliton if and only if $\beta = 0$ (i.e., the trivial case). As a corollary of our main results, non-trivial gradient Ricci almost solitons on a type of warped products are also introduced.

2. Almost Kenmotsu manifolds

If on a $(2n+1)$-dimensional smooth manifold M^{2n+1} there exist a $(1,1)$-type tensor field ϕ, a global vector field ξ and a 1-form η such that

$$\phi^2 = -\mathrm{id} + \eta \otimes \xi, \ \eta(\xi) = 1, \tag{2.1}$$

where id denotes the identity endomorphism, then we say that the triplet (ϕ, ξ, η) is an *almost contact structure* on M^{2n+1} and ξ is called the *characteristic* or the *Reeb vector field* (see Blair [2]). It follows from relation (2.1) that $\phi(\xi) = 0$, $\eta \circ \phi = 0$ and $\mathrm{rank}(\phi) = 2n$. In general, a smooth manifold M^{2n+1} endowed with an almost contact structure is called an *almost contact manifold* and therefore it is denoted by $(M^{2n+1}, \phi, \xi, \eta)$. It is well-known that a smooth manifold M^{2n+1} admits an almost contact structure if and only if the structure group of the tangent bundle of M^{2n+1} reduces to $U(n) \times 1$.

If on an almost contact manifold $(M^{2n+1}, \phi, \xi, \eta)$ there exists a Riemannian metric g satisfying

$$g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y) \tag{2.2}$$

for any vector fields X, Y, then the metric is said to be *compatible* with the almost contact structure. A smooth manifold furnished with an almost contact structure and a compatible Riemannian metric is said to be an *almost contact metric manifold* and it is denoted by $(M^{2n+1}, \phi, \xi, \eta, g)$.

The *fundamental* 2-*form* Φ of an almost contact metric manifold M^{2n+1} is defined by $\Phi(X, Y) = g(X, \phi Y)$ for any vector fields X and Y. We define an almost complex structure J on the product manifold $M^{2n+1} \times \mathbb{R}$ by

$$J\left(X, f\frac{\mathrm{d}}{\mathrm{d}t}\right) = \left(\phi X - f\xi, \eta(X)\frac{\mathrm{d}}{\mathrm{d}t}\right),$$

where X denotes the vector field tangent to M^{2n+1}, t is the coordinate of $\mathbb{R}$ and f is a smooth function defined on the product manifold $M^{2n+1} \times \mathbb{R}$. An almost contact structure is said to be *normal* if the above almost complex structure is integrable. According to Blair [2], the *normality* of an almost contact structure is expressed by $[\phi, \phi] = -2d\eta \otimes \xi$, where $[\phi, \phi]$ denotes the Nijenhuis tensor of ϕ which is defined by

$$[\phi, \phi](X, Y) = \phi^2[X, Y] + [\phi X, \phi Y] - \phi[\phi X, Y] - \phi[X, \phi Y]$$

for any vector fields X and Y on M^{2n+1}.

According to Janssens and Vanhecke [14], an almost contact metric manifold satisfying $\mathrm{d}\eta = 0$ and $\mathrm{d}\Phi = 2\eta \wedge \Phi$ is called an *almost Kenmotsu manifold*. An almost Kenmotsu manifold with a normal almost contact structure is said to be a *Kenmotsu manifold*.

On an almost Kenmotsu manifold M^{2n+1}, we set $l = R(\cdot,\xi)\xi$, $h = \frac{1}{2}\mathcal{L}_\xi\phi$ and $h' = h \circ \phi$, where R denotes the curvature tensor of M^{2n+1} and $\mathcal{L}$ is the Lie differentiation. The above these tensor fields play key roles in the studies of geometry of almost Kenmotsu manifolds. According to [7, 8], the three $(1,1)$-type tensor fields l, h and h' are all symmetric and satisfy the following equations.

$$h\xi = l\xi = 0, \quad \mathrm{tr}h = \mathrm{tr}h' = 0, \quad h\phi + \phi h = 0, \tag{2.3}$$

$$\nabla_X\xi = X - \eta(X)\xi + h'X, \tag{2.4}$$

$$\phi l\phi - l = 2(h^2 - \phi^2), \tag{2.5}$$

for any vector field X on M^{2n+1}, where ∇ and tr denote the Levi-Civita connection of g and the trace operator, respectively.

