SOME RESULTS ON COMMON BEST PROXIMITY POINT AND COMMON FIXED POINT THEOREM IN PROBABILISTIC MENGER SPACE

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Abstract. In this paper, we define the concepts of commute proximally, dominate proximally, weakly dominate proximally, proximal generalized φ-contraction and common best proximity point in probabilistic Menger space. We prove some common best proximity point and common fixed point theorems for dominate proximally and weakly dominate proximally mappings in probabilistic Menger space under certain conditions. Finally we show that proximal generalized φ-contractions have best proximity point in probabilistic Menger space. Our results generalize many known results in metric space.

1. Introduction and preliminaries

An interesting and important generalization of the notion of metric space was introduced by, K. Menger [18] in 1942 under the name of statistical metric space, which is now called probabilistic metric space (abbreviated, PM-space). The idea of K. Menger was to use distribution functions instead of nonnegative real numbers as values of the metric. The notion of PM-space corresponds to situations when we do not know exactly the distance between two points, but we know probabilities of possible values of this distance. In fact the study of such spaces received an impetus with the pioneering works of Schweizer and Sklar [23, 24].

Schweizer and Sklar [23] developed the study of fixed point theory in probabilistic metric space. Recently, the study of fixed point theorems in probabilistic metric spaces is also a topic of recent interest and forms an active direction of research. Sehgal et al. [25] made the first ever effort in this direction. Since then

several authors have already studied fixed point and common fixed point theorems in PM-spaces, we refer to [16, 19, 20] and others have recently initiated work along these lines.

In 1972, Sehgal and Bharucha-Reid [25] studied the Banach contraction principle of metric space into the complete Menger space. In an interesting paper [14], Hicks observed that fixed point theorems for certain contraction mappings on a Menger space endowed with a triangular t-norm may be obtained from corresponding results in metric spaces.

We first bring notation, definitions and known results, which are related to our work. For more details, we refer the reader to [7, 12, 13].

Definition 1.1. A distribution function is a function $F : (-\infty, \infty) \to [0, 1]$, that is nondecreasing and left continuous on $\mathbb{R}$, moreover, $\inf_{t \in \mathbb{R}} F(t) = 0$ and $\sup_{t \in \mathbb{R}} F(t) = 1$.

The set of all the distribution functions is denoted by Δ, and the set of those distribution functions such that $F(0) = 0$ is denoted by Δ^+. The space Δ^+ is partially ordered by the usual pointwise ordering of functions, and has a maximal element ϵ_0, defined by

$$\epsilon_0(x) = \begin{cases} 0 & x \leq 0, \\ 1 & x > 0. \end{cases}$$

Definition 1.2. A probabilistic metric space (abbreviated, PM-space) is an ordered pair (X, F), where X is a nonempty set and $F : X \times X \to \Delta^+$ ($F(p, q)$ is denoted by $F_{p,q}$) satisfies the following conditions:

(PM1) $F_{p,q} = \epsilon_0$, if and only if $p = q$,
(PM2) $F_{p,q}(t) = F_{q,p}(t)$,
(PM3)   If $F_{p,q}(t) = 1$ and $F_{q,r}(s) = 1$, then $F_{p,r}(t + s) = 1$,

for every $p, q, r \in X$ and $t, s \geq 0$.

Lemma 1.3. *Let (X, F) be a PM-space. If there exists $q \in (0, 1)$ such that for all $t \geq 0$, $F_{x,y}(qt) \geq F_{z,w}(t) \geq F_{x,y}(t)$ where $x, y, z, w \in X$, then $x = y$ and $z = w$.*

Proof. As $F_{x,y}(qt) \geq F_{x,y}(t)$, we have

$$F_{x,y}(t) \geq F_{x,y}(q^{-1}t) \geq F_{x,y}(q^{-2}t) \geq \cdots \geq F_{x,y}(q^{-n}t) \geq \cdots.$$

Taking the limit as $n \to \infty$, we get $F_{x,y}(t) = 1$ for all $t > 0$, or in other words, $F_{x,y} = \epsilon_0 = F_{z,w}$, so by the condition (PM1), $x = y$ and $z = w$. $\square$

Definition 1.4. A mapping $* : [0, 1] \times [0, 1] \to [0, 1]$ is called a triangular norm (abbreviated, t-norm) if the following conditions are satisfied:

  (i) $a * b = b * a$,
  (ii) $a * (b * c) = (a * b) * c$,
  (iii) $a * b \geq c * d$,   whenever $a \geq c$ and $b \geq d$,
  (iv) $a * 1 = a$,

for every $a, b, c, d \in [0,1]$. Two typical examples of continuous t-norm are $a *_M b = \min\{a, b\}$ and $a *_P b = ab$.

An arbitrary t-norm can be extended (by (iii)) in a unique way to an n-ary operation. For $(a_1, \ldots, a_n) \in [0,1]^n$ $(n \in \mathbb{N})$, the value $*^n(a_1, \ldots, a_n)$ is defined by $*^1(a_1) = a_1$ and $*^n(a_1, \ldots, a_n) = *^{n-1}(a_1, \ldots, a_{n-1}) * a_n$. For each $a \in [0,1]$, the sequence $(*^n(a))$ is defined by $*^n(a) = *^n(a, \ldots, a)$.

Definition 1.5. A t-norm $*$ is said to be of Hadžić type (abbreviated, H-type) if the sequence of functions $(*^n(a))$ is equicontinuous at $a = 1$, that is

$$\forall \, \varepsilon \in (0,1), \ \ \exists \, \delta \in (0,1): \ a > 1 - \delta \Rightarrow *^n(a) > 1 - \varepsilon \ \ (n \in \mathbb{N}).$$

The t-norm $*_M$ is a trivial example of a t-norm of H-type, but there are t-norms $*$ of H-type with $* \neq *_M$, see [13]. It is easy to see that if $*$ is of H-type, then $*$ satisfies $\sup_{a \in (0,1)} a * a = 1$.

Definition 1.6. A probabilistic Menger space is a triplet $(X, F, *)$, where (X, F) is PM-space and $*$ is a t-norm such that for all $p, q, r \in X$ and for all $t, s \geq 0$,
$$F_{p,r}(t + s) \geq F_{p,q}(t) * F_{q,r}(s).$$

Definition 1.7. Let $(X, F, *)$ be a probabilistic Menger space. An open ball with center x and radius λ $(0 < \lambda < 1)$ in X is the set $U_x(\varepsilon, \lambda) = \{y \in X : F_{x,y}(\varepsilon) > 1 - \lambda\}$, for all $\varepsilon > 0$. It is easy to see that $\mathfrak{U} = \{U_x(\varepsilon, \lambda) : x \in X, \ \varepsilon > 0, \ \lambda \in (0,1)\}$ determines a Hausdorff topology for X [23].

Lemma 1.8 ([27]). *In a probabilistic Menger space $(X, F, *)$, $a * a \geq a$, for all $a \in [0,1]$, if and only if $* = *_M$.*

Definition 1.9. Let (X, F) be a PM-space. For each $\lambda \in (0,1)$, define the function
$$d_\lambda : X \times X \to \mathbb{R},$$
by
$$d_\lambda(x, y) = \sup_{t \in \mathbb{R}}\{t \in \mathbb{R} \ : \ F_{x,y}(t) \leq 1 - \lambda\}.$$
Since $F_{x,y}$ is nondecreasing, left continuous with
$$\inf_{t \in \mathbb{R}} F_{x,y}(t) = 0, \quad \sup_{t \in \mathbb{R}} F_{x,y}(t) = 1,$$
then $d_\lambda(x, y)$ is finite.

