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REGULARITY FOR FRACTIONAL ORDER RETARDED NEUTRAL DIFFERENTIAL EQUATIONS IN HILBERT SPACES

Seong Ho Cho, Jin-Mun Jeong, and Yong Han Kang

Abstract. In this paper, we study the existence of solutions and L^2-regularity for fractional order retarded neutral functional differential equations in Hilbert spaces. We no longer require the compactness of structural operators to prove the existence of continuous solutions of the nonlinear differential system, but instead we investigate the relation between the regularity of solutions of fractional order retarded neutral functional differential systems with unbounded principal operators and that of its corresponding linear system excluded by the nonlinear term. Finally, we give a simple example to which our main result can be applied.

1. Introduction

Let H and V be two complex Hilbert spaces such that V is a dense subspace of H. In this paper, we study the existence of solutions and L^2-regularity for the following fractional order retarded neutral functional differential equation:

(1.1)
$$\begin{cases} \frac{d^\alpha}{dt^\alpha}[x(t)+g(t,x_t)] = Ax(t) + \int_{-h}^0 a_1(s)A_1x(t+s)ds + (Fx)(t) + k(t), \ t>0, \\ x(0)=\phi^0, \quad x(s)=\phi^1(s), \quad -h\le s<0, \end{cases}$$

where $1/2<\alpha<1$, $h>0$, $a_1(\cdot)$ is Hölder continuous, k is a forcing term, and g, f, are given functions satisfying some assumptions. Moreover, $A : D(A)\subset H\to H$ is unbounded but A_1 is bounded. For each $s\in[0,T]$, we define $x_s : [-h,0]\to H$ as $x_s(r)=x(s+r)$ for $r\in[-h,0]$ and $(\phi^0,\phi^1)\in H\times L^2(-h,0;V)$.

This kind of systems arises in many practical mathematical models arising in dynamic systems, economy, physics, biological and engineering problems, etc. (see [5, 6, 17, 18]). There has been a significant development in fractional

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differential equations in recent years, see [9, 14, 16, 20] and the references therein.

In [4, 7, 8], the authors have discussed the existence of solutions for mild solutions for the neutral differential systems with state-dependence delay. Most studies about the neutral initial value problems governed by retarded semilinear parabolic equation have been devoted to the control problems. As for the retarded differential equations, Jeong et al. [12, 13], Sukavanam et al. [22], and Wang [25], have discussed the regularity of solutions and controllability of the semilinear retarded systems, and see [12, 13, 22, 25] and references therein for the linear retarded systems.

Recently, the existence of mild solutions for fractional neutral evolution equations has been studied in [10, 14], the existence of solutions of inhomogeneous fractional diffusion equations with a forcing function in Baeumer et al. [2], and the existence and approximation of solutions to fractional evolution equation in Muslim [19]. In addition, Sukavanam et al. [21] studied approximate controllability of fractional order semilinear delay systems.

In this paper, we propose a different approach of the earlier works used properties of the relative compactness. Our approach is that regularity results of general retarded linear systems of Di Blasio et al. [3] and semilinear systems of [13] remain valid under the above formulation of fractional order retarded neutral differential system (1.1) even though the system (1.1) contains unbounded principal operators, delay term, and local Lipschitz continuity of the nonlinear term. The methods of the functional analysis concerning an analytic semigroup of operators and some fixed point theorems are applied effectively.

The paper is organized as follows. In Section 2, we deal with properties of the analytic semigroup constructing the strict solution of the corresponding linear systems excluded by the nonlinear term and introduce basic properties. In Section 3, by using properties of the strict solutions in dealt in Section 2, we will obtain the L^2-regularity of solutions of (1.1), and a variation of constant formula of solutions of (1.1). Finally, we also give an example to illustrate the applications of the abstract results.

2. Preliminaries and lemmas

The inner product and norm in H are denoted by $(\cdot,\cdot)$ and $|\cdot|$, respectively. V is another Hilbert space densely and continuously embedded in H. The notations $||\cdot||$ and $||\cdot||_*$ denote the norms of V and V^* as usual, respectively. For brevity we may regard that

$$||u||_* \le |u| \le ||u||, \quad u \in V. \tag{2.1}$$

Let $a(\cdot,\cdot)$ be a bounded sesquilinear form defined in $V \times V$ and satisfying Gårding's inequality

$$\text{Re } a(u,u) \ge c_0||u||^2 - c_1|u|^2, \quad c_0 > 0, \ c_1 \ge 0. \tag{2.2}$$

Let A be the operator associated with the sesquilinear form $-a(\cdot,\cdot)$:

$$((c_1 - A)u, v) = -a(u,v), \quad u,\ v \in V.$$

It follows from (2.2) that for every $u \in V$

$$\mathrm{Re}\,(Au, u) \geq c_0 ||u||^2.$$

Then A is a bounded linear operator from V to V^* according to the Lax-Milgram theorem, and its realization in H which is the restriction of A to

$$D(A) = \{u \in V; Au \in H\}$$

is also denoted by A. Then A generates an analytic semigroup $S(t) = e^{tA}$ in both H and V^* as in Theorem 3.6.1 of [23]. Moreover, there exists a constant C_0 such that

$$||u|| \leq C_0 ||u||_{D(A)}^{1/2} |u|^{1/2} \tag{2.3}$$

for every $u \in D(A)$, where

$$||u||_{D(A)} = (|Au|^2 + |u|^2)^{1/2}$$

is the graph norm of $D(A)$. Thus we have the following sequence

$$D(A) \subset V \subset H \subset V^* \subset D(A)^*,$$

where each space is dense in the next one and continuous injection.

Lemma 2.1. *With the notations* (2.1), (2.3), *we have*

$$(V, V^*)_{1/2,2} = H,$$
$$(D(A), H)_{1/2,2} = V,$$

where $(V,V^*)_{1/2,2}$ *denotes the real interpolation space between* V *and* V^* (*see Section 1.3.3 of* [24]).

If X is a Banach space and $1 < p < \infty$, $L^p(0,T;X)$ is the collection of all strongly measurable functions from $(0,T)$ into X the p-th powers of norms are integrable. $\mathcal{L}(X,Y)$ is the collection of all bounded linear operators from X into Y, and $\mathcal{L}(X,X)$ is simply written as $\mathcal{L}(X)$.

For the sake of simplicity we assume that the semigroup $S(t)$ generated by A is uniformly bounded, that is, There exists a constant M_0 such that

$$||S(t)||_{\mathcal{L}(H)} \leq M_0, \quad ||AS(t)||_{\mathcal{L}(H)} \leq \frac{M_0}{t}. \tag{2.4}$$

The following lemma is from [23, Lemma 3.6.2].