3. Gradient Ricci almost solitons on $(k,\mu)'$-almost Kenmotsu manifolds with $k < -1$

By a $(k,\mu)'$*-almost Kenmotsu manifold*, we mean an almost Kenmotsu manifold $(M^{2n+1},\phi,\xi,\eta,g)$ with the characteristic vector field ξ belonging to the $(k,\mu)'$-nullity distribution (see Dileo and Pastore [8]), that is,

$$R(X,Y)\xi = k(\eta(Y)X - \eta(X)Y) + \mu(\eta(Y)h'X - \eta(X)h'Y) \tag{3.1}$$

for any vector fields X and $Y \in \mathfrak{X}(M)$, where $k,\mu \in \mathbb{R}$, $h' = h \circ \phi$ and $\mathfrak{X}(M)$ denotes the Lie algebra of all vector fields on M^{2n+1}. Also, we denote by $\mathcal{D}$ the distribution defined by $\mathcal{D} = \ker\eta$. In relation (3.1), substituting Y with ξ gives that $l = -k\phi^2 + \mu h'$. Putting this relation into equation (2.5) and using (2.3), we obtain

$$h'^2 = (k+1)\phi^2. \tag{3.2}$$

By (2.1), it follows from relation (3.2) that $k \leq -1$. Obviously, by (3.2), we see that the tensor field h' vanishes if and only if $k = -1$. In the present section, we aim to investigate the existences of the gradient Ricci almost solitons on a $(k,\mu)'$-almost Kenmotsu manifold with $k < -1$. According to [8, Proposition 4.1], on a $(k,\mu)'$-almost Kenmotsu manifold with $k < -1$, we have $\mu = -2$. We also observe from (3.2) that $k < -1$ if and only if $h \neq 0$ (or, equivalently, $h' \neq 0$). The following result is deduced directly from Dileo and Pastore [8, Proposition 4.2].

Lemma 3.1 ([25, Lemma 3.2])**.** *Let $(M^{2n+1},\phi,\xi,\eta,g)$ be a $(k,\mu)'$-almost Kenmotsu manifold with $h' \neq 0$. Then the Ricci operator of M^{2n+1} is given by*

$$Q = -2n\mathrm{id} + 2n(k+1)\eta\otimes\xi - 2nh', \tag{3.3}$$

where $k < -1$. Moreover, the scalar curvature of M^{2n+1} is $2n(k-2n)$.

Now we consider an almost Kenmotsu manifold $(M^{2n+1}, \phi, \xi, \eta, g)$ which admits a gradient Ricci almost soliton (g, f, λ), i.e.,

$$\nabla_X Df = -QX + \lambda X \tag{3.4}$$

for any $X \in \mathfrak{X}(M)$, where λ is the soliton function on M^{2n+1}. Then, it follows from (3.4) that

$$\begin{aligned} R(X,Y)Df &= \nabla_X \nabla_Y Df - \nabla_Y \nabla_X Df - \nabla_{[X,Y]} Df \\ &= (\nabla_Y Q)X - (\nabla_X Q)Y - Y(\lambda)X + X(\lambda)Y \end{aligned} \tag{3.5}$$

for any $X, Y \in \mathfrak{X}(M)$, where Df denotes the gradient of the potential function f. Applying Lemma 3.1, relations (3.4) and (3.5), we obtain the following result.

Theorem 3.1. *Let $(M^{2n+1}, \phi, \xi, \eta, g)$ be a $(k, \mu)'$-almost Kenmotsu manifold with $k < -1$ which admits a gradient Ricci almost soliton (g, f, λ). Then, the soliton is expanding with $\lambda = -4n$ and M^{2n+1} is locally isometric to a rigid gradient Ricci soliton $\mathbb{H}^{n+1}(-4) \times \mathbb{R}^n$. Moreover, the potential vector field is tangential to the Euclidean factor $\mathbb{R}^n$.*

Proof. In case of $k < -1$, taking the covariant derivative of equation (3.3) along any vector field $Y \in \mathfrak{X}(M)$ and using (2.4) we have

$$\begin{aligned} (\nabla_Y Q)X = {} & 2n(k+1)\eta(X)(Y + h'Y) - 2n(\nabla_Y h')X \\ & + 2n(k+1)(g(X,Y) - 2\eta(X)\eta(Y) + g(h'X, Y))\xi \end{aligned}$$

for any $X, Y \in \mathfrak{X}(M)$. Using the above equation in (3.5) we obtain

$$\begin{aligned} R(X,Y)Df = {} & 2n(k+1)\eta(X)(Y + h'Y) - 2n(k+1)\eta(Y)(X + h'X) \\ & - 2n(\nabla_Y h')X + 2n(\nabla_X h')Y - Y(\lambda)X + X(\lambda)Y \end{aligned} \tag{3.6}$$