Proposition 1.10 ([6]). *Let $(X, F, *_M)$ be a probabilistic Menger space. Then the function d_λ is a pseudometric for each $\lambda \in (0,1)$. Furthermore $d_\lambda(x, y) = 0$ for all $\lambda \in (0,1)$ if and only if $x = y$.*

Theorem 1.11 ([6]). *Let $(X, F, *_M)$ be a probabilistic Menger space. Then the topology on X generated by the family of pseudometrics associated with the probabilistic metric F is the same as the topology induced by F.*

Theorem 1.12 ([6])**.** *Suppose X is a Hausdorff space with a topology generated by a family of pseudometrics $d_\lambda : \lambda \in (0,1)$ such that for $x, y \in X$, $d_\lambda(x,y)$ is a nonincreasing left continuous function of λ such that $d_\lambda(x,y) = 0$ for all $\lambda \in (0,1)$ if and only if $x = y$. Then there is a probabilistic metric F on X such that d_λ is the family of pseudometrics associated with it.*

Definition 1.13. A sequence (x_n) in a probabilistic Menger space $(X, F, *)$ is said to be convergent to a point $x \in X$ if and only if for every $\varepsilon > 0$ and $\lambda \in (0,1)$, there exists $n_0(\varepsilon, \lambda) \in \mathbb{N}$ such that $F_{x_n, x}(\varepsilon) > 1 - \lambda$ for all $n \geq n_0(\varepsilon, \lambda)$ or for every $\lambda \in (0,1)$, $d_\lambda(x_n, x) \to 0$ or $\lim_{n \to \infty} F_{x_n, x}(t) = 1$ for all $t > 0$, in this case we say that limit of the sequence (x_n) is x.

Definition 1.14. A sequence (x_n) in a probabilistic Menger space $(X, F, *)$ is said to be Cauchy sequence if and only if for every $\varepsilon > 0$ and $\lambda \in (0,1)$, there exists $n_0(\varepsilon, \lambda) \in \mathbb{N}$ such that $F_{x_{n+p}, x_n}(\varepsilon) > 1 - \lambda$ for all $n \geq n_0(\varepsilon, \lambda)$ and every $p \in \mathbb{N}$ or for every $\lambda \in (0,1)$, $d_\lambda(x_{n+p}, x_n) \to 0$ for all $p \in \mathbb{N}$, or $\lim_{n \to \infty} F_{x_n, x_{n+p}}(t) = 1$, for all $t > 0$ and $p \in \mathbb{N}$.

Also, a probabilistic Menger space $(X, F, *)$ is said to be complete if and only if every Cauchy sequence in X, is convergent.

The concept of Cauchy sequence is inspired from that of G-Cauchy sequence (it belongs to Grabiec [11]).

Proposition 1.15. *The limit of a convergent sequence in a probabilistic Menger space $(X, F, *)$ is unique.*

Proof. It is obvious. $\qquad \square$

Proposition 1.16. *Let $(X, F, *)$ be a probabilistic Menger space and (x_n) be a sequence in X. If sequence (x_n) converges to $x \in X$, then $F_{x,x}(t) = 1$ for all $t > 0$.*

Proof. It is obvious. $\qquad \square$

Lemma 1.17 ([15])**.** *Let $n \in \mathbb{N}$, $g_n : (0, \infty) \to (0, \infty)$ and $F_n, F : \mathbb{R} \to [0,1]$. Assume that $\sup\{F(t) : t > 0\} = 1$ and for any $t > 0$, $\lim_{n \to \infty} g_n(t) = 0$ and $F_n(g_n(t)) \geq F(t)$. If each F_n is nondecreasing, then $\lim_{n \to \infty} F_n(t) = 1$ for any $t > 0$.*

Lemma 1.18. *Let $(X, F, *)$ be a probabilistic Menger space and $\varphi : (0, \infty) \to (0, \infty)$ be a mapping such that $\lim_{n \to \infty} \varphi^n(t) = 0$. If $x, y \in X$ and $F_{x,y}(\varphi(t)) \geq F_{x,y}(t)$ for all $t > 0$. Then $x = y$.*

Proof. By using the above lemma, the result follows. $\qquad \square$

Definition 1.19. Let $(X, F, *)$ be a probabilistic Menger space, $A \subseteq X$ and $T : A \to A$ be a mapping. The mapping T is said to be isometry if for all $x, y \in X$, we have

$$F_{Tx, Ty}(t) = F_{x,y}(t), \quad \forall t \geq 0.$$

It is easy to see that every isometry mappings are injective mappings.

In nonlinear analysis, the theory of fixed points is an essential instrument to solve the equation $Tx = x$ for a self-mapping T defined on a subset of an abstract space such as a metric space, a normed linear space or a topological vector space. If T is a non-self-mapping from A to B, then the aforementioned equation does not necessarily admit a solution. However, in such circumstances, it may be speculated to determine an element x for which the error $d(x, Tx)$ is minimum, where d is the distance function, in which case x and Tx are in close proximity to each other. In fact, best approximation theorems and best proximity point theorems are applicable for solving such problems. In view of the fact that $d(x, Tx)$ is at least $d(A, B)$, a best proximity point theorem guarantees the global minimization of $d(x, Tx)$ by the requirement that an approximate solution x satisfies the condition $d(x, Tx) = d(A, B)$. Such optimal approximate solutions are called best proximity points of the mapping T. Further, it is interesting to observe that best proximity theorems also emerge as a natural generalization of fixed point theorems, for a best proximity point reduces to a fixed point if the mapping under consideration is a self mapping. Investigation of several variants of contractions for the existence of a best proximity point can be found in [1, 2, 3, 4, 8, 10, 17, 21, 22]. Eldred et al. [9] have established a best proximity point theorem for relatively non-expansive mappings. Further, Anuradha and Veeramani have focussed on best proximity point theorems for proximal pointwise contraction mappings [5].

Recently, Su and Zhang [26] presented some definitions and basic concepts of best proximity point in a new class of probabilistic metric spaces and to proved the best proximity point theorems for the contractive mappings and weak contractive mappings.

In this paper, we establish some definitions and basic concepts of the common best proximity point in the framework of probabilistic metric spaces.

Definition 1.20. Let A and B be nonempty subsets of a PM-space (X, F). Let
$$F_{A,B}(t) = \sup_{x \in A, y \in B} F_{x,y}(t), \quad t \geq 0,$$
which is said to be the probabilistic distance of A, B.

Definition 1.21. Let A and B be nonempty subsets of a PM-space (X, F). We define the following sets:
$$A_0 = \{x \in A : \exists\, y \in B \text{ s.t } \forall\, t \geq 0,\ F_{x,y}(t) = F_{A,B}(t)\},$$
$$B_0 = \{y \in B : \exists\, x \in A \text{ s.t } \forall\, t \geq 0,\ F_{x,y}(t) = F_{A,B}(t)\}.$$

Definition 1.22. Let A and B be nonempty subsets of a PM-space (X, F) and $T, S : A \to B$ be non-self mappings. We say that an element $x \in A$ is a common best proximity of the mappings if
$$F_{x,Sx}(t) = F_{A,B}(t) = F_{x,Tx}(t)$$
for all $t \geq 0$.

It is clear that the notion of a common fixed point coincided with the notion of a common best proximity point when the underlying mapping is a self mapping. Also, it can be noticed that common best proximity point is an element at which both function $x \to F_{x,Sx}(t)$ and $x \to F_{x,Tx}(t)$ for all $t \geq 0$, attain global supremum.

Definition 1.23. Let A and B be nonempty subsets of a PM-space (X, F) and $T, S : A \to B$ be non-self mappings. We say that T, S are commute proximally if

$$F_{u,Sx}(t) = F_{A,B}(t) = F_{v,Tx}(t)$$

for all $t \geq 0$, then $Sv = Tu$, where $x, u, v \in A$.

Example 1.24. Let (X, F) be a PM-space and $T, S : X \to X$ be two mappings such that $TS = ST$. Clearly $F_{X,X}(t) = 1$ for all $t \geq 0$ and so if

$$F_{u,Sx}(t) = F_{X,X}(t) = F_{v,Tx}(t) \quad (x, u, v \in X, \ t \geq 0),$$

then by the hypothesis, $u = Sx$ and $v = Tx$. Therefore $Sv = STx = TSx = Tu$, hence T, S are commute proximally.