Lemma 2.2. *There exists a constant* M_0 *such that the following inequalities hold*:

$$||S(t)||_{\mathcal{L}(V,H)} \leq t^{-1/2} M_0, \tag{2.5}$$

$$||S(t)||_{\mathcal{L}(V^*,V)} \leq t^{-1} M_0, \tag{2.6}$$

$$||AS(t)||_{\mathcal{L}(H,V)} \leq t^{-3/2} M_0. \tag{2.7}$$

The following initial value problem for the abstract linear parabolic equation

$$(2.8)\qquad \begin{cases} \frac{dx(t)}{dt} = Ax(t) + \int_{-h}^{0} a_1(s)A_1x(t+s)ds + k(t), & 0 < t \le T, \\ x(0) = \phi^0, \quad x(s) = \phi^1(s) \ s \in [-h, 0). \end{cases}$$

Then the mild solution $x(t)$ is represented by

$$\begin{aligned} x(t) &= S(t)\phi^0 + \int_0^t S(t-s)\{\int_{-h}^0 a_1(\tau)A_1x(s+\tau)d\tau + f(s, x(s))\}ds \\ &\quad + \int_0^t S(t-s)k(s)ds, \\ x(0) &= \phi^0, \quad x(s) = \phi^1(s) \quad s \in [-h, 0). \end{aligned}$$

By virtue of Theorem 2.1 of [11] or [3], we have the following result on the corresponding linear equation of (2.8).

Lemma 2.3. (1) *For* $(\phi^0, \phi^1) \in V \times L^2(-h, 0; D(A))$ *and* $k \in L^2(0, T; H)$, $T > 0$*, there exists a unique solution* x *of* (2.8) *belonging to*

$$L^2(0, T; D(A)) \cap W^{1,2}(0, T; H) \subset C([0, T]; F)$$

and satisfying

$$(2.9)\quad ||x||_{L^2(0,T;D(A))\cap W^{1,2}(0,T;H)} \le C_1(||\phi^0||_F + ||\phi^1||_{L^2(-h,0;D(V))} + ||k||_{L^2(0,T;H)}),$$

where C_1 *is a constant depending on* T *and*

$$||x||_{L^2(0,T;D(A))\cap W^{1,2}(0,T;H)} = \max\{||x||_{L^2(0,T;D(A)),||x||^{1,2}_W(0,T;H)}\}.$$

(2) *Let* $(\phi^0, \phi^1) \in H \times L^2(-h, 0; V)$ *and* $k \in L^2(0, T; V^*)$, $T > 0$*. Then there exists a unique solution* x *of* (2.8) *belonging to*

$$L^2(0, T; V) \cap W^{1,2}(0, T; V^*) \subset C([0, T]; H)$$

and satisfying

$$(2.10)\quad ||x||_{L^2(0,T;V)\cap W^{1,2}(0,T;V^*)} \le C_1(|\phi^0| + ||\phi^1||_{L^2(-h,0;V)} + ||k||_{L^2(0,T;V^*)}),$$

where C_1 *is a constant depending on* T.

Let the solution spaces $\mathcal{W}_0(T)$ and $\mathcal{W}_1(T)$ of strong solutions be defined by

$$\mathcal{W}_0(T) = L^2(0, T; D(A)) \cap W^{1,2}(0, T; H),$$
$$\mathcal{W}_1(T) = L^2(0, T; V) \cap W^{1,2}(0, T; V^*).$$

Here, we note that by using interpolation theory, we have

$$\mathcal{W}_0(T) \subset C([0, T]; V), \quad \mathcal{W}_1(T) \subset C([0, T]; H).$$

Thus, there exists a constant $c_1 > 0$ such that

$$(2.11)\qquad ||x||_{C([0,T];V)} \le c_1||x||_{\mathcal{W}_0(T)}, \quad ||x||_{C([0,T];H)} \le c_1||x||_{\mathcal{W}_1(T)}.$$

In what follows in this section, we assume $c_1 = 0$ in (2.2) without any loss of generality. So we have that $0 \in \rho(A)$ and the closed half plane $\{\lambda : \mathrm{Re}\,\lambda \ge 0\}$

is contained in the resolvent set of A. In this case, it is possible to define the fractional power A^α for $\alpha > 0$. The subspace $D(A^\alpha)$ is dense in H and the expression

$$||x||_\alpha = ||A^\alpha x||, \quad x \in D(A^\alpha)$$

defines a norm on $D(A^\alpha)$. It is also well known that A^α is a closed operator with its domain dense and $D(A^\alpha) \supset D(A^\beta)$ for $0 < \alpha < \beta$. Due to the well known fact that $A^{-\alpha}$ is a bounded operator, we can assume that there is a constant $C_{-\alpha} > 0$ such that

$$||A^{-\alpha}||_{\mathcal{L}(H)} \le C_{-\alpha}, \quad ||A^{-\alpha}||_{\mathcal{L}(V^*,V)} \le C_{-\alpha}. \tag{2.12}$$

3. Existence of solutions

Consider the following fractional order retarded neutral differential system:

(3.1)
$$\begin{cases} \frac{d^\alpha}{dt^\alpha}[x(t) + g(t,x_t)] = Ax(t) + \int_{-h}^0 a_1(s)A_1x(t+s)ds + (Fx)(t) + k(t), \ t > 0, \\ x(0) = \phi^0, \quad x(s) = \phi^1(s), \quad -h \le s < 0, \end{cases}$$

where $0 < \alpha < 1$ and A and A_1 are the linear operators defined as in Section 2. For each $s \in [0,T]$, we define $x_s : [-h,0] \to H$ as

$$x_s(r) = x(s+r), \quad -h \le r \le 0.$$

We will set

$$\Pi = L^2(-h,0;V).$$

Definition. The fractional integral of order $\alpha > 0$ with the lower limit 0 from a function f is defined as

$$I^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t \frac{f(s)}{(t-s)^{1-\alpha}} ds, \quad t > 0,$$

provided the right hand side is pointwise defined on $[0,\infty)$, Γ is the Gamma function.

The fractional derivative of order $\alpha > 0$ in the Caputo sense with the lower limit 0 from a function $f \in C^n[0,\infty)$ is defined as

$$\frac{d^\alpha f(t)}{dt^\alpha} = \frac{1}{\Gamma(n-\alpha)} \int_0^t \frac{f^{(n)}(s)}{(t-s)^{1+\alpha-n}} ds = I^{n-\alpha} f^{(n)}(t), \quad t > 0, \ n-1 < \alpha < n.$$

For the basic results about fractional integrals and fractional derivative, one can refer to [20].

The mild solution of the system (3.1) is represented as (see [10, 26]):

(3.2)
$$x(t) = S(t)[\phi^0 + g(0,\phi^1)] - g(t,x_t) + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{(\alpha-1)} AS(t-s)g(s,x_s)ds$$
$$+ \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{(\alpha-1)} S(t-s)$$

$$\times \{ \int_{-h}^{0} a_1(\tau)A_1 x(s+\tau)d\tau + (Fx)(s) + k(s)\}ds.$$

To establish our results, we introduce the following assumptions on system (3.1).