for any $X, Y \in \mathfrak{X}(M)$. In view of equations (2.3), (2.4) and (3.2), it follows from (3.6) that

$$g(R(X,Y)Df, \xi) = -\eta(X)Y(\lambda) + \eta(Y)X(\lambda) \tag{3.7}$$

for any $X, Y \in \mathfrak{X}(M)$. On the other hand, using (3.1) we have

$$\begin{aligned} g(R(X,Y)\xi, Df) = {} & k\eta(Y)X(f) - k\eta(X)Y(f) \\ & - 2\eta(Y)g(h'Df, X) + 2\eta(X)g(h'Df, Y) \end{aligned} \tag{3.8}$$

for any $X, Y \in \mathfrak{X}(M)$, where we have used $\mu = -2$. Comparing (3.7) with (3.8) yields that

$$\eta(Y)X(kf + \lambda) - \eta(X)Y(kf + \lambda) - 2\eta(Y)g(h'Df, X) + 2\eta(X)g(h'Df, Y) = 0$$

for any $X, Y \in \mathfrak{X}(M)$, and substituting Y with ξ in this equation we get

$$D\lambda = -kDf + \xi(\lambda)\xi + k\xi(f)\xi + 2h'Df. \tag{3.9}$$

According to Corollary 4.1 of Dileo and Pastore [8], we obtain that a $(k, \mu)'$-almost Kenmotsu manifold M^{2n+1} with $k < -1$ is CR-integrable (i.e., the induced almost complex structure from ϕ on $\mathcal{D}$ is integrable). In addition, it follows from relation (3.3) that $Q\xi = 2nk\xi$. Thus, applying Lemma 3.4 of Wang and Liu [24] we have that $\mathrm{tr}(\nabla_X h') = 0$ and $(\mathrm{div} h')X = 2n(k+1)\eta(X)$ for any $X \in \mathfrak{X}(M)$. Therefore, contracting Y in (3.6) gives that

$$S(X, Df) = -2nX(\lambda)$$

for any $X \in \mathfrak{X}(M)$, and comparing this relation with (3.3) we obtain

$$D\lambda = Df - (k+1)\xi(f)\xi + h'Df. \tag{3.10}$$

Obviously, it follows (3.9) and (3.10) that

$$(k+1)Df - (2k+1)\xi(f)\xi - \xi(\lambda)\xi - h'Df = 0.$$

Taking into account the assumption $k < -1$ and equation (3.2), then the action of h' on the above equation gives that

$$h'Df = -Df + \xi(f)\xi. \tag{3.11}$$

In view of the action of h' on (3.11), using again equation (3.2) we obtain

$$(k+1)(Df - \xi(f)\xi) = h'Df.$$

Comparing the above relation with (3.11) gives that either $k = -2$ or $Df = \xi(f)\xi$. Next, we discuss the above two cases as follows.

Case i: $Df = \xi(f)\xi$. Taking the covariant derivative of $Df = \xi(f)\xi$ along any vector field $X \in \mathfrak{X}(M)$ and using (2.4) we get

$$\nabla_X Df = X(\xi(f))\xi + \xi(f)X - \xi(f)\eta(X)\xi + \xi(f)h'X$$

for any $X \in \mathfrak{X}(M)$. Putting the above equation into (3.4) yields that

$$QX = (\lambda - \xi(f))X + (\xi(f)\eta(X) - X(\xi(f)))\xi - \xi(f)h'X \tag{3.12}$$

for any $X \in \mathfrak{X}(M)$. Comparing (3.12) with (3.3) gives that

$$\begin{aligned}(2n + \lambda - \xi(f))X &+ (2n - \xi(f))h'X \\ &+ (\xi(f)\eta(X) - X(\xi(f)) - 2n(k+1)\eta(X))\xi = 0\end{aligned} \tag{3.13}$$

for any $X \in \mathfrak{X}(M)$. By the action of h' on equation (3.13) and making use of (2.3) and (3.2), we obtain

$$(2n + \lambda - \xi(f))h'X - (k+1)(2n - \xi(f))X + (k+1)(2n - \xi(f))\eta(X)\xi = 0$$

for any $X \in \mathfrak{X}(M)$. Contracting X in the above equation and using (2.3) we have

$$2n(k+1)(2n - \xi(f)) = 0$$

and hence by the assumption $k < -1$ we obtain $\xi(f) = 2n$. Using $\xi(f) = 2n$ in (3.13) gives

$$\lambda X - 2nk\eta(X)\xi = 0 \tag{3.14}$$

for any $X \in \mathfrak{X}(M)$. Thus, by considering a vector field X orthogonal to ξ in equation (3.14) we have that $\lambda = 0$. Therefore, (3.14) becomes $2nk\eta(X)\xi = 0$

for any $X \in \mathfrak{X}(M)$. It follows that $k = 0$, this contradicts the assumption $k < -1$.