Definition 1.25. Let A and B be nonempty subsets of a PM-space (X, F) and $T, S : A \to B$ be non-self mappings. We say that the mapping T is to dominate the mapping S proximally if

$$F_{u_1,Sx_1}(t) = F_{u_2,Sx_2}(t) = F_{A,B}(t) = F_{v_1,Tx_1}(t) = F_{v_2,Tx_2}(t)$$

for all $t \geq 0$, then there exists a $\alpha \in (0, 1)$ such that for all $t \geq 0$,

$$F_{u_1,u_2}(\alpha t) \geq F_{v_1,v_2}(t),$$

where $u_1, u_2, v_1, v_2, x_1, x_2 \in A$.

Example 1.26. Let $X = [0, 2]$ and $F_{x,y}(t) = \epsilon_0(t - |x - y|)$ for all $x, y \in X$, it is easy to see that (X, F) is a PM-space. Define self mappings S and T on X as

$$Sx = \frac{1}{8}x, \quad Tx = -\frac{1}{2}x \quad (x \in X).$$

It is easy to see that $F_{X,X}(t) = 1$. If

$$F_{u_1,Sx_1}(t) = F_{u_2,Sx_2}(t) = F_{X,X}(t) = 1 = F_{v_1,Tx_1}(t) = F_{v_2,Tx_2}(t) \quad (t \geq 0),$$

where $u_1, u_2, v_1, v_2, x_1, x_2 \in A$. Then $u_i = Sx_i$ and $v_i = Tx_i$ $(i = 1, 2)$ and so for $\alpha = 1/4$ we have $F_{u_1,u_2}(\alpha t) = F_{v_1,v_2}(t)$, hence T dominates S proximally for $\alpha = 1/4$.

Definition 1.27. Let A and B be nonempty subsets of a PM-space (X, F) and $T, S : A \to B$ be non-self mappings. We say that the mapping T is to weakly dominate the mapping S proximally if

$$F_{u_1,Sx_1}(t) = F_{u_2,Sx_2}(t) = F_{A,B}(t) = F_{v_1,Tx_1}(t) = F_{v_2,Tx_2}(t)$$

for all $t \geq 0$, then there exists a $\alpha \in (0, 1)$ such that for all $t \geq 0$,

$$F_{u_1,u_2}(\alpha t) \geq \min\{F_{v_1,v_2}(t), F_{v_1,u_1}(t), F_{v_1,u_2}(t), F_{v_2,u_1}(t)\},$$

where $u_1, u_2, v_1, v_2, x_1, x_2 \in A$.

Obviously, if T dominates S proximally, then T weakly dominates S proximally. The following example shows that the converse is not true, in general.

Example 1.28. Let $X = [0,1] \times [0,1]$ and $d : X \times X \rightarrow [0,\infty)$ be given by $d((x_1, x_2), (y_1, y_2)) = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2}$ and define $F : (-\infty, \infty) :\rightarrow [0,1]$ by

$$F_{(x_1,x_2),(y_1,y_2)}(t) = \frac{t}{t + d((x_1,x_2),(y_1,y_2))}.$$

Clearly, (X, F) is a PM-space. Let $A = \{(0, x) : x \in [0,1]\}$, $B = \{(1, x) : x \in [0,1]\}$ and $S, T : A \rightarrow B$ be defined as $T(0, x) = (1, x)$ for all $x \in [0,1]$ and

$$S(0, x) = \begin{cases} (1, \frac{1}{3}) ; & x < 1, \\ (1, \frac{1}{2}) ; & x = 1, \end{cases} \quad (\forall \, x \in [0,1]).$$

It is easy to see that $F_{A,B}(t) = \frac{t}{t+1}$. We show that T does not dominate S proximally. To show the claim, suppose that there exists $\alpha \in (0,1)$ such that for all $t \geq 0$,

$$F_{U_1,U_2}(\alpha t) \geq F_{V_1,V_2}(t),$$

where $U_1 = (0, u_1)$, $U_2 = (0, u_2)$, $V_1 = (0, v_1)$, $V_2 = (0, v_2)$, $X_1 = (0, x_1)$ and $X_2 = (0, x_2)$ be elements in A satisfying

(1) $\qquad F_{U_1,SX_1}(t) = F_{U_2,SX_2}(t) = F_{A,B}(t) = F_{V_1,TX_1}(t) = F_{V_2,TX_2}(t)$

for all $t \geq 0$. Let $U_1 = (0, \frac{1}{3})$, $U_2 = (0, \frac{1}{2})$, $V_1 = (0, x)$, $V_2 = (0, 1)$, $X_1 = (0, x)$ and $X_2 = (0, 1)$ where $0 \leq x < 1$. Then U_1, U_2, V_1, V_2, X_1 and X_2 satisfy (1) and then, we have

$$F_{U_1,U_2}(\alpha t) = \frac{t}{t + \frac{1}{\alpha 6}} \geq F_{V_1,V_2}(t) = \frac{t}{t + (1 - x)} \quad (\forall \, x \in [0,1)),$$

a contradiction. Then we show that T weakly dominates S proximally for $\alpha = 1/4$, to verify this, let $x_1, x_2, u_1, u_2, v_1, v_2 \in [0,1]$ and

$$F_{(0,u_1),S(0,x_1)}(t) = F_{(0,u_2),S(0,x_2)}(t) = F_{A,B}(t)$$
$$= F_{(0,v_1),T(0,x_1)}(t) = F_{(0,v_2),T(0,x_2)}(t).$$

Now we need to consider several possible cases.

Case 1. Let $x_1, x_2 \in [0,1)$. Then $u_1 = u_2 = \frac{1}{3}$ and

$$F_{(0,u_1),(0,u_2)}(\frac{1}{4}t) = 1 \geq \min\{F_{(0,v_1),(0,v_2)}(t), F_{(0,v_1),(0,u_1)}(t),$$
$$F_{(0,v_1),(0,u_2)}(t), F_{(0,v_2),(0,u_1)}(t)\}.$$

Case 2. Let $x_1 = 1 = x_2$. Then $u_1 = u_2 = \frac{1}{2}$ and

$$F_{(0,u_1),(0,u_2)}(\frac{1}{4}t) = 1 \geq \min\{F_{(0,v_1),(0,v_2)}(t), F_{(0,v_1),(0,u_1)}(t),$$
$$F_{(0,v_1),(0,u_2)}(t), F_{(0,v_2),(0,u_1)}(t)\}.$$

Case 3. Let $x_1 \in [0,1)$ and $x_2 = 1$. Then $u_1 = \frac{1}{3}$, $u_2 = \frac{1}{2}$, $v_2 = 1$ and $F_{(0,u_1),(0,u_2)}(\frac{1}{4}t) = \frac{t}{t+\frac{2}{3}} = F_{(0,v_2),(0,u_1)}(t)$, so

$$F_{(0,u_1),(0,u_2)}(\frac{1}{4}t) \geq \min\{F_{(0,v_1),(0,v_2)}(t), F_{(0,v_1),(0,u_1)}(t),$$
$$F_{(0,v_1),(0,u_2)}(t), F_{(0,v_2),(0,u_1)}(t)\}.$$

Case 4. Let $x_1 = 1$ and $x_2 \in [0,1)$. Then $u_1 = \frac{1}{2}$, $u_2 = \frac{1}{3}$, $v_1 = 1$ and $F_{(0,u_1),(0,u_2)}(\frac{1}{4}t) = \frac{t}{t+\frac{2}{3}} = F_{(0,v_1),(0,u_2)}(t)$, so

$$F_{(0,u_1),(0,u_2)}(\frac{1}{4}t) \geq \min\{F_{(0,v_1),(0,v_2)}(t), F_{(0,v_1),(0,u_1)}(t),$$
$$F_{(0,v_1),(0,u_2)}(t), F_{(0,v_2),(0,u_1)}(t)\}.$$

Definition 1.29. Let (A, B) be a pair of nonempty subsets of a PM-space (X, F) and $T : A \to B$ be a mapping. We say that the mapping T is to proximal generalized φ-contraction, if there exists a function $\varphi : (0, \infty) \to (0, \infty)$ such that
$$F_{u,Tx}(t) = F_{A,B}(t) = F_{v,Ty}(t) \implies F_{u,v}(\varphi(t)) \geq F_{x,y}(t)$$
for all $u, v, x, y \in A$ and $t > 0$.