Assumption (A). We assume that $a_1(\cdot)$ is Hölder continuous of order ρ:

$$|a_1(0)| \le H_1, \quad |a_1(s) - a_1(\tau)| \le H_1(s-\tau)^\rho.$$

Assumption (F). F is a nonlinear mapping of $L^2(0,T;V)$ into $L^2(0,T;H)$ satisfying following:

(i) There exists a function $L_f : \mathbb{R}_+ \to \mathbb{R}$ such that

$$||Fx - Fy||_{L^2(0,T;H)} \le L_f(r)||x-y||_{L^2(0,T;V)}, \quad t \in [0,T]$$

hold for $||x||_{L^2(0,T;V)} \le r$ and $||y||_{L^2(0,T;V)} \le r$.

(ii) The inequality

$$||Fx||_{L^2(0,T;H)} \le L_f(r)(||x||_{L^2(0,T;V)} + 1)$$

holds for every $t \in [0,T]$ and $||x||_{L^2(0,T;V)} \le r$.

Assumption (G). Let $g : [0,T] \times \Pi \to H$ be a nonlinear mapping such that there exists a constant L_g satisfying the following conditions hold:

(i) For any $x \in \Pi$, the mapping $g(\cdot, x)$ is strongly measurable;

(ii) There exists a positive constant $\beta > 1 - 2\alpha/3$ such that

$$|A^\beta g(t,0)| \le L_g, \quad |A^\beta g(t,x) - A^\beta g(t,\hat{x})| \le L_g||x - \hat{x}||_\Pi$$

for all $t \in [0,T]$, and $x, \hat{x} \in \Pi$.

Lemma 3.1. *Let $x \in L^2(-h,T;V)$. Then the mapping $s \mapsto x_s$ belongs to $C([0,T];\Pi)$, and*

$$||x_t||_\Pi \le ||x||_{L^2(-h,t;V)} (t > 0), \tag{3.3}$$

$$||x.||_{L^2(0,T;\Pi)} \le \sqrt{T}||x||_{L^2(-h,T;V)}. \tag{3.4}$$

Proof. The first paragraph is easy to verify. Moreover, we have

$$||x_t||_\Pi = \big[\int_{-h}^{0} ||x(s+\tau)||^2 d\tau\big]^{1/2} \le \big[\int_{-h}^{t} ||x(\tau)||^2 d\tau\big]^{1/2} \le ||x||_{L^2(-h,t;V)},\ t > 0,$$

and

$$\begin{aligned} ||x.||^2_{L^2(0,T;\Pi)} &\le \int_0^T ||x_s||^2_\Pi ds \le \int_0^T \int_{-h}^{0} ||x(s+r)||^2 dr ds \\ &\le \int_0^T ds \int_{-h}^{T} ||x(r)||^2 dr \le T||x||^2_{L^2(-h,T;V)}. \end{aligned}$$

□

One of the main useful tools in the proof of existence theorems for nonlinear functional equations is the following fixed point theorem.

Lemma 3.2 (Krasnoselski [15]). *Suppose that Σ is a closed convex subset of a Banach space X. Assume that K_1 and K_2 are mappings from Σ into X such that the following conditions are satisfied:*

(i) $(K_1+K_2)(\Sigma)\subset\Sigma$,
(ii) K_1 *is a completely continuous mapping,*
(iii) K_2 *is a contraction mapping.*

Then the operator K_1+K_2 has a fixed point in Σ.

From now on, we establish the following results on the solvability of the equation (3.1).

Theorem 3.3. *Let Assumptions* (A), (F) *and* (G) *be satisfied. Assume that $(\phi^0,\phi^1)\in H\times\Pi$ and $k\in L^2(0,T;V^*)$ for $T>0$. Then, there exists a solution x of the system* (3.1) *such that*

$$x\in\mathcal{W}_1(T)=L^2(0,T;V)\cap W^{1,2}(0,T;V^*)\hookrightarrow C([0,T];H).$$

Moreover, there is a constant C_2 independent of the initial data (ϕ^0,ϕ^1) and the forcing term k such that

$$||x||_{L^2(-h,T;V)}\le C_2(1+|\phi^0|+||\phi^1||_\Pi+||k||_{L^2(0,T;V^*)}). \tag{3.5}$$

Proof. Let

$$r:=2\big[C_1|\phi^0|+C_1C_{-\alpha}L_g(||\phi^1||+1)\big],$$

and

$$\begin{aligned}N:=&\;C_{-\alpha}L_g\big(||\phi^1||_\Pi+||x||_{L^2(0,T_1;V)}+1\big)\\&+\frac{C_1(2\alpha)^{-1/2}(2\alpha-1)^{-1/2}}{\Gamma(\alpha)}\\&\times\big(|\phi^0|+||\phi^1||_{L^2(-h,0;V)}+L_f(r)(||x||_{L^2(0,T_1;V)}+1)+||k||_{L^2(0,T_1;V)}\big)\\&+\frac{C_{1-\beta}L_g}{\big(\alpha-3(1-\beta)/2\big)\big(2\alpha+3\beta-2\big)^{1/2}\Gamma(\alpha)}(||\phi^1||_\Pi+||x||_{L^2(0,T_1;V)}+1),\end{aligned}$$

where C_1 is the constants in Lemma 2.3 and $\beta>1-2\alpha/3$ in Assumption (G). Let

$$T_1^\gamma:=\max\{T_1^{1/2},T_1^{(2\alpha+3\beta-2)/2}\}$$

and choose $0<T_1<T$ such that

$$T_1^\gamma N\le\frac{r}{2}=\big[C_1|\phi^0|+C_1C_{-\alpha}L_g(||\phi^1||+1)\big], \tag{3.6}$$

and

$$\begin{aligned}\hat N:=&\;T_1^\gamma\Big\{C_{-\alpha}L_g+\frac{C_{1-\beta}L_g}{\big(\alpha-3(1-\beta)/2\big)\big(2\alpha+3\beta-2\big)^{1/2}\Gamma(\alpha)}\\&+\frac{C_1(2\alpha)^{-1/2}(2\alpha-1)^{-1/2}L_f(r)}{\Gamma(\alpha)}\Big\}\\&<1.\end{aligned} \tag{3.7}$$

Let J be the operator on $L^2(0,T_1;V)$ defined by

$$
\begin{aligned}
&(Jx)(t)\\
&= S(t)[\phi^0 + g(0,\phi^1)] - g(t,x_t) + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{(\alpha-1)} AS(t-s)g(s,x_s)ds\\
&\quad + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{(\alpha-1)} S(t-s)\{\int_{-h}^0 a_1(\tau)A_1x(s+\tau)d\tau + (Fx)(s) + k(s)\}ds.
\end{aligned}
$$

Let

$$\Sigma = \{x \in L^2(-h,T;V) : x(0) = \phi^0, \text{ and } x(s) = \phi^1(s)(s \in [-h,0))\}$$

and

$$\Sigma_r = \{x \in \Sigma : ||x||_{L^2(0,T_1;V)} \le r\},$$

which is a bounded closed subset of $L^2(0,T_1;V)$.

Now, in order to show that the operator J has a fixed point in $\Sigma_r \subset L^2(0,T_1;V)$, we take the following steps according to the process of Lemma 3.2.