Case ii: $k = -2$. In view of $k = \mu = -2$, according to Corollary 4.2 and Proposition 4.1 of Dileo and Pastore [8] we obtain that M^{2n+1} is locally isometric to the Riemannian product $\mathbb{H}^{n+1}(-4) \times \mathbb{R}^n$. In fact, from Petersen and Wylie [17, 18] we state that the product $\mathbb{H}^{n+1}(-4) \times \mathbb{R}^n$ is a rigid gradient Ricci soliton. Next, we need to show that λ is a constant. Using $k = -2$ and (3.11) in equation (3.9) we obtain

$$D\lambda = \xi(\lambda)\xi. \tag{3.15}$$

Putting (3.15) and (3.11) into (3.10) gives that

$$\xi(2f - \lambda) = 0. \tag{3.16}$$

In this context, it follows from equations (3.3), (3.4) and (3.16) that

$$\lambda = S(\xi, \xi) + g(\nabla_\xi Df, \xi) = -4n + \frac{1}{2}\xi(\xi(\lambda)). \tag{3.17}$$

By relation $h'^2 = -\phi^2$, we shall denote by $[1]'$ and $[-1]'$ the distributions of the eigenvectors of h' orthogonal to ξ with eigenvalues 1 and -1, respectively. Also, from $h'^2 = -\phi^2$ we may consider a local orthonormal ϕ-frame $\{\xi, e_i, \phi e_i\}$ for $1 \leq i \leq n$ with $e_i \in [1]'$ and $\phi e_i \in [-1]'$. From (3.11), we see that Df has no components on the distribution $[1]'$. Thus, we write $Df = \sum_{i=1}^n \beta_i \phi e_i + \xi(f)\xi$, where β_i, $1 \leq i \leq n$, are smooth functions on M^{2n+1}. Using this and equation (3.16) in (3.4) we have

$$\begin{aligned} QX = \left(\lambda - \frac{1}{2}\xi(\lambda)\right) X - \sum_{i=1}^n X(\beta_i)\phi e_i - \sum_{i=1}^n \beta_i \nabla_X \phi e_i \\ - \frac{1}{2}\xi(\lambda) h'X + \frac{1}{2}(\xi(\lambda)\eta(X) - X(\xi(\lambda)))\xi \end{aligned}$$

for any $X \in \mathfrak{X}(M)$. Combining the above equation with (3.3) yields that

$$\begin{aligned} &\left(2n + \lambda - \frac{1}{2}\xi(\lambda)\right) X - \sum_{i=1}^n X(\beta_i)\phi e_i - \sum_{i=1}^n \beta_i \nabla_X \phi e_i \\ &+ \left(2n - \frac{1}{2}\xi(\lambda)\right) h'X + \frac{1}{2}(4n\eta(X) + \xi(\lambda)\eta(X) - X(\xi(\lambda)))\xi = 0 \end{aligned} \tag{3.18}$$

for any $X \in \mathfrak{X}(M)$. According to the proof of Proposition 4.1 of [8], we have that $\nabla_{e_j} \phi e_i \in [-1]'$ for any $e_j \in [1]'$, $1 \leq j \leq n$. Consequently, substituting X with $e_j \in [1]'$ in (3.18) we obtain that

$$\lambda = \xi(\lambda) - 4n. \tag{3.19}$$

Finally, applying (3.19) in (3.17) we have $\lambda = -4n$, and this means that the gradient Ricci soliton is expanding. By using $\lambda = -4n$ in equation (3.16) we get $\xi(f) = 0$ and hence from (3.11) we have $h'Df = -Df$. Actually, by Theorem 4.2 of [8], on product space $\mathbb{H}^{n+1}(-4) \times \mathbb{R}^n$ the factor $\mathbb{R}^n$ is the integral submanifold of the distribution $[-1]'$. Then, $h'Df = -Df$ implies that the gradient of the potential function Df is tangential to the Euclidean factor $\mathbb{R}^n$. This completes the proof. $\square$

Remark 3.1. Theorem 1.2 of Wang et al. [22] is a direct corollary of the above Theorem 3.1.

Remark 3.2. We observe that some sufficient conditions for a compact manifold to be a rigid gradient Ricci soliton were presented by Petersen and Wylie [17, 18]. However, in our Theorem 3.1, the strictly almost Kenmotsu manifold $\mathbb{H}^{n+1}(-4)\times\mathbb{R}^n$ (a rigid gradient Ricci soliton) is non-compact due to $\mathrm{div}\xi = 2n$ (this can be deduced from equations (2.3) and (2.4)).