Example 1.30. Let $X = [0, 2]$ and T be a self mapping on X as $Tx = \frac{1}{8}x$. If $F_{x,y}(t) = \frac{t}{t+|x-y|}$, then it is easy to see that $F_{X,X}(t) = 1$. If $F_{u,Tx}(t) = 1 = F_{v,Ty}(t)$, then for $\varphi(t) = \frac{1}{8}t$, we have $F_{u,v}(\varphi(t)) = F_{x,y}(t)$, where $u, v, x, y \in X$. Therefore T is a proximal generalized φ-contraction.

In this article, we introduce a new class of non-self mappings, called weakly dominate proximally in probabilistic Menger space. We provide sufficient conditions for the existence and uniqueness of common best proximity points and common fixed points for weakly dominate proximally non-self mappings in probabilistic Menger space. Finally we show that proximal generalized φ-contractions have best proximity point in probabilistic Menger space. Our results generalize many known results in metric space. Examples are given to support our main results.

2. Main results

Now we state and prove our main theorem about existence and uniqueness of a common best proximity for dominate proximally and weakly dominate proximally non-self-mappings in probabilistic Menger space under certain conditions.

Theorem 2.1. *Let A and B be nonempty subsets of a complete probabilistic Menger space $(X, F, *_M)$, A_0 and B_0 are nonempty and A_0 is closed. If the mappings $T, S : A \to B$ satisfy the following conditions:*

   (i) *T weakly dominates S proximally,*
   (ii) *S and T commute proximally,*

(iii) S and T are continuous,
(iv) $S(A_0) \subset B_0$ and $S(A_0) \subset T(A_0)$.

Then, there exists a unique element $x \in A$ such that

$$F_{x,Sx}(t) = F_{A,B}(t) = F_{x,Tx}(t)$$

for all $t \geq 0$.

Proof. First, suppose that there exists an element $u \in A_0$ such that $Su = Tu$. By the hypothesis, there exists an element $x \in A_0$ such that

(2) $$F_{x,Su}(t) = F_{A,B}(t) = F_{x,Tu}(t), \quad \forall t \geq 0,$$

so, $Sx = Tx$. Once again, by the hypothesis, there exists an element $v \in A_0$ such that

(3) $$F_{v,Sx}(t) = F_{A,B}(t) = F_{v,Tx}(t), \quad \forall t \geq 0.$$

Since T weakly dominates S, then from (2) and (3), we get

$$F_{x,v}(\alpha t) \geq \min\{F_{x,v}(t), F_{x,x}(t), F_{x,v}(t), F_{v,x}(t)\} = F_{x,v}(t), \quad \forall t \geq 0,$$

which implies $x = v$, by Lemma 1.3. Therefore, it follows that

$$F_{x,Sx}(t) = F_{v,Sx}(t) = F_{A,B}(t) = F_{v,Tx}(t) = F_{x,Tx}(t), \quad \forall t \geq 0.$$

So, x is a common best proximity point of the mappings S and T. If x' is another common best proximity point of the mappings S and T, in other words

$$F_{x',Sx'}(t) = F_{A,B}(t) = F_{x',Tx'}(t), \quad \forall t \geq 0,$$

then by using the same argument as above we can show that $x = x'$.

Second, we claim that there exists an element $u \in A_0$ such that $Su = Tu$. To support the claim, let x_0 be a fixed point element in A_0. By the hypothesis, there exists an element $x_1 \in A_0$ such that $Sx_0 = Tx_1$. This process can be carried on. Having chosen $x_n \in A$, by the hypothesis, we can find an element $x_{n+1} \in A_0$ such that $Sx_n = Tx_{n+1}$. By the condition (iv), there exists an element $u_n \in A_0$ such that $F_{u_n,Sx_n}(t) = F_{A,B}(t)$ for all $t \geq 0$ and $n \in \mathbb{N}$.

Further, it follows from the choice x_n and u_n that

$$F_{u_{n+1},Sx_{n+1}}(t) = F_{A,B}(t) = F_{u_n,Tx_{n+1}}(t), \quad \forall\, t \geq 0.$$

So, by the condition (i), we have

$$F_{u_n,u_{n+1}}(\alpha t) \geq \min\{F_{u_{n-1},u_n}(t), F_{u_{n-1},u_n}(t), F_{u_{n-1},u_{n+1}}(t), F_{u_n,u_n}(t)\}$$

for all $t \geq 0$. Thus, we have

(4) $$F_{u_n,u_{n+1}}(\alpha t) \geq \min\{F_{u_{n-1},u_n}(t), F_{u_{n-1},u_{n+1}}(t)\}$$

for all $t \geq 0$. In the following we show by induction that for each $n \in \mathbb{N}$ and for each $t \geq 0$, there exists $1 \leq m \leq n+1$ such that

(5) $$F_{u_n,u_{n+1}}(t) \geq F_{u_0,u_m}(\alpha^{-n}t).$$

If $n = 1$, then by (4), we have

$$F_{u_1,u_2}(\alpha t) \geq \min\{F_{u_0,u_1}(t), F_{u_0,u_2}(t)\}$$
$$= F_{u_0,u_m}(t)$$

for some $1 \leq m \leq 2$ and for all $t \geq 0$. Thus (5) holds for $n = 1$. Assume towards a contradiction that (5) is not true and take $n_0 > 1$, be the least natural number such that (5) does not hold. So there exists $t_0 > 0$, such that for all $1 \leq m \leq n_0 + 1$, we have

(6) $$F_{u_{n_0},u_{n_0+1}}(t_0) < F_{u_0,u_m}(\alpha^{-n_0}t_0).$$

If $\min\{F_{u_{n_0-1},u_{n_0}}(\alpha^{-1}t_0), F_{u_{n_0-1},u_{n_0+1}}(\alpha^{-1}t_0)\} = F_{u_{n_0-1},u_{n_0}}(\alpha^{-1}t_0)$, then by the hypothesis we have

$$F_{u_{n_0},u_{n_0+1}}(t_0) \geq F_{u_{n_0-1},u_{n_0}}(\alpha^{-1}t_0) \geq F_{u_0,u_m}(\alpha^{-n_0}t_0)$$

for some $1 \leq m \leq n_0$, a contradiction. Thus

$$\min\{F_{u_{n_0-1},u_{n_0}}(\alpha^{-1}t_0), F_{u_{n_0-1},u_{n_0+1}}(\alpha^{-1}t_0)\} = F_{u_{n_0-1},u_{n_0+1}}(\alpha^{-1}t_0),$$

and form (4), we have

(7) $$F_{u_{n_0},u_{n_0+1}}(t_0) \geq F_{u_{n_0-1},u_{n_0+1}}(\alpha^{-1}t_0).$$

By the condition (i), we get

$$F_{u_{n_0-1},u_{n_0+1}}(\alpha^{-1}t) \geq \min\{F_{u_{n_0-2},u_{n_0}}(\alpha^{-2}t), F_{u_{n_0-2},u_{n_0-1}}(\alpha^{-2}t),$$
$$F_{u_{n_0-2},u_{n_0+1}}(\alpha^{-2}t), F_{u_{n_0},u_{n_0-1}}(\alpha^{-2}t)\}$$

for all $t \geq 0$. If

$$\min\{F_{u_{n_0-2},u_{n_0}}(\alpha^{-2}t_0), F_{u_{n_0-2},u_{n_0-1}}(\alpha^{-2}t_0),$$
$$F_{u_{n_0-2},u_{n_0+1}}(\alpha^{-2}t_0), F_{u_{n_0},u_{n_0-1}}(\alpha^{-2}t_0)\} = F_{u_{n_0},u_{n_0-1}}(\alpha^{-2}t_0),$$

then from (7) and the above, we have

$$F_{u_{n_0},u_{n_0+1}}(t_0) \geq F_{u_{n_0-1},u_{n_0+1}}(\alpha^{-1}t_0) \geq F_{u_{n_0},u_{n_0-1}}(\alpha^{-2}t_0)$$
$$= F_{u_{n_0-1},u_{n_0}}(\alpha^{-2}t_0) \geq F_{u_0,u_m}(\alpha^{-(n_0+1)}t_0)$$
$$\geq F_{u_0,u_m}(\alpha^{-n_0}t_0)$$