Step 1. J maps Σ_r into Σ_r.

By (2.10), (2.14) and Assumption (G), and noting $x_0 = \phi^1$, we know

$$
\begin{aligned}
&||S(\cdot)g(0,x_0)||_{L^2(0,T_1;V)}\\
&= C_1|g(0,\phi^1)|\\
&= C_1||A^{-\beta}||_{\mathcal{L}(H)}\big(|A^\beta g(0,\phi^1) - A^\beta g(0,0)| + |A^\beta g(0,0)|\big)\\
&\le C_1C_{-\alpha}L_g(||\phi^1|| + 1).
\end{aligned}
\tag{3.8}
$$

From (2.10) of Lemma 2.3 it follows

$$||S(t)\phi^0||_{L^2(0,T_1;V)} \le C_1|\phi^0|, \tag{3.9}$$

and by using Hölder inequality

$$
\begin{aligned}
&\int_0^t (t-s)^{\alpha-1}\big\|S(t-s)\{\int_{-h}^0 a_1(\tau)A_1x(s+\tau)d\tau + (Fx)(s) + k(s)\}\big\|ds\\
&\le (2\alpha-1)^{-1/2}t^{(2\alpha-1)/2}C_1(|\phi^0| + ||\phi^1||_{L^2(-h,0;V)} + ||Fx||_{L^2(0,t;V^*)}\\
&\qquad + ||k||_{L^2(0,t;V^*)}).
\end{aligned}
\tag{3.10}
$$

Define the operator I_1 from $L^2(0,T_1;V)$ to itself by

$$(I_1x)(t) = \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}S(t-s)\big\{\int_{-h}^0 a_1(\tau)A_1x(s+\tau)d\tau + (Fx)(s) + k(s)\big\}ds.$$

Then according to (3.10) we obtain the following inequality

$$
\begin{aligned}
||I_1||_{L^2(0,T_1;V)} \le \frac{C_1(2\alpha)^{-1/2}(2\alpha-1)^{-1/2}T_1^\alpha}{\Gamma(\alpha)}\big(|\phi^0| + ||\phi^1||_{L^2(-h,0;V)}\\
+ L_f(r)(||x||_{L^2(0,T_1;V)} + 1) + ||k||_{L^2(0,T_1;V)}\big).
\end{aligned}
\tag{3.11}
$$

By using Assumption (G) and Lemma 3.1, we have

$$
\begin{aligned}
(3.12) \qquad ||g(\cdot, x.)||_{L^2(0,T_1;V)} &= \Big(\int_0^{T_1} \big\| A^{-\beta} A^{\beta} g(t,x_t) \big\|^2 dt \Big)^{1/2} \\
&\le C_\alpha \Big(\int_0^{T_1} \big\| A^{\beta} g(t,x_t) \big\|^2 dt \Big)^{1/2} \\
&\le C_{-\alpha} L_g \sqrt{T_1} \big(||x_t||_\Pi + 1 \big) \\
&\le C_{-\alpha} L_g \sqrt{T_1} \big(||\phi^1||_\Pi + ||x||_{L^2(0,T_1;V)} + 1 \big).
\end{aligned}
$$

Here, we note

$$
(3.13) \qquad ||x_t||_\Pi \le ||x||_{L^2(-h,T_1;V)} \le ||\phi^1||_\Pi + ||x||_{L^2(0,T_1;V)}.
$$

Again we define the operator I_2 from $L^2(0,T_1;V)$ to itself by

$$
(I_2 x)(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{(\alpha-1)} A S(t-s) g(s,x_s) ds.
$$

From Lemma 2.6 and Assumption (G) we have

$$
\begin{aligned}
& |(t-s)^{(\alpha-1)} A S(t-s) g(s,x_s)| \\
&= (t-s)^{(\alpha-1)} ||A^{1-\beta} S(t-s)||_{\mathcal{L}(H,V)} |A^{\beta}(g(s,x_s)| \\
&\le \frac{C_{1-\beta}}{(t-s)^{1-\alpha+3(1-\beta)/2}} |A^{\beta}(g(s,x_s)| \\
&\le \frac{C_{1-\beta}}{(t-s)^{1-\alpha+3(1-\beta)/2}} L_g (||\phi^1||_\Pi + ||x||_{L^2(0,T_1;V)} + 1),
\end{aligned}
$$

and hence, by using Hölder inequality and Assumption (G),

$$
\begin{aligned}
(3.14) \quad & ||I_2 x||_{L^2(0,T_1;V)} \\
&= \Big[\int_0^{T_1} \Big\| \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{(\alpha-1)} A S(t-s) g(s,x_s) ds \Big\|^2 dt \Big]^{1/2} \\
&\le \frac{1}{\Gamma(\alpha)} C_{1-\beta} L_g (||\phi^1||_\Pi + ||x||_{L^2(0,T_1;V)} + 1) \\
&\qquad \Big[\int_0^{T_1} \Big(\int_0^t \frac{1}{(t-s)^{1-\alpha+3(1-\beta)/2}} ds \Big)^2 dt \Big]^{1/2} \\
&\le \frac{C_{1-\beta} L_g T_1^{(2\alpha+3\beta-2)/2}}{\big(\alpha - 3(1-\beta)/2\big)\big(2\alpha+3\beta-2\big)^{1/2} \Gamma(\alpha)} (||\phi^1||_\Pi + ||x||_{L^2(0,T_1;V)} + 1).
\end{aligned}
$$

Thus, from (3.8)-(3.14) it follows that

$$
\begin{aligned}
& ||Jx||_{L^2(0,T_1;V)} \\
&\le C_1 |\phi^0| + C_1 C_{-\alpha} L_g (||\phi^1|| + 1) + C_{-\alpha} L_g \sqrt{T_1} \big(||\phi^1||_\Pi + ||x||_{L^2(0,T_1;V)} + 1 \big) \\
&\quad + \frac{C_1 (2\alpha)^{-1/2} (2\alpha-1)^{-1/2} T_1^{\alpha}}{\Gamma(\alpha)}
\end{aligned}
$$

$$\times \big(|\phi^0| + ||\phi^1||_{L^2(-h,0;V)} + L_f(r)(||x||_{L^2(0,T_1;V)} + 1) + ||k||_{L^2(0,T_1;V)}\big)$$
$$+ \frac{C_{1-\beta} L_g T_1^{(2\alpha+3\beta-2)/2}}{\big(\alpha - 3(1-\beta)/2\big)\big(2\alpha+3\beta-2\big)^{1/2}\Gamma(\alpha)}(||\phi^1||_\Pi + ||x||_{L^2(0,T_1;V)} + 1),$$
$$\le C_1|\phi^0| + C_1C_{-\alpha}L_g(||\phi^1|| + 1) + T_1^\gamma N \le \frac{r}{2} + \frac{r}{2} \le r.$$

Therefore, J maps Σ_r into Σ_r.