4. Gradient Ricci almost solitons on three dimensional Kenmotsu manifolds

It is proved in Dileo and Pastore [7] that the almost contact metric structure of an almost Kenmotsu manifold is normal if and only if the foliations of the distribution $\mathcal{D}$ are Kählerian and the $(1,1)$-type tensor field h vanishes. In particular, as a consequence, we get immediately that a three dimensional almost Kenmotsu manifold is a Kenmotsu manifold if and only if $h = 0$. In this section, we aim to investigate the existences of gradient Ricci almost solitons on a three dimensional Kenmotsu manifold.

On a Kenmotsu manifold of dimension 3, by using $h = 0$ in equation (2.4) we have $\nabla\xi = -\phi^2$, and this implies that

$$R(X,Y)\xi = -\eta(Y)X + \eta(X)Y \tag{4.1}$$

for any $X, Y \in \mathfrak{X}(M)$ and hence by contracting Y in (4.1) we get $Q\xi = -2\xi$.

We present the following useful result with its proof as follows.

Lemma 4.1. *Let $(M^3, \phi, \xi, \eta, g)$ be a three dimensional Kenmotsu manifold. Then we have*

$$\xi(r) = -2(r+6), \tag{4.2}$$

where r denotes the scalar curvature of M^3.

Proof. On any three dimensional Riemannian manifold (M^3, g), since the Weyl conformal tensor vanishes, then the following formula holds.

$$\begin{aligned} R(X,Y)Z = {} & g(Y,Z)QX - g(X,Z)QY + S(Y,Z)X - S(X,Z)Y \\ & - \frac{r}{2}\{g(Y,Z)X - g(X,Z)Y\} \end{aligned} \tag{4.3}$$

for any vector fields X, Y, Z on M^3. Replacing $Y = Z$ by ξ in the above equation and making use of (4.1) we have

$$Q = \left(\frac{r}{2} + 1\right)\mathrm{id} - \left(\frac{r}{2} + 3\right)\eta \otimes \xi. \tag{4.4}$$

This means that M^3 is an η-Einstein manifold. Taking the covariant derivative of relation (4.4) along any vector field and using $Q\xi = -2\xi$ we get

$$\begin{aligned}(\nabla_X Q)Y = {}& \frac{1}{2}X(r)Y - \left(\frac{r}{2}+3\right)\eta(Y)X - \frac{1}{2}X(r)\eta(Y)\xi \\ & - \left(\frac{r}{2}+3\right)g(X,Y)\xi + (r+6)\eta(X)\eta(Y)\xi\end{aligned} \tag{4.5}$$

for any $X, Y \in \mathfrak{X}(M)$. Next, let us recall that the following well-known formula holds on any Riemannian manifolds:

$$\mathrm{div}Q = \frac{1}{2}\mathrm{grad}r.$$

Making using of equation (4.5) in the above formula and taking into account $\nabla\xi = -\phi^2$ we obtain

$$\xi(r)\eta(Y) = -2(r+6)\eta(Y)$$

for any vector field Y on M^3. Substituting Y with ξ in the above equation we obtain (4.2). This completes the proof. □

Lemma 4.2. *Let $(M^3, \phi, \xi, \eta, g)$ be a three dimensional Kenmotsu manifold which admits a Ricci almost soliton. Then we have*

$$\Delta\lambda = 4(\lambda+2) - \xi(\xi(\lambda)), \tag{4.6}$$

where Δ denote the Laplacian operator.

Proof. Suppose that on a three dimensional Kenmotsu manifold M^3 there exists a Ricci almost soliton (g, V, λ), where λ is the soliton function on M^3. It follows from (1.1) and (4.4) that

$$\mathcal{L}_V g = (2\lambda - r - 2)g + (r+6)\eta\otimes\eta. \tag{4.7}$$

Taking the covariant derivative of the above equation along any vector field $X \in \mathfrak{X}(M)$, we obtain

$$\nabla_X\mathcal{L}_V g = X(2\lambda - r)g + X(r)\eta\otimes\eta + (r+6)((\nabla_X\eta)\otimes\eta + \eta\otimes(\nabla_X\eta)) \tag{4.8}$$

for any $X \in \mathfrak{X}(M)$. According to Yano [26], we also have

$$\begin{aligned}&(\mathcal{L}_V\nabla_X g - \nabla_X\mathcal{L}_V g - \nabla_{[V,X]}g)(Y,Z) \\ &= -\, g((\mathcal{L}_V\nabla)(X,Y),Z) - g((\mathcal{L}_V\nabla)(X,Z),Y)\end{aligned}$$