for some $1 \leq m \leq n_0 \leq n_0 + 1$, a contradiction. Therefore

$$F_{u_{n_0-1},u_{n_0+1}}(\alpha^{-1}t_0) \geq \min\{F_{u_{n_0-2},u_{n_0}}(\alpha^{-2}t_0), F_{u_{n_0-2},u_{n_0-1}}(\alpha^{-2}t_0),$$
$$F_{u_{n_0-2},u_{n_0+1}}(\alpha^{-2}t_0)\},$$

from (7) and the above, we get

(8) $$\begin{aligned} F_{u_{n_0},u_{n_0+1}}(t_0) \geq &F_{u_{n_0-1},u_{n_0+1}}(\alpha^{-1}t_0) \\ \geq &\min\{F_{u_{n_0-2},u_{n_0}}(\alpha^{-2}t_0), F_{u_{n_0-2},u_{n_0-1}}(\alpha^{-2}t_0), \\ &F_{u_{n_0-2},u_{n_0+1}}(\alpha^{-2}t_0)\} \\ = &F_{u_{n_0-2},u_m}(\alpha^{-2}t_0) \end{aligned}$$

for some $1 \leq m \in \{n_0 - 1, n_0, n_0 + 1\} \leq n_0 + 1$. Again by the condition (i), we get

$$F_{u_{n_0-2}, u_m}(\alpha^{-2} t) \geq \min\{F_{u_{n_0-3}, u_{m-1}}(\alpha^{-3} t), F_{u_{n_0-3}, u_{n_0-2}}(\alpha^{-3} t),$$
$$F_{u_{n_0-3}, u_m}(\alpha^{-3} t), F_{u_{n_0-2}, u_{m-1}}(\alpha^{-3} t)\}$$

for all $t \geq 0$. If $m = n_0 - 1$, then

$$\min\{F_{u_{n_0-3}, u_{m-1}}(\alpha^{-3} t_0), F_{u_{n_0-3}, u_{n_0-2}}(\alpha^{-3} t_0),$$
$$F_{u_{n_0-3}, u_m}(\alpha^{-3} t_0), F_{u_{n_0-2}, u_{m-1}}(\alpha^{-3} t_0)\}$$
$$= \min\{F_{u_{n_0-3}, u_{m-1}}(\alpha^{-3} t_0), F_{u_{n_0-3}, u_{n_0-2}}(\alpha^{-3} t_0), F_{u_{n_0-3}, u_m}(\alpha^{-3} t_0)\}.$$

If $m = n_0$, $m \neq n_0 - 1$ and

$$\min\{F_{u_{n_0-3}, u_{m-1}}(\alpha^{-3} t_0), F_{u_{n_0-3}, u_{n_0-2}}(\alpha^{-3} t_0),$$
$$F_{u_{n_0-3}, u_m}(\alpha^{-3} t_0), F_{u_{n_0-2}, u_{m-1}}(\alpha^{-3} t_0)\}$$
$$= F_{u_{n_0-2}, u_{m-1}}(\alpha^{-3} t_0) = F_{u_{n_0-2}, u_{n_0-1}}(\alpha^{-3} t_0),$$

then from (8) and the above, we have

$$F_{u_{n_0}, u_{n_0+1}}(t_0) \geq F_{u_{n_0-2}, u_m}(\alpha^{-2} t_0) \geq F_{u_{n_0-2}, u_{n_0-1}}(\alpha^{-3} t_0)$$
$$\geq F_{u_0, u_{m'}}(\alpha^{-(n_0-2)}(\alpha^{-3} t_0))$$
$$\geq F_{u_0, u_{m'}}(\alpha^{-n_0} t_0)$$

for some $1 \leq m' \leq n_0 + 1$, a contradiction. Therefore

$$\min\{F_{u_{n_0-3}, u_{m-1}}(\alpha^{-3} t_0), F_{u_{n_0-3}, u_{n_0-2}}(\alpha^{-3} t_0),$$
$$F_{u_{n_0-3}, u_m}(\alpha^{-3} t_0), F_{u_{n_0-2}, u_{m-1}}(\alpha^{-3} t_0)\}$$
$$= \min\{F_{u_{n_0-3}, u_{m-1}}(\alpha^{-3} t_0), F_{u_{n_0-3}, u_{n_0-2}}(\alpha^{-3} t_0), F_{u_{n_0-3}, u_m}(\alpha^{-3} t_0)\}.$$

If $m = n_0 + 1$, $m \neq n_0 - 1$ and $m \neq n_0$, then

$$F_{u_{n_0-2}, u_{n_0}}(\alpha^{-2} t_0) \geq F_{u_{n_0-2}, u_m}(\alpha^{-2} t_0).$$

Now if

$$\min\{F_{u_{n_0-3}, u_{m-1}}(\alpha^{-3} t_0), F_{u_{n_0-3}, u_{n_0-2}}(\alpha^{-3} t_0)$$
$$F_{u_{n_0-3}, u_m}(\alpha^{-3} t_0), F_{u_{n_0-2}, u_{m-1}}(\alpha^{-3} t_0)\}$$
$$= F_{u_{n_0-2}, u_{m-1}}(\alpha^{-3} t_0) = F_{u_{n_0-2}, u_{n_0}}(\alpha^{-3} t_0),$$

then from the above, we have

$$F_{u_{n_0-2}, u_{n_0}}(\alpha^{-2} t_0) \geq F_{u_{n_0-2}, u_m}(\alpha^{-2} t_0)$$
$$\geq F_{u_{n_0-2}, u_{n_0}}(\alpha^{-3} t_0)$$
$$\geq F_{u_{n_0-2}, u_{n_0}}(\alpha^{-2} t_0),$$

a contradiction, since if the above inequality becomes equality, then we can
assume that $m = n_0$. Therefore from the above, we get

$$
\begin{aligned}
F_{u_{n_0}, u_{n_0+1}}(t_0) &\geq F_{u_{n_0-2}, u_m}(\alpha^{-2} t_0) \\
&\geq \min\{F_{u_{n_0-3}, u_{m-1}}(\alpha^{-3} t_0), F_{u_{n_0-3}, u_{n_0-2}}(\alpha^{-3} t_0), \\
&\quad\quad F_{u_{n_0-3}, u_m}(\alpha^{-3} t_0)\} \\
&= F_{u_{n_0-3}, u_{m'}}(\alpha^{-3} t_0)
\end{aligned}
$$

for some $1 \leq m' \leq n_0 + 1$. Therefore by continuing this process, we see that
for each $1 \leq k \leq n_0$, there exists $1 \leq m \leq n_0 + 1$ such that

$$
(9) \quad\quad\quad F_{u_{n_0}, u_{n_0+1}}(t_0) \geq F_{u_{n_0-k}, u_m}(\alpha^{-k} t_0).
$$

If $k = n_0$ in (9), then this is a contradiction by (6). So (5) holds for all $n \in \mathbb{N}$.
Suppose that $(d_\lambda)_{\lambda \in (0,1)}$ is the family of pseudometrics in Definition 1.9, by
Theorem 1.11 the family of pseudometrics $(d_\lambda)_{\lambda \in (0,1)}$ generates the topology
induced by F on X. We obtain by (5) that for u_n and every $\lambda \in (0,1)$,

$$
(10) \quad\quad\quad d_\lambda(u_n, u_{n+1}) \leq \alpha^n \max_{1 \leq m \leq n+1} \{d_\lambda(u_0, u_m)\}.
$$

Indeed, if $\max_{1 \leq m \leq n+1}\{d_\lambda(u_0, u_m)\} < r$, then $F_{u_0, u_m}(r) > 1 - \lambda$ for all
$m \in \{1, \ldots, n+1\}$ and (5) implies $F_{u_n, u_{n+1}}(\alpha^n r) > 1 - \lambda$, which means that
$d_\lambda(u_n, u_{n+1}) < \alpha^n r$. From (10) we get

$$
d_\lambda(u_n, u_{n+1}) \leq \alpha^n \left(d_\lambda(u_0, u_1) + d_\lambda(u_1, u_2) + \cdots + d_\lambda(u_n, u_{n+1})\right).
$$