Define mapping $K_1 + K_2$ on $L^2(0, T_1; V)$ by the formula

$$(Jx)(t) = (K_1x)(t) + (K_2x)(t),$$

where

$$(K_1x)(t) = \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{(\alpha-1)} S(t-s)\int_0^s a_1(\tau - s)A_1x(\tau)d\tau ds,$$

and

$$\begin{aligned}(K_2x)(t) = {}& S(t)[\phi^0 + g(0,x_0)] - g(t,x_t)\\ &+ \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{(\alpha-1)} AS(t-s)g(s,x_s)ds\\ &+ \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{(\alpha-1)} S(t-s)\\ &\times \{\int_{s-h}^0 a_1(\tau-s)A_1\phi^1(\tau)d\tau + F(x)(s) + k(s)\}ds.\end{aligned}$$

Step 2. K_1 is a completely continuous mapping.

We can now employ Lemma 3.2 with Σ_r. Assume that a sequence $\{x_n\}$ of $L^2(0,T_1;V)$ converges weakly to an element $x_\infty \in L^2(0,T_1;V)$, i.e., $w - \lim_{n\to\infty} x_n = x_\infty$. Then we will show that

$$\lim_{n\to\infty} ||K_1x_n - K_1x_\infty||_{L^2(0,T_1;V)} = 0, \tag{3.15}$$

which is equivalent to the completely continuity of K_1 since $L^2(0,T_1;V)$ is reflexive. For a fixed $t \in [0,T_1]$, let $x_t^*(x) = (K_1x)(t)$ for every $x \in L^2(0,T_1;V)$. Then $x_t^* \in L^2(0,T_1;V^*)$ and we have $\lim_{n\to\infty} x_t^*(x_n) = x_t^*(x_\infty)$ since $w - \lim_{n\to\infty} x_n = x_\infty$. Hence,

$$\lim_{n\to\infty}(K_1x_n)(t) = (K_1x_\infty)(t), \quad t \in [0,T_1].$$

By using Hölder inequality, we obtain easily the following inequality:

$$\begin{aligned}&|\int_0^s a_1(\tau-s)A_1x(\tau)d\tau| \qquad (3.16)\\ &= |\int_0^s (a_1(\tau-s) - a_1(0) + a_1(0))A_1x(\tau)d\tau|\\ &\le \{\big((2\rho+1)^{-1}s^{2\rho+1}\big)^{1/2} + \sqrt{s}\}H_1||A_1||_{\mathcal{L}(H)}\big(\int_0^s ||x(\tau)||^2 d\tau\big)^{1/2}.\end{aligned}$$

Thus, by (2.5) and (3.16) it holds

$$\begin{aligned}
&||(K_1x)(t)|| \\
&= \Big\|\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{(\alpha-1)}S(t-s)\int_0^s a_1(\tau-s)A_1x(\tau)d\tau ds\Big\| \\
&\le \frac{H_1||A_1||_{\mathcal{L}(H)}||x||_{L^2(0,t;V)}}{\Gamma(\alpha)}\Big\|\int_0^t \frac{1}{(t-s)^{1/2-\alpha}}\big\{((2\rho+1)^{-1}s^{(2\rho+1)/2}+\sqrt{s}\big\}ds\Big\| \\
&\le \frac{H_1||A_1||_{\mathcal{L}(H)}||x||_{L^2(0,t;V)}}{\Gamma(\alpha)}\big\{(2\rho+1)^{-1}B(1/2+\alpha,(2\rho+3)/2)t^{\rho+1} \\
&\qquad\qquad + B(1/2+\alpha,3/2)t\big\}. \\
&:= c_2||x||_{L^2(0,t;V)},
\end{aligned}$$

where c_2 is a constant and $B(\cdot,\cdot)$ is the Beta function, that is,

$$B(1/2+\alpha,(2\rho+3)/2)t^{\rho+1}=\int_0^t (t-s)^{\alpha-1/2}s^{(2\rho+1)/2}ds.$$

And we know

$$\sup_{0\le t\le T_1}||(K_1x)(t)||\le c_2||x||_{M^2(0,T_1;V)}\le\infty.$$

Therefore, by Lebesgue's dominated convergence theorem it holds

$$\lim_{n\to\infty}\Big(\int_0^{T_1}||(K_1x_n)(t)||^2dt\Big)=\Big(\int_0^{T_1}||(K_1x_\infty)(t)||^2dt\Big),$$

i.e., $\lim_{n\to\infty}||K_1x_n||_{L^2(0,T_1;V)}=||K_1x_\infty||_{L^2(0,T_1;V)}$. Since $L^2(0,T_1;V)$ is a reflexive space, it holds (3.15).

Step 3. K_2 is a contraction mapping.

For every x_1 and $x_2\in\Sigma_r$, we have

$$\begin{aligned}
(K_2x_1)(t)-(K_2x_2)(t) &= g(t,x_{2_t})-g(t,x_{1_t}) \\
&\quad -\int_0^t AS(t-s)\big(g(t,x_{1_s})-g(t,x_{2_s})\big)ds \\
&\quad +\int_0^t S(t-s)\{F(x_1)(s)-F(x_2)(s)\}dW.
\end{aligned}$$

By the similar way to (3.8)-(3.14), we have

$$\begin{aligned}
&||K_2x_1-K_2x_2||_{L^2(0,T_1;V)} \\
&\le \Big\{C_{-\alpha}L_g\sqrt{T_1}+\frac{C_{1-\beta}L_gT_1^{(2\alpha+3\beta-2)/2}}{\big(\alpha-3(1-\beta)/2\big)\big(2\alpha+3\beta-2\big)^{1/2}\Gamma(\alpha)} \\
&\quad +\frac{C_1(2\alpha)^{-1/2}(2\alpha-1)^{-1/2}L_f(r)T_1^{\alpha}}{\Gamma(\alpha)} \\
&\le \hat{N}||x_1-x_2||_{M^2(0,T_1;V)}.
\end{aligned}$$

So by virtue of the condition (3.7) the contraction mapping principle gives that the solution of (3.1) exists uniquely in $L^2(0,T_1;V)$. This has proved the local existence and uniqueness of the solution of (3.1).

Step 4. We drive a priori estimate of the solution.