for any vector fields $X, Y, Z \in \mathfrak{X}(M)$. In view of the parallelism of the Riemannian metric g, we get from the above relation that

$$(\nabla_X\mathcal{L}_V g)(Y,Z) = g((\mathcal{L}_V\nabla)(X,Y),Z) + g((\mathcal{L}_V\nabla)(X,Z),Y)$$

for any vector fields $X, Y, Z \in \mathfrak{X}(M)$. Taking into account the symmetry of $\mathcal{L}_V\nabla$ (that is, $(\mathcal{L}_V\nabla)(X,Y) = (\mathcal{L}_V\nabla)(Y,X)$), it follows from the above equation that

$$2g((\mathcal{L}_V\nabla)(X,Y),Z) = (\nabla_X\mathcal{L}_V g)(Y,Z) + (\nabla_Y\mathcal{L}_V g)(Z,X) - (\nabla_Z\mathcal{L}_V g)(X,Y)$$

for any $X, Y, Z \in \mathfrak{X}(M)$. Using (4.8) in the above equation we have

$$\begin{aligned}
&(\mathcal{L}_V\nabla)(X,Y)\\
&= X\left(\lambda - \frac{1}{2}r\right)Y + Y\left(\lambda - \frac{1}{2}r\right)X - g(X,Y)D\lambda + \frac{1}{2}g(X,Y)Dr\\
&\quad + \frac{1}{2}X(r)\eta(Y)\xi + \frac{1}{2}Y(r)\eta(X)\xi - \frac{1}{2}\eta(X)\eta(Y)Dr\\
&\quad + (r+6)g(X,Y)\xi - (r+6)\eta(X)\eta(Y)\xi
\end{aligned} \tag{4.9}$$

for any $X, Y \in \mathfrak{X}(M)$. Substituting Y with ξ in (4.9) and using (4.2) gives that

$$(\mathcal{L}_V\nabla)(X,\xi) = X(\lambda)\xi + \xi(\lambda)X - \eta(X)D\lambda + (r+6)(X - \eta(X)\xi) \tag{4.10}$$

for any $X \in \mathfrak{X}(M)$. Clearly, from equations (4.9) and (4.10) and Lemma 4.1 we obtain

$$\begin{aligned}
&(\nabla_Y\mathcal{L}_V\nabla)(X,\xi)\\
=\ & -\frac{3}{2}Y(r)\eta(X)\xi - \frac{1}{2}X(r)\eta(Y)\xi - 2(r+6)(g(X,Y) - \eta(X)\eta(Y))\xi\\
&+ g(X,\nabla_Y D\lambda)\xi + \left(\frac{3}{2}Y(r) + (r+6)\eta(Y) + \eta(\nabla_Y D\lambda)\right)X - (r+6)\eta(X)Y\\
&+ \frac{1}{2}X(r)Y - \eta(X)\nabla_Y D\lambda - \frac{1}{2}g(X,Y)Dr + \frac{1}{2}\eta(X)\eta(Y)Dr
\end{aligned}$$

for any $X, Y \in \mathfrak{X}(M)$. Substituting the above equation into the following relation (see Yano [26])

$$(\mathcal{L}_V R)(X,Y)Z = (\nabla_X\mathcal{L}_V\nabla)(Y,Z) - (\nabla_Y\mathcal{L}_V\nabla)(X,Z)$$

and using $g(\nabla_X D\lambda, Y) = g(\nabla_Y D\lambda, X)$, then we immediately obtain

$$\begin{aligned}
(\mathcal{L}_V R)(X,Y)\xi = {} & Y(r)\eta(X)\xi - X(r)\eta(Y)\xi - \eta(Y)\nabla_X D\lambda + \eta(X)\nabla_Y D\lambda\\
& + (X(r) + 2(r+6)\eta(X) + \eta(\nabla_X D\lambda))Y\\
& - (Y(r) + 2(r+6)\eta(Y) + \eta(\nabla_Y D\lambda))X
\end{aligned} \tag{4.11}$$

for any $X, Y \in \mathfrak{X}(M)$. On the other hand, from (4.7) we have

$$g(\nabla_\xi V, Y) + g(\nabla_Y V, \xi) = 2(\lambda+2)\eta(Y)$$

for any $Y \in \mathfrak{X}(M)$. In view of the above equation, by taking the Lie derivative of relation (4.1) along the potential vector field V and making use of equations (2.4) and (4.1), we obtain