Let $a_n = d_\lambda(u_{n-1}, u_n)$ and let $s_n = \sum_{i=1}^n a_n$. So we have

$$
(11) \quad\quad\quad a_n \leq \alpha^{n-1} s_n.
$$

We now show that $\sum_{n=1}^\infty a_n = \lim_{n\to\infty} s_n < \infty$. Assume towards a contradic-
tion that $\lim_{n\to\infty} s_n = \infty$. By the hypothesis we can assume without loss of
generality that $s_n \neq 0$ for all $n \in \mathbb{N}$. So by the hypothesis the series

$$
(12) \quad\quad\quad \sum_{n=1}^\infty \frac{a_n}{s_n} < \infty
$$

is convergent. From (12), we get there exists $n \in \mathbb{N}$ such that for every $m \in \mathbb{N}$,

$$
1 - \frac{s_n}{s_{n+m}} = \frac{s_{n+m} - s_n}{s_{n+m}} = \frac{a_{n+1} + \cdots + a_{n+m}}{s_{n+m}} \leq \sum_{j=1}^m \frac{a_{n+j}}{s_{n+j}} < \frac{1}{2},
$$

taking the limit as $m \to \infty$, we get $1 \leq \frac{1}{2}$, a contradiction. Therefore for every
$\lambda \in (0,1)$ and $p \in \mathbb{N}$, we have

$$
\lim_{n\to\infty} d_\lambda(u_n, u_{n+p}) = 0.
$$

Then (u_n) is a Cauchy sequence and by the hypothesis there exists some element
$u \in A_0$ such that $\lim_{n\to\infty} u_n = u$. By the hypothesis it is easy to see that
$Su_n = Tu_{n+1}$, for all $n \in \mathbb{N}$, now by the continuity of the mappings S and T
we get $Su = Tu$, so the desired result is obtained. $\qquad\square$

The following corollary, is immediate.

Corollary 2.2. *Let A and B be nonempty subsets of a complete probabilistic Menger space $(X, F, *_M)$, A_0 and B_0 are nonempty and A_0 is closed. If the mappings $T, S : A \to B$ satisfy the following conditions*:

   (i) *T dominates S proximally,*
  (ii) *S and T commute proximally,*
 (iii) *S and T are continuous,*
 (iv) *$S(A_0) \subset B_0$ and $S(A_0) \subset T(A_0)$.*

Then, there exists a unique element $x \in A$ such that

$$F_{x,Sx}(t) = F_{A,B}(t) = F_{x,Tx}(t)$$

for all $t \geq 0$.

Corollary 2.3. *Let $(X, F, *_M)$ be a complete probabilistic Menger space, S be a self map on X and $T : X \to X$ be a continuous mapping such that commutes with S. If $S(X) \subseteq T(X)$ and there exists a constant $\alpha \in (0, 1)$ such that*

$$(13) \qquad F_{Sx,Sy}(\alpha t) \geq \min\{F_{Tx,Ty}(t), F_{Tx,Sx}(t), F_{Tx,Sy}(t), F_{Ty,Sx}(t)\}$$

for every $x, y \in X$ and $t \geq 0$. Then S and T have a unique common fixed point.

Proof. We used the assumption of continuity of S in Theorem 2.1 to show that

$$\lim_{n\to\infty} u_n = u, \quad Tu_n = Su_{n-1}, \quad \& \quad \lim_{n\to\infty} Tu_n = Tu, \quad \forall n \in \mathbb{N},$$
$$\Rightarrow \lim_{n\to\infty} Su_{n-1} = Su.$$

By (13), we have

$$F_{Su_n,Su}(\alpha t) \geq \min\{F_{Tu_n,Tu}(t), F_{Tu_n,Su_n}(t), F_{Tu_n,Su}(t), F_{Tu,SU_n}(t)\}, \quad \forall\, t \geq 0.$$

Since $\lim_{n\to\infty} Su_n = \lim_{n\to\infty} Tu_{n+1} = Tu$, then $F_{Tu,Su}(\alpha t) \geq F_{Tu,Su}(t)$ for all $t \geq 0$. By Lemma 1.3, we have $Tu = Su$ or

$$\lim_{n\to\infty} Su_n = \lim_{n\to\infty} Tu_{n+1} = Tu = Su.$$

Also, it is easy to see that $F_{X,X}(t) = 1$ for all $t \geq 0$, $X_0 = X$, S and T satisfy the condition (i) and (ii) of Theorem 2.1. So there exists $x \in X$ such that

$$F_{x,Sx}(t) = F_{x,Tx}(t) = F_{X,X}(t) = 1$$

for all $t \geq 0$ or $Sx = x = Tx$, as required. $\qquad\square$

If we take T to be the identity mapping in the above corollary, we get the following:

Corollary 2.4. *Let $(X, F, *_M)$ be a complete probabilistic Menger space and $S : X \to X$ be a mapping. If there exists a constant $\alpha \in (0, 1)$ such that*

$$(14) \qquad F_{Sx,Sy}(\alpha t) \geq \min\{F_{x,y}(t), F_{x,Sx}(t), F_{x,Sy}(t), F_{y,Sx}(t)\}$$

for every $x, y \in X$ and $t \geq 0$. Then S has a unique fixed point.

Theorem 2.5. *Let (A, B) be a pair of nonempty subsets of a complete probabilistic Menger space $(X, F, *)$ such that $*$ is a t-norm of H-type and A_0 is a nonempty closed set. Let a function $\varphi : (0, \infty) \to (0, \infty)$ be such that, for any $t > 0$,*

$$0 < \varphi(t) < t, \quad \text{and} \quad \lim_{n \to \infty} \varphi^n(t) = 0.$$

If $T : A \to B$ is a proximal generalized φ-contraction such that $T(A_0) \subseteq B_0$. Then there exists a unique $x \in A_0$ such that $F_{x, Tx}(t) = F_{A,B}(t)$ for all $t > 0$.

Proof. Since A_0 is nonempty and $T(A_0) \subseteq B_0$, then there exist $x_1, x_0 \in A_0$ such that $F_{x_1, Tx_0}(t) = F_{A,B}(t)$. Since $Tx_1 \in B_0$, then there exists $x_2 \in A_0$ such that $F_{x_2, Tx_1}(t) = F_{A,B}(t)$. Continuing this process, we obtain a sequence $(x_n) \subseteq A_0$ such that $F_{x_{n+1}, Tx_n}(t) = F_{A,B}(t)$, for all $n \in \mathbb{N}$ and $t > 0$. Since for all $n \in \mathbb{N}$,

$$F_{x_n, Tx_{n-1}}(t) = F_{A,B}(t) = F_{x_{n+1}, Tx_n}(t), \quad (t > 0),$$

and T is a proximal generalized φ-contraction, then we have

$$(15) \qquad\qquad F_{x_{n+1}, x_n}(\varphi(t)) \geq F_{x_n, x_{n-1}}(t), \quad (t > 0).$$

Observe that, for any $t > 0$, the sequence $(F_{x_{n+1}, x_n}(\varphi^n(t)))$ is nondecreasing. Indeed, given $n \in \mathbb{N}$, so by (15), we get

$$F_{x_{n+1}, x_n}(\varphi^n(t)) = F_{x_{n+1}, x_n}(\varphi(\varphi^{n-1}(t))) \geq F_{x_n, x_{n-1}}(\varphi^{n-1}(t)), \quad (t > 0).$$

Hence, we infer that $F_{x_{n+1}, x_n}(\varphi^n(t)) \geq F_{x_1, x_0}(t)$, so, by Lemma 1.17,

$$(16) \qquad\qquad \lim_{n \to \infty} F_{x_{n+1}, x_n}(t) = 1 \quad \text{for any } t > 0.$$

Now let $n \in \mathbb{N}$ and $t > 0$. We show by induction that, for any $k \in \mathbb{N} \cup \{0\}$,

$$(17) \qquad\qquad F_{x_{n+k}, x_n}(t) \geq *^k(F_{x_{n+1}, x_n}(t - \varphi(t))).$$