To prove the global existence, we establish a variation of constant formula (3.5) of solution of (3.1). Let x be a solution of (3.1) and $\phi^0 \in H$. Then we have that from (3.8)-(3.14) it follows that

$$\begin{aligned}
&||x||_{L^2(0,T_1;V)} \\
&\le C_1|\phi^0| + C_1C_{-\alpha}L_g(||\phi^1||+1) + C_{-\alpha}L_g\sqrt{T_1}\big(||\phi^1||_\Pi + ||x||_{L^2(0,T_1;V)}+1\big) \\
&\quad + \frac{C_1(2\alpha)^{-1/2}(2\alpha-1)^{-1/2}T_1^\alpha}{\Gamma(\alpha)} \\
&\quad \times \big(|\phi^0| + ||\phi^1||_{L^2(-h,0;V)} + L_f(r)(||x||_{L^2(0,T_1;V)}+1) + ||k||_{L^2(0,T_1;V)}\big) \\
&\quad + \frac{C_{1-\beta}L_gT_1^{(2\alpha+3\beta-2)/2}}{\big(\alpha-3(1-\beta)/2\big)\big(2\alpha+3\beta-2\big)^{1/2}\Gamma(\alpha)}(||\phi^1||_\Pi + ||x||_{L^2(0,T_1;V)}+1), \\
&= \hat{N}||x||_{L^2(0,T_1;V)} + \hat{N}_1,
\end{aligned}$$

where $\hat{N}$ is the constant of (3.7) and

$$\begin{aligned}
\hat{N}_1 &= C_1|\phi^0| + C_1C_{-\alpha}L_g(||\phi^1||+1) + C_{-\alpha}L_g\sqrt{T_1}\big(||\phi^1||_\Pi + 1\big) \\
&\quad + \frac{C_1(2\alpha)^{-1/2}(2\alpha-1)^{-1/2}T_1^\alpha}{\Gamma(\alpha)}\big(|\phi^0| + ||\phi^1||_{L^2(-h,0;V)}+1) + ||k||_{L^2(0,T_1;V)}\big) \\
&\quad + \frac{C_{1-\beta}L_gT_1^{(2\alpha+3\beta-2)/2}}{\big(\alpha-3(1-\beta)/2\big)\big(2\alpha+3\beta-2\big)^{1/2}\Gamma(\alpha)}(||\phi^1||_\Pi + 1).
\end{aligned}$$

Taking into account (3.7) there exists a constant C_2 such that

$$\begin{aligned}
(3.17)\qquad ||x||_{L^2(0,T_1;V)} &\le (1-\hat{N})^{-1}\hat{N}_1 \\
&\le C_2\big(1 + E(|\phi^0|^2) + ||\phi^1||_\Pi + ||k||_{M^2(0,T_1;V^*)}\big),
\end{aligned}$$

which obtain the inequality (3.5).

Now we will prove that $|x(T_1)| < \infty$ in order that the solution can be extended to the interval $[T_1, 2T_1]$. From (2.11) and Lemma 2.3 it follows that

$$\begin{aligned}
(3.18)\qquad |S(T_1)[\phi^0 + g(0,x_0)]| &\le c_1||S(\cdot)[\phi^0 + g(0,x_0)||_{\mathcal{W}_1(T_1)} \\
&\le c_1C_1|\phi^0 + g(0,\phi^1)| \\
&\le c_1C_1\big\{|\phi^0| + C_{-\alpha}L_g(||\phi^1||_\Pi + 1)\big\} := I,
\end{aligned}$$

and by using Assumption (G) we have

$$\begin{aligned}
(3.19)\qquad |g(T_1, x_{T_1})| &\le \big\|A^{-\beta}A^\beta g(t, x_{T_1})\big\|, \\
&\le C_{-\alpha}L_g\big(||x_{T_1}||_\Pi + 1\big)
\end{aligned}$$

$$\leq C_{-\alpha} L_g\big(||\phi^1||_\Pi + ||x||_{L^2(0,T_1;V)} + 1\big) := II.$$

By (3.10), we have

(3.20)

$$\begin{aligned}
&|(I_1x)(T_1)| \\
&\leq \frac{1}{\Gamma(\alpha)}\Big\|\int_0^{T_1}(T_1-s)^{\alpha-1}S(T_1-s) \\
&\quad\times\Big\{\int_{-h}^0 a_1(\tau)A_1x(s+\tau)d\tau + (Fx)(s) + k(s)\Big\}ds\Big\| \\
&\leq (2\alpha-1)^{-1/2}\Gamma(\alpha)^{-1}T_1{}^{(2\alpha-1)/2} \\
&\quad\times C_1(|\phi^0| + ||\phi^1||_{L^2(-h,0;V)} + L_f(r)(||x||_{L^2(0,T_1;V)}+1) + ||k||_{L^2(0,T_1;V^*)}) \\
&:= III.
\end{aligned}$$

From Lemma 2.4 and Assumption (G) we have

$$\begin{aligned}
&|(T_1-s)^{(\alpha-1)}AS(T_1-s)g(s,x_s)| \\
&\leq (T_1-s)^{(\alpha-1)}|A^{1-\beta}S(T_1-s)|_{\mathcal{L}(H)}|A^\beta(g(s,x_s)| \\
&\leq \frac{C_{1-\beta}}{(T_1-s)^{1-\alpha+(1-\beta)}}|A^\beta(g(s,x_s)| \\
&\leq \frac{C_{1-\beta}}{(T_1-s)^{2-\alpha-\beta}}L_g(||\phi^1||_\Pi + ||x||_{L^2(0,T_1;V)}+1),
\end{aligned}$$

and so

(3.21)

$$\begin{aligned}
|(I_2x)(T_1)| &= \Big|\frac{1}{\Gamma(\alpha)}\int_0^{T_1}(T_1-s)^{(\alpha-1)}AS(T_1-s)g(s,x_s)ds\Big| \\
&\leq C_{1-\beta}\big(\alpha+\beta-1\big)^{-1}T_1^{\alpha+\beta-1}L_g(||\phi^1||_\Pi + ||x||_{L^2(0,T_1;V)}+1) \\
&:= IV.
\end{aligned}$$

Thus, by (3.17)-(3.21) we have

$$\begin{aligned}
|x(T_1)| &= \big|S(T_1)[\phi^0 + g(0,x_0)] - g(T_1,x_{T_1}) + (I_1x)(T_1) + (I_2x)(T_1)\big| \\
&\leq I + II + III + IV < \infty.
\end{aligned}$$

Hence we can solve the equation in $[T_1, 2T_1]$ with the initial $(x(T_1), x_{T_1})$ and an analogous estimate to (3.4). Since the condition (3.6) is independent of initial values, the solution can be extended to the interval $[0, nT_1]$ for any natural number n, and so the proof is complete. □

Remark 3.4. Thanks for Lemma 2.3, we note that the solution of (3.1) under conditions of Theorem 3.1 with $(\phi^0,\phi^1) \in V \times L^2(0,T;D(A))$ and $k \in L^2(0,T;H)$ for $T > 0$ belongs to

$$\mathcal{W}_0(T) = L^2(0,T;D(A))) \cap W^{1,2}(0,T;H) \hookrightarrow C([0,T];V).$$

Moreover, there is a constant C_2 independent of the initial data (ϕ^0, ϕ^1) and the forcing term k such that

$$||x||_{L^2(-h,T;D(A))} \le C_2(1 + ||\phi^0|| + ||\phi^1||_{L^2(0,T;D(A))} + ||k||_{L^2(0,T;H)}).$$

Now, we obtain that the solution mapping is Lipschitz continuous in the following result, which is useful for the control problem and physical applications of the given equation.