$$\begin{aligned}
&(\mathcal{L}_V R)(X,Y)\xi + R(X,Y)\mathcal{L}_V\xi\\
&= -(g(\mathcal{L}_V\xi, Y) + 2(\lambda+2)\eta(Y))X + (g(\mathcal{L}_V\xi, X) + 2(\lambda+2)\eta(X))Y
\end{aligned} \tag{4.12}$$

for any $X, Y \in \mathfrak{X}(M)$. It follows from equation (4.1) that

$$R(\xi, X)Y = -g(X,Y)\xi + \eta(Y)X$$

for any $X, Y \in \mathfrak{X}(M)$. Clearly, using the above relation and comparing equation (4.11) with (4.12) (substituting X with ξ and applying Lemma 4.1), then we obtain

$$\begin{aligned}\nabla_Y D\lambda &= \eta(Y)\nabla_\xi D\lambda - 2(\lambda+2)\eta(Y)\xi \\ &\quad + \eta(\nabla_Y D\lambda)\xi + 2(\lambda+2)Y - \eta(\nabla_\xi D\lambda)Y\end{aligned}$$

for any $Y \in \mathfrak{X}(M)$. It follows from the above equation that

$$\Delta\lambda = \mathrm{div} D\lambda = -\eta(\nabla_\xi D\lambda) + 4(\lambda+2).$$

This completes the proof. □

Corollary 4.1. *A Ricci soliton on any three dimensional Kenmotsu manifold is expanding with* $\lambda = -2$.

Proof. This corollary follows directly from relation (4.6). □

Theorem 4.1. *Let* $(M^3, \phi, \xi, \eta, g)$ *be a three dimensional Kenmotsu manifold which admits a gradient Ricci almost soliton* (g, f, λ). *Then, either* M^3 *is of constant sectional curvature* -1 *or the potential vector field is pointwise colinear with the Reeb vector field which is locally characterized by* (4.23).

Proof. Firstly, putting (4.5) into (3.5) gives that

$$\begin{aligned}&R(X,Y)Df \\ &= \frac{1}{2}Y(r)X - \frac{1}{2}X(r)Y - \left(\frac{r}{2}+3\right)\eta(X)Y + \left(\frac{r}{2}+3\right)\eta(Y)X \\ &\quad - \frac{1}{2}Y(r)\eta(X)\xi + \frac{1}{2}X(r)\eta(Y)\xi - Y(\lambda)X + X(\lambda)Y\end{aligned} \tag{4.13}$$

for any $X, Y \in \mathfrak{X}(M)$. By (4.13), we see easily that

$$g(R(X,Y)Df, \xi) = X(\lambda)\eta(Y) - Y(\lambda)\eta(X) \tag{4.14}$$

for any $X, Y \in \mathfrak{X}(M)$. On the other hand, by a simple calculation we obtain from relation (4.1) that

$$g(R(X,Y)\xi, Df) = Y(f)\eta(X) - X(f)\eta(Y) \tag{4.15}$$

for any $X, Y \in \mathfrak{X}(M)$.

Comparing (4.14) with (4.15) gives an equation and replacing Y by ξ in the resulting equation we obtain

$$\mathrm{d}(f-\lambda) = \xi(f-\lambda)\eta,$$

where d is the exterior differentiation. This means that $f - \lambda$ is invariant along the distribution $\mathcal{D}$, i.e., $X(f-\lambda) = 0$ for any vector field $X \in \mathcal{D}$.

Contracting Y in (4.13) and applying Lemma 4.1 we obtain

$$S(X, Df) = \frac{1}{2}X(r) - 2X(\lambda)$$

for any $X \in \mathfrak{X}(M)$. Clearly, comparing the above equation with (4.4) yields

$$4X(\lambda) - X(r) + (r+2)X(f) - (r+6)\eta(X)\xi(f) = 0 \tag{4.16}$$

for any $X \in \mathfrak{X}(M)$. Replacing X by ξ in (4.16) we have $\xi(\lambda - f) = -(\frac{r}{2} + 3)$, and using this in $\mathrm{d}(f - \lambda) = \xi(f - \lambda)\eta$ we obtain that

$$\mathrm{d}(f - \lambda) = \left(\frac{r}{2} + 3\right)\eta. \tag{4.17}$$

Applying the well-known Poincare lemma and using the fact $\mathrm{d}\eta = 0$ on the above equation, we get $\mathrm{d}r \wedge \eta = 0$ and hence by using (4.2) we have

$$Dr = -2(r + 6)\xi. \tag{4.18}$$

Suppose that X in (4.16) is orthogonal to ξ. Taking into account $\lambda - f$ being a constant along $\mathcal{D}$ and using (4.17) and (4.18), then we get $(r + 6)X(f) = 0$ for any $X \in \mathcal{D}$. This implies that either $r = -6$ or

$$Df = \xi(f)\xi. \tag{4.19}$$

Now we discuss the above two cases as follows.