This is obvious for $k = 0$, since $F_{x_n, x_n}(t) = 1$. Assume that (17) holds for some k. Hence, by (15) and the monotonicity of $*$, we have

$$\begin{aligned}
F_{x_{n+k+1}, x_n}(t) &= F_{x_{n+k+1}, x_n}((t - \varphi(t)) + \varphi(t)) \\
&\geq F_{x_{n+k+1}, x_{n+1}}(\varphi(t)) * F_{x_{n+1}, x_n}(t - \varphi(t)) \\
&\geq F_{x_{n+k}, x_n}(t) * F_{x_{n+1}, x_n}(t - \varphi(t)) \\
&\geq *^k(F_{x_{n+1}, x_n}(t - \varphi(t))) * F_{x_{n+1}, x_n}(t - \varphi(t)) \\
&= *^{k+1}(F_{x_{n+1}, x_n}(t - \varphi(t))),
\end{aligned}$$

which completes the induction. We show that (x_n) is a Cauchy sequence, let $t > 0$ and $\varepsilon > 0$. Since $*$ is a t-norm of H-type and $*^n(1) = 1$, so there is $\delta > 0$ such that

$$(18) \qquad\qquad a > 1 - \delta \Rightarrow *^n(a) > 1 - \varepsilon \quad (n \in \mathbb{N}).$$

Since, by (16), $\lim_{n \to \infty} F_{x_{n+1}, x_n}(t - \varphi(t)) = 1$, there is $n_0 \in \mathbb{N}$ such that, for any $n \geq n_0$, $F_{x_{n+1}, x_n}(t - \varphi(t)) > 1 - \delta$. Hence, by (17) and (18), we get $F_{x_{n+k}, x_n}(t) > 1 - \varepsilon$ for any $k \in \mathbb{N} \cup \{0\}$. This proves the Cauchy condition

for (x_n). By the hypothesis, the sequence (x_n) converges to some $x \in A_0$. With use of the assumption $T(A_0) \subseteq B_0$ again, $Tx \in B_0$. Then there exists an element $u \in A_0$ such that $F_{u,Tx}(t) = F_{A,B}(t)$ for all $t > 0$. Since for all $n \in \mathbb{N}$,

$$F_{u,Tx}(t) = F_{A,B}(t) = F_{x_{n+1},Tx_n}(t), \quad (t > 0),$$

then by the hypothesis we have

$$F_{u,x_{n+1}}(t) \geq F_{u,x_{n+1}}(\varphi(t)) \geq F_{x,x_n}(t), \quad (t > 0).$$

Letting $n \to \infty$ shows that $x_n \to u$ and thus $x = u$, so $F_{x,Tx}(t) = F_{A,B}(t)$. Suppose that there is another element y such that $F_{y,Ty}(t) = F_{A,B}(t)$. Since T is a proximal generalized φ-contraction, we have $F_{x,y}(\varphi(t)) \geq F_{x,y}(t)$ which implies that x and y are identical by Lemma 1.18. $\square$

If $A = B$ in the above theorem, then we get the following:

Corollary 2.6. *Let A be a nonempty closed subset of a complete probabilistic Menger space $(X, F, *)$ such that $*$ is a t-norm of H-type. Let a function $\varphi : (0, \infty) \to (0, \infty)$ be such that, for any $t > 0$,*

$$0 < \varphi(t) < t, \quad \text{and} \quad \lim_{n \to \infty} \varphi^n(t) = 0.$$

If $T : A \to A$ is a proximal generalized φ-contraction, then T has a unique fixed point.

Proposition 2.7. *Let (A, B) be a pair of nonempty subsets of a probabilistic Menger space $(X, F, *)$ such that A_0 is a nonempty set. If $T : A \to B$ is a proximal generalized φ-contraction such that $T(A_0) \subseteq B_0$ and $g : A \to A$ is an isometry mapping such that $A_0 \subseteq g(A_0)$. Denote $G = g(A)$ and*

$$G_0 = \{z \in G : \exists \, y \in B \ s.t. \ \forall t \geq 0, \ F_{z,y}(t) = F_{G,B}(t)\}.$$

Then Tg^{-1} is a proximal generalized φ-contraction and $G_0 = A_0$.

Proof. Since $G \subseteq A$, so $F_{G,B}(t) \leq F_{A,B}(t)$ for all $t > 0$. Assume that $x \in A_0 \subseteq g(A_0)$, then $x = g(x')$ for some $x' \in A_0$ and so there exists $y \in B$ such that $F_{A,B}(t) = F_{g(x'),y}(t) \leq F_{G,B}(t)$ for all $t > 0$. Thus $F_{A,B}(t) = F_{G,B}(t)$ for all $t > 0$. Now we show that Tg^{-1} is a proximal generalized φ-contraction, to do this, suppose that $u, v, x, y \in G$ such that

$$F_{u,Tg^{-1}x}(t) = F_{G,B}(t) = F_{A,B}(t) = F_{v,Tg^{-1}y}(t), \quad (t > 0).$$

Since T is a proximal generalized φ-contraction and g is an isometry, we have

$$F_{u,v}(\varphi(t)) \geq F_{g^{-1}x,g^{-1}y}(t) = F_{gg^{-1}x,gg^{-1}y}(t) = F_{x,y}(t), \quad (t > 0).$$

Therefore Tg^{-1} is a proximal generalized φ-contraction. If $x \in G_0$, then $x \in G \subseteq A$ and there exists $y \in B$ such that $F_{x,y}(t) = F_{G,B}(t) = F_{A,B}(t)$ for all $t > 0$, so $x \in A_0$. If $x \in A_0 \subseteq g(A_0) \subseteq g(A) = G$, then there exists $y \in B$ such that $F_{x,y}(t) = F_{A,B}(t) = F_{G,B}(t)$ for all $t > 0$. Therefore $x \in G_0$. $\square$

Corollary 2.8. *Let (A, B) be a pair of nonempty subsets of a complete probabilistic Menger space $(X, F, *)$ such that $*$ is a t-norm of H-type and A_0 is a nonempty closed set. Let a function $\varphi : (0, \infty) \to (0, \infty)$ be such that, for any $t > 0$,*

$$0 < \varphi(t) < t, \quad \text{and} \quad \lim_{n \to \infty} \varphi^n(t) = 0.$$

If $T : A \to B$ is a proximal generalized φ-contraction such that $T(A_0) \subseteq B_0$ and $g : A \to A$ is an isometry mapping such that $A_0 \subseteq g(A_0)$. Then there exists a unique $x \in A_0$ such that $F_{gx, Tx}(t) = F_{A,B}(t)$.

Proof. By Proposition 2.7, $Tg^{-1} : G = g(A) \to B$ is proximal generalized φ-contraction and $Tg^{-1}(G_0) = Tg^{-1}(A_0) \subseteq T(A_0) \subseteq B_0$. Now by Theorem 2.5, there exists a unique $x' \in A_0$ such that $F_{x', Tg^{-1}x'}(t) = F_{A,B}(t)$. Since $A_0 \subseteq g(A_0)$, then there exists $x \in A_0$ such that $x' = g(x)$, so $F_{g(x), Tx}(t) = F_{A,B}(t)$. Note that g is injective mapping, therefore by Theorem 2.5, x is unique and hence the result follows. $\qquad\square$

In what follows, we present some illustrative examples which demonstrate the validity of the hypotheses and degree of utility of our results proved in this paper.

Example 2.9. Consider $X = [-1, 1]$ and define $F_{x,y}(t) = \epsilon_0(t - |x - y|)$ for all $x, y \in X$. Then $(X, F, *_M)$ is a complete probabilistic Menger space. Define continuous self mappings S and T on X as

$$S(x) = \frac{x}{4}, \qquad T(x) = -\frac{x}{2}, \qquad (x \in X).$$

One can verify all the conditions in Theorem 2.1, thus there exist unique element $x \in X$ such that

$$F_{x, Sx}(t) = F_{X, X}(t) = 1 = F_{x, Tx}(t)$$

for all $t \geq 0$.