Theorem 3.5. *Let Assumptions* (A), (F) *and* (G) *be satisfied. Assuming that the initial data* $(\phi^0, \phi^1) \in H \times \Pi$ *and the forcing term* $k \in M^2(0,T;V^*)$. *Then the solution* x *of the equation* (3.1) *belongs to* $x \in L^2(0,T;V)$ *and the mapping*

$$H \times \Pi \times L^2(0,T;V^*) \ni (\phi^0, \phi^1, k) \mapsto x \in L^2(0,T;V) \tag{3.22}$$

is Lipschitz continuous.

Proof. From Theorem 3.1, it follows that if $(\phi^0, \phi^1, k) \in L^2(\Omega, H) \times \Pi \times M^2(0,T;V^*)$, then x belongs to $M^2(0,T;V)$. Let $(\phi_i^0, \phi_i^1, k_i)$ and x^i be the solution of (3.1) with $(\phi_i^0, \phi_i^1, k_i)$ in place of (ϕ^0, ϕ^1, k) for $i = 1,\ 2$. Let $x_i (i = 1, 2) \in \Sigma_r$. Then it holds

$$\begin{aligned}
&x^1(t) - x^2(t)\\
&= S(t)[(\phi_1^0 - \phi_2^0) + (g(0, x_0^1) - g(0, x_0^2))]\\
&\quad - (g(t, x_t^1) - g(t, x_t^2)) + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{(\alpha-1)} AS(t-s)(g(s, x_s^1) - g(t, x_s^2))ds\\
&\quad + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{(\alpha-1)} S(t-s)\{ \int_{-h}^0 a_1(\tau) A_1 (x^1(s+\tau) - x^2(s+\tau)) d\tau ds\\
&\quad + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{(\alpha-1)} S(t-s)\{((Fx^1)(s) - (Fx^2)(s)) + (k_1(s) - k_2(s))\} ds.\\
&\quad + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{(\alpha-1)} S(t-s)(k_1(s) - k_2(s)) ds.
\end{aligned}$$

Hence, by applying the same argument as in the proof of Theorem 3.1, we have

$$||x_1 - x_2||_{L^2(0,T_1;V)} \le \hat{N} ||x_1 - x_2||_{L^2(0,T_1;V)} + \hat{N}_2,$$

where

$$\begin{aligned}
\hat{N}_2 = {} & C_1 |\phi_1^0 - \phi_2^0| + C_1 C_{-\alpha} L_g (||\phi_1^1 - \phi_2^1||_\Pi) + C_{-\alpha} L_g \sqrt{T_1} ||\phi_1^1 - \phi_2^1||_\Pi\\
& + \frac{C_1 (2\alpha)^{-1/2} (2\alpha - 1)^{-1/2} T_1^\alpha}{\Gamma(\alpha)}\\
& \times \big(|\phi_1^0 - \phi_2^0| + ||\phi_1^1 - \phi_2^1||_{L^2(-h,0;V)} + ||k_1 - k_2||_{L^2(0,T_1;V^*)} \big)\\
& + \frac{C_{1-\beta} L_g T_1^{(2\alpha+3\beta-2)/2}}{\big(\alpha - 3(1-\beta)/2\big)\big(2\alpha + 3\beta - 2\big)^{1/2} \Gamma(\alpha)} ||\phi_1^1 - \phi_2^1||_\Pi
\end{aligned}$$

which implies

$$||x||_{M^2(0,T_1;V)} \leq \hat{N}_2(1-\hat{N})^{-1}.$$

Therefore, it implies the inequality (3.22). □

Corollary 3.6. *For a forcing term $k \in L^2(0,T;V^*)$ let x_k be the solution of equation* (3.1). *Let us assume that the embedding $V \subset H$ is compact. Then the mapping $k \mapsto x_k$ is compact from $L^2(0,T;V^*)$ to $L^2(0,T;H)$.*

Proof. If $k \in L^2(0,T;V^*)$, then in view of Theorem 3.1

$$||x_k||_{\mathcal{W}_1(T)} \leq C_3(1+|g^0|+||g^1||_{L^2(-h,0;V)}+||k||_{L^2(0,T;V^*)}).$$

Hence if k is bounded in $L^2(0,T;V^*)$, then so is x_k in $L^2(0,T;V)) \cap W^{1,2}(0,T;V^*)$. Since V is compactly embedded in H by assumption, the embedding

$$L^2(0,T;V) \cap W^{1,2}(0,T;V^*) \hookrightarrow L^2(0,T;H)$$

is compact in view of Theorem 2 of J. P. Aubin [1]. □

4. Example

Let

$$H = L^2(0,\pi),\ V = H_0^1(0,\pi),\ V^* = H^{-1}(0,\pi).$$

Consider the following retarded neutral stochastic differential system in Hilbert space H:

$$\text{(4.1)} \quad \begin{cases} \frac{d^\alpha}{dt^\alpha}[x(t,y)+g(t,x_t(t,y))] = Ax(t,y) + \int_{-h}^0 a_1(s)A_1x(t+s,y)ds \\ \qquad\qquad + f^{'}(|x(t,y)|^2)x(t,y)+k(t,y), \quad (t,y) \in [0,T]\times[0,\pi], \\ x(0,y) = \phi^0(y), \quad x(s,y) = \phi^1(s,y), \quad (s,y) \in [-h,0)\times[0,\pi], \end{cases}$$

where $h > 0$, $a_1(\cdot)$ is Hölder continuous, and $A_1 \in \mathcal{L}(H)$. Let

$$a(u,v) = \int_0^\pi \frac{du(y)}{dy}\overline{\frac{dv(y)}{dy}}dy.$$

Then

$$A = \partial^2/\partial y^2 \quad \text{with} \quad D(A) = \{x \in H^2(0,\pi) : x(0) = x(\pi) = 0\}.$$

The eigenvalue and the eigenfunction of A are $\lambda_n = -n^2$ and $z_n(y) = (2/\pi)^{1/2} \sin ny$, respectively. Moreover,

(a1) $\{z_n : n \in N\}$ is an orthogonal basis of H and

$$S(t)x = \sum_{n=1}^\infty e^{n^2t}(x,z_n)z_n, \quad \forall x \in H,\ t > 0.$$

Moreover, there exists a constant M_0 such that $||S(t)||_{\mathcal{L}(H)} \leq M_0$.

(a2) Let $0 < \alpha < 1$. Then the fractional power $A^\alpha : D(A^\alpha) \subset H \to H$ of A is given by

$$A^\alpha x = \sum_{n=1}^{\infty} n^{2\alpha}(x, z_n)z_n, \ D(A^\alpha) := \{x : A^\alpha x \in H\}.$$

In particular,

$$A^{-1/2}x = \sum_{n=1}^{\infty} \frac{1}{n}(x, z_n)z_n, \ \text{and} \ ||A^{-1/2}|| = 1.$$

The nonlinear mapping f is a real valued function belong to $C^2([0,\infty))$ which satisfies the conditions

(f1) $f(0) = 0$, $f(r) \geq 0$ for $r > 0$,
(f2) $|f^{'}(r) \leq c(r+1)$ and $|qf''(r)| \leq c$ for $r \geq 0$ and $c > 0$.