Case i: $r = -6$. By using this in equation (4.4) we see that g is an Einstein metric, i.e., $Q = -2\mathrm{id}$. Therefore, by using equation (4.3) we conclude that M^3 is of constant sectional curvature -1. Moreover, it follows from relation (4.17) that $f - \lambda$ is a constant. Using this in relation (3.4) we see that the gradient of the potential function is a conformal vector field.

Case ii: $r \neq -6$. It follows from equations (4.17) and (4.19) that

$$D\lambda = \xi(\lambda)\xi. \tag{4.20}$$

Following a straightforward calculation, from equations (2.4) and (4.20) we get $\Delta\lambda = 2\xi(\lambda) + \xi(\xi(\lambda))$. Comparing this with (4.6) gives

$$\xi(\lambda) + \xi(\xi(\lambda)) = 2(\lambda + 2). \tag{4.21}$$

By relation (4.4), it follows from (3.4) that $\lambda = g(\nabla_\xi Df, \xi) + S(\xi, \xi) = \xi(\xi(f)) - 2$ and using (4.2), (4.21) and (4.17) in this equation we have

$$\xi(\lambda) = \lambda - r - 4 \text{ and } \xi(f) = \lambda - \frac{r}{2} - 1. \tag{4.22}$$

It is well-known [15] that a Kenmotsu manifold of dimension $2n+1$ is locally isometric to the warped product $(-\epsilon, \epsilon) \times_{ce^t} N$, where N is a Kählerian manifold of dimension $2n$ and $(-\epsilon, \epsilon)$ is an open interval. Locally, we may write $\xi = \frac{\partial}{\partial_t}$, where t is the coordinate of the interval. Thus, using (4.18) we get $r = \alpha_1 e^{-2t} - 6$ for certain non-zero constant α_1. Hence, using this in (4.22) we obtain

$$\lambda = \frac{1}{3}\alpha_1 e^{-2t} + \alpha_2 e^t - 2 \text{ and } f = \frac{1}{12}\alpha_1 e^{-2t} + \alpha_2 e^t + \alpha_3 \tag{4.23}$$

for certain non-zero constants α_2 and α_3. This completes the proof. $\square$

According to Kenmotsu [15, Proposition 3], we see that the warped product manifold $R \times_{ce^t} N^{2n}$ admits a Kenmotsu structure, where N^{2n} is a Kählerian manifold and c is a constant. Thus, as a corollary of the above Theorem 4.1 we get the following result.

Corollary 4.2. *Suppose that the warped product* $R \times_{ce^t} N^2$ *admits a gradient Ricci almost soliton* (g, f, λ). *Then either it is locally a hyperbolic space* $\mathbb{H}^3(-1)$ *or the soliton is non-trivial and it is locally characterized in Theorem* 4.1.

Remark 4.1. Some sufficient conditions for a warped product to be a non-trivial gradient Ricci soliton were shown in Pigola et al. [19, Section 2].

Suppose that $(g, \beta\xi, \lambda)$ (where β is a variable function and λ a constant) on a three dimensional Kenmotsu manifold M^3 is a Ricci soliton. Then, according to Theorem 1 of Ghosh [9] we see that M^3 is of constant sectional curvature -1 and the soliton is expanding with $\lambda = -2$. Hence, by relation (1.1), we have that $\beta\xi$ is a Killing vector field. Using (2.4) then we get

$$(\mathcal{L}_{\beta\xi}g)(X, Y) = 2X(\beta)\eta(Y) + 2\beta g(X, Y) - 2\beta\eta(X)\eta(Y) = 0$$

for any vector fields X, Y. By considering $X = Y \in \mathcal{D}$ in the above equation we obtain $\beta = 0$.

Remark 4.2. On a three dimensional Kenmotsu manifold, $(g, \beta\xi, \lambda)$ (where β is a variable function) is a trivial Ricci soliton and (g, ξ, λ) is never a Ricci soliton.

Acknowledgement. This work was supported by the Research Foundation for the Doctoral Program of Henan Normal University (No. qd14145) and the Youth Science Foundation of Henan Normal University (No. 2014QK01). I would like to thank the referee for his or her careful reading. This work was supported by the National Natural Science Foundation of China (No. 11526080) and Key Scientific Research Program in Universities of Henan Province (No. 16A110004).

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