Example 2.10. Let $X = [0, 1] \times [0, 1]$ and $d : X \times X \to [0, \infty)$ be given by $d((x_1, x_2), (y_1, y_2)) = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2}$ and define $F : (-\infty, \infty) :\to [0, 1]$ by

$$F_{(x_1, x_2), (y_1, y_2)}(t) = \frac{t}{t + d((x_1, x_2), (y_1, y_2))}.$$

Clearly, $(X, F, *_M)$ is a complete probabilistic Menger space. Let $A = \{(0, x) : x \in [0, 1]\}$, $B = \{(1, x) : x \in [0, 1]\}$ and $S, T : A \to B$ be defined as $T(0, x) = (1, x)$ for all $x \in [0, 1]$ and

$$S(0, x) = \begin{cases} (1, \frac{1}{3}) \; ; & x < 1, \\ (1, \frac{1}{2}) \; ; & x = 1, \end{cases} \quad (\forall \, x \in [0, 1]).$$

It is easy to see that $A_0 = A$, $B_0 = B$, S, T commute proximally and by Example (1.28), T weakly dominates S proximally for $\alpha = 1/4$. Therefore, all

the hypothesis of Corollary 2.3 are satisfied, then there exist unique element $x \in X$ such that

$$F_{x,Sx}(t) = F_{A,B}(t) = \frac{t}{t+1} = F_{x,Tx}(t)$$

for all $t \geq 0$.

Example 2.11. Consider $X = [0,3]$ and define $F_{x,y}(t) = \epsilon_0(t - |x - y|)$ for all $x, y \in X$. Then $(X, F, *_M)$ is a complete probabilistic Menger space. Define continuous self mappings S and T on X as

$$Sx = \frac{1}{6}x + 1, \quad Tx = \frac{1}{3}(x + \frac{12}{5}), \quad (x \in X).$$

It is easy to see that $ST = TS$ and so S and T commute proximally, $F_{X,X}(t) = 1$ and $X_0 = X$. If

$$F_{u_1,Sx_1}(t) = F_{u_2,Sx_2}(t) = F_{X,X}(t) = 1 = F_{v_1,Tx_1}(t) = F_{v_2,Tx_2}(t), \quad (t \geq 0),$$

where $u_1, u_2, v_1, v_2, x_1, x_2 \in A$. Then $u_i = Sx_i$ and $v_i = Tx_i$ $(i = 1, 2)$ and so for $\alpha = 1/2$ we have $F_{u_1,u_2}(\alpha t) = F_{v_1,v_2}(t)$, hence T dominates S proximally for $\alpha = 1/2$. Therefore, all the hypothesis of Corollary 2.2 are satisfied, then there exist unique element $x \in X$ such that

$$F_{x,Sx}(t) = F_{X,X}(t) = 1 = F_{x,Tx}(t)$$

for all $t \geq 0$.

Example 2.12. Consider $X = [-1, 1]$ and define $F_{x,y}(t) = \epsilon_0(t - |x - y|)$ for all $x, y \in X$. Then $(X, F, *_M)$ is a complete probabilistic Menger space. Define self mapping S on X as follows:

$$Sx = \begin{cases} 0 \; ; & -1 \leq x < 0, \\ \frac{x}{16(1+x)} \; ; & 0 \leq x < \frac{4}{5} \text{ or } \frac{7}{8} < x \leq 1, \quad (\forall \, x \in [0,1]). \\ \frac{x}{16} \; ; & \frac{4}{5} \leq x \leq \frac{7}{8}, \end{cases}$$

To verify condition (14) in Corollary 2.4 we need to consider several possible cases.

Case 1. Let $x, y \in [-1, 0)$. Then

$$d(Sx, Sy) = |Sx - Sy| = 0 \leq \frac{1}{8}|x - y| = \frac{1}{8}d(x, y).$$

Case 2. Let $x \in [-1, 0)$ and $y \in [0, \frac{4}{5}) \cup (\frac{7}{8}, 1]$. Then

$$d(Sx, Sy) = |Sx - Sy| = \frac{y}{16(1+y)} \leq \frac{1}{8}|y - 0| = \frac{1}{8}d(y, Sx).$$

Case 3. Let $x \in [-1, 0)$ and $y \in [\frac{4}{5}, \frac{7}{8}]$. Then

$$d(Sx, Sy) = |Sx - Sy| = \frac{y}{16} \leq \frac{1}{8}|y - 0| = \frac{1}{8}d(y, Sx).$$

Case 4. Let $x, y \in [0, \frac{4}{5}) \cup (\frac{7}{8}, 1]$. Then

$$d(Sx, Sy) = |Sx - Sy| = |\frac{x}{16(1+x)} - \frac{y}{16(1+y)}| \leq \frac{1}{8}|x - y| = \frac{1}{8}d(x, y).$$

Case 5. Let $x \in [0, \frac{4}{5}) \cup (\frac{7}{8}, 1]$ and $y \in [\frac{4}{5}, \frac{7}{8}]$. Then

$$d(Sx, Sy) = |Sx - Sy| = |\frac{x}{16(1+x)} - \frac{y}{16}| \leq \frac{1}{16}(\frac{x}{1+x} + y) \leq \frac{1}{16}(\frac{1}{2} + \frac{7}{8}) \leq \frac{11}{128},$$

and

$$\frac{123}{160} = \frac{4}{5} - \frac{1}{16}\frac{1}{2} \leq y - \frac{x}{16(1+x)} \leq |y - \frac{x}{16(1+x)}| = d(y, Sx).$$

Thus

$$d(Sx, Sy) \leq \frac{11}{128} \leq \frac{123}{1280} = \frac{1}{8} \times \frac{123}{160} \leq \frac{1}{8}d(y, Sx).$$

Case 6. Let $x, y \in [\frac{4}{5}, \frac{7}{8}]$. Then

$$d(Sx, Sy) = |Sx - Sy| = |\frac{x}{16} - \frac{y}{16}| \leq \frac{1}{8}|x - y| = \frac{1}{8}d(x, y).$$

Hence, we obtain

$$d(Sx, Sy) \leq \frac{1}{8}\max\{d(x, y), d(x, Sx), d(x, Sy), d(y, Sx)\}, \quad (x, y \in [-1, 1]),$$

or in other words

$$F_{Sx, Sy}(\frac{1}{8}t) \geq \min\{F_{x,y}(t), F_{x,Sx}(t), F_{x,Sy}(t), F_{y,Sx}(t)\}$$

for every $x, y \in X$ and $t \geq 0$. Then S has a unique fixed point 0 in X, by Corollary 2.4.

Example 2.13. Let $X = \mathbb{R}^2$, $A = \{(0, y) : y \in \mathbb{R}\}$ and $B = \{(1, y) : y \in \mathbb{R}\}$. Suppose that $T : A \to B$ is defined by $T(0, y) = (1, \frac{y}{4})$, $g : A \to A$ is defined by $g(0, y) = (0, -y)$ and $F_{(x,x'),(y,y')}(t) = \frac{t}{t+|x-y|+|x'-y'|}$. It is easy to see that $(X, F, *_M)$ is a complete probabilistic Menger space, $F_{A,B}(t) = \frac{t}{t+1}$, $A_0 = A$, $B_0 = B$, $T(A_0) \subseteq B_0$ and

$$F_{g(0,x),g(0,y)}(t) = F_{(0,-x),(0,-y)}(t) = \frac{t}{t+|x-y|} = F_{(0,x),(0,y)}(t).$$

If $(0, u), (0, x), (0, v), (0, y) \in A$ such that

$$\frac{t}{t+1+|u-\frac{x}{4}|} = F_{(0,u),T(0,x)}(t) = F_{A,B}(t) = F_{(0,v),T(0,y)}(t) = \frac{t}{t+1+|v-\frac{y}{4}|},$$

then $u = \frac{x}{4}$ and $v = \frac{y}{4}$, so

$$F_{(0,u),(0,v)}(t) = F_{(0,\frac{x}{4}),(0,\frac{y}{4})}(t) = \frac{t}{t+\frac{1}{4}|x-y|} = F_{(0,x),(0,y)}\left(\frac{t}{\frac{1}{4}}\right).$$

Therefore all the hypothesis of Corollary 2.8 are satisfied, and also we have

$$F_{(0,0),T(0,0)}(t) = F_{(0,0),(1,0)}(t) = \frac{t}{t+1} = F_{A,B}(t).$$

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