If we present

$$F(t, x(t, y)) = f^{'}(|x(t, y)|^2)x(t, y),$$

then it is well known that F is a locally Lipschitz continuous mapping from the whole V into H by Sobolev's imbedding theorem (see [23, Theorem 6.1.6]). As an example of q in the above, we can choose $q(r) = \mu^2 r + \eta^2 r^2/2$ (μ and η is constants).

Define $g : [0, T] \times \Pi \to H$ as

$$g(t, x_t) = \sum_{n=1}^{\infty} \int_0^T e^{n^2 t}(\int_{-h}^0 a_2(s)x(t+s, y)ds)dt, \quad t > 0.$$

Then it can be checked that Assumption (G) is satisfied. Indeed, for $x \in \Pi$, we know

$$Ag(t, x_t) = (S(t) - I) \int_{-h}^0 a_2(s)x(t+s)ds,$$

where I is the identity operator form H to itself and, we assume that

$$|a_2(0)| \leq \rho, \quad |a_2(s) - a_2(\tau)| \leq \rho(s-\tau)^\kappa, \quad s, \ \tau \in [-h, 0]$$

for a constant $\kappa > 0$. Hence we have

$$\begin{aligned}|Ag(t, x_t)| &\leq (M_0+1)\big\{\big|\int_{-h}^0 (a_2(s) - a_2(0))x(t+s)d\tau\big| + \big|\int_{-h}^0 a_2(0)x(t+s)d\tau\big|\big\} \\ &\leq (M_0+1)\rho\big\{(2\kappa+1)^{-1}h^{2\kappa+1} + h\big\}||x_t||_\Pi.\end{aligned}$$

It is immediately seen that Assumption (G) has been satisfied. Thus, all the conditions stated in Theorem 3.1 have been satisfied for the equation (4.1), and so there exists a solution of (4.1) belongs to $\mathcal{W}_1(T) = L^2(0, T; V)) \cap W^{1,2}(0, T; V^*) \hookrightarrow C([0, T]; H)$.

References

[1] J. P. Aubin, *Un thèoréme de compacité*, C. R. Acad. Sci. **256** (1963), 5042–5044.

[2] B. Baeumer, S. Kurita, and M. M. Meerschaert, *Inhomogeneous fractional diffusion equations*, Fract. Calc. Appl. Anal. **8** (2005), no. 4, 371–386.

[3] G. Di Blasio, K. Kunisch, and E. Sinestrari, *L^2-regularity for parabolic partial integrodifferential equations with delay in the highest-order derivatives*, J. Math. Anal. Appl. **102** (1984), no. 1, 38–57.

[4] X. Fu, *Controllability of neutral functional differential systems in abstract space*, Appl. Math. Comput. **141** (2003), no. 2-3, 281–296.

[5] L. Gaul, P. Klein, and S. Kempfle, *Damping description involving fractional operators*, Mech. Syst. Signal Process. **5** (1991), 81–88.

[6] W. G. Glockle and T. F. Nonnenmacher, *A fractional calculus approach of self-similar protein dynamics*, Biophys. J. **68** (1995), 46–53.

[7] E. Hernández and M. Mckibben, *On state-dependent delay partial neutral functional differential equations*, Appl. Math. Comput. **186** (2007), no. 1, 294–301.

[8] E. Hernández, M. Mckibben, and H. Henrŕquez, *Existence results for partial neutral functional differential equations with state-dependent delay*, Math. Comput. Modelling **49** (2009), no. 5-6, 1260–1267.

[9] R. Hilfer, *Applications of Fractional Calculus in Physics*, World Scientific, Singapore, 2000.

[10] O. K. Jaradat, A. Al-Omari, and S. Momani, *Existence of the mild solution for fractional semilinear initial value problems*, Nonlinear Anal. **69** (2008), no. 9, 3153–3159.

[11] J. M. Jeong, *Stabilizability of retarded functional differential equation in Hilbert space*, Osaka J. Math. **28** (1991), no. 2, 347–365.

[12] J. M. Jeong and H. G. Kim, *Controllability for semilinear functional integrodifferential equations*, Bull. Korean Math. Soc. **46** (2009), no. 3, 463–475.

[13] J. M. Jeong, Y. C. Kwun, and J. Y. Park, *Approximate controllability for semilinear retarded functional differential equations*, J. Dynam. Control Systems **5** (1999), no. 3, 329–346.

[14] A. A. Kilbas, H. M. Srivastava, and J. Juan Trujillo, *Theory and Applications of Fractional Differential Equations*, in: North-Holland Mathematics Studies, vol. 204, Elsevier Science B.V, Amsterdam, 2006.

[15] M. A. Krannoselski, *Topological Methods in the Theory of Nonlinear Integral Equations*, Pergamon Press, New York, 1964.

[16] V. Lakshmikantham, S. Leela, and J. V. Devi, *Theory of Fractional Dynamic Systems*, Cambridge Scientific Publishers, 2009.

[17] F. Mainardi, *Fractional calculus: Some basic problems in continuum and statistical mechanics*, Fractals and fractional calculus in continuum mechanics (Udine, 1996), 291–348, CISM Courses and Lectures, 378, Springer, Vienna, 1997.

[18] K. S. Miller and B. Ross, *An Introduction to the Fractional Calculus and Differential Equations*, John Wiley, New York, 1993.

[19] M. Muslim, *Existence and approximation of solutions to fractional differential equations*, Math. Comput. Modelling **49** (2009), no. 5-6, 1164–1172.

[20] I. Podlubny, *Fractional Differential Equations*, Academic Press, San Diego, 1999.

[21] N. Sukavanam and S. Kumar, *Approximate controllability of fractional order semilinear delay systems*, J. Optim. Theory Appl. **151** (2011), no. 2, 373–384.

[22] N. Sukavanam and N. K. Tomar, *Approximate controllability of semilinear delay control system*, Nonlinear Funct. Anal. Appl. **12** (2007), no. 1, 53–59.

[23] H. Tanabe, *Equations of Evolution*, Pitman-London, 1979.

[24] H. Triebel, *Interpolation Theory, Function Spaces, Differential Operators*, North-Holland, 1978.

[25] L. Wang, *Approximate controllability for integrodifferential equations and multiple delays*, J. Optim. Theory Appl. **143** (2009), no. 1, 185–206.

[26] Y. Zhou and F. Jiao, *Existence of mild solutions for fractional neutral evolution equations*, Comput. Math. Appl. **59** (2010), no. 3, 1063–1077.

Seong Ho Cho
Department of Applied Mathematics
Pukyong National University
Busan 608-737, Korea
E-mail address: dopp500@nate.com

Jin-Mun Jeong
Department of Applied Mathematics
Pukyong National University
Busan 608-737, Korea
E-mail address: jmjeong@pknu.ac.kr

Yong Han Kang
Institute of Liberal Education
Catholic University of Daegu
Kyeongsan 680-749, Korea
E-mail address: yonghann@cu.ac.kr