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GENERALIZED GOLDEN SHAPED HYPERSURFACES IN LORENTZ SPACE FORMS

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Abstract. In this paper, we define the generalized golden shaped hypersurfaces in Lorentz space forms. Based on the classification of proper semi-Riemannian hypersurfaces in semi-Riemannian real space forms, we obtain the whole families of the generalized golden shaped hypersurfaces in Lorentz space forms.

1. Introduction

The Golden ratio has many applications in many parts of mathematics, for example, natural sciences, music, art, philosophies and computational science [8]. In the past few years, the Golden ratio has played a more and more significant role in modern physical research and atomic physics [4]. The Golden ratio also has interesting properties in topology of four-manifolds, in conformal field theory, in mathematical probability theory, in Cantorian spacetime [7] and in differential geometry.

The notion of *golden structure* on a manifold M was introduced in [2, 5] as a $(1,1)$-tensor field on M which satisfies the equation: $J^2 = J + I$, where I is the usual Kronecker tensor field of M. It attracts many authors' attentions to focus on a class of well-known objects namely hypersurfaces in real space forms. Recently, the golden shaped hypersurfaces in real space forms were defined and the whole families of the golden shaped hypersurfaces were obtained in [3]. The golden shaped hypersurfaces in Lorentz space forms were defined and the whole families of the golden shaped hypersurfaces were obtained in [9]. In this paper, we define the generalized golden shaped hypersurfaces in Lorentz space forms. Based on the classification of proper semi-Riemannian hypersurfaces in semi-Riemannian real space forms, we obtain the whole families of the generalized golden shaped hypersurfaces in Lorentz space forms.

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2. Preliminaries

Let $\mathbb{R}_1^n$ be an n-dimensional real vector space together with an inner product given by

$$\langle x, x\rangle = -x_1^2 + \sum_{i=2}^{n+1} x_i^2,$$

where $x = (x_1, \ldots, x_n)$ is the natural coordinate of $\mathbb{R}_1^n$.

$$S_1^n(c) = \{(x_1, \ldots, x_{n+1}) \in \mathbb{R}_1^{n+1} \mid -x_1^2 + \sum_{i=2}^{n+1} x_i^2 = \frac{1}{c}\}\ (c > 0),$$

$$H_1^n(c) = \{(x_1, \ldots, x_{n+1}) \in \mathbb{R}_1^{n+1} \mid -\sum_{i=1}^{2} x_1^2 + \sum_{i=3}^{n+1} x_i^2 = \frac{1}{c}\}\ (c < 0).$$

These spaces are complete and of constant curvature c. In general relativity, the Lorentz manifolds $\mathbb{R}_1^n$, $S_1^n(c)$, $H_1^n(c)$ are respectably known as the Minkowski, de Sitter and anti-de Sitter space, which is called Lorentz space form and is denoted by $N_1^n(c)$.

Let M be a hypersurface in Lorentz space form $N_1^{n+1}(c)$. For a certain normal vector field N, we put $\epsilon = \langle N, N\rangle$. Let $A = A_N$ be the associated shape operator and $\mu_1, \ldots, \mu_n$ be the principal curvatures of M. If A can be expressed by a real diagonal matrix with respect to an orthonormal frame at each point of the semi-Riemannian manifold M, then A is said to be proper. M is said to be proper if A is proper for a unit normal vector N at each point of M.

Definition 1. A hypersurface M in Lorentz space form $N_1^{n+1}(c)$ is called *golden shaped hypersurface* if A is a golden structure, i.e., $A^2 = A + I$.

In this paper, we give the definition of generalized golden-shaped hypersurfaces:

Definition 2. A proper hypersurface M in Lorentz space form $N_1^{n+1}(c)$ is called *generalized golden shaped hypersurface* if A is a generalized golden structure, i.e., $A^2 = aA + bI$, where a, b are constants satisfying $a^2 + 4b \geq 0$.

3. The classification of golden shaped hypersurfaces

Let M be a generalized golden shaped hypersurface. Then the principal curvatures of M, $\mu_1, \ldots, \mu_n$, which are the eigenvalues of A, are $\lambda_1 = \frac{a+\sqrt{a^2+4b}}{2}$ and $\lambda_2 = \frac{a-\sqrt{a^2+4b}}{2}$ if $a^2 + 4b > 0$. Especially, the principal curvatures of M, which are the eigenvalues of A, are $\lambda_1 = \lambda_2 = \frac{a}{2}$ if $a^2 + 4b = 0$. According to [6], the manifold M is an *isoparametric hypersurface* and based on [1], we get the following theorems.

Theorem 1. *The only generalized golden shaped hypersurfaces in Minkowski space* $\mathbb{R}^{n+1}_1$ *are as follows*:

Case 1: $a^2 + 4b = 0$:

(1) *If* $a \neq 0$,

(i) $S^n_1(\frac{a^2}{4}) = \{x \in \mathbb{R}^{n+1}_1 \mid -x_1^2 + \sum_{i=2}^{n+1} x_i^2 = \frac{4}{a^2}\}$ *with* $A = \frac{a}{2}I$ *and* $\varepsilon = 1$.

(ii) $H^n(-\frac{a^2}{4}) = \{x \in \mathbb{R}^{n+1}_1 \mid -x_1^2 + \sum_{i=2}^{n+1} x_i^2 = -\frac{4}{a^2}\}$ *with* $A = \frac{a}{2}I$ *and* $\varepsilon = -1$.

(2) *If* $a = 0$,

(i) $\mathbb{R}^n_1 = \{x \in \mathbb{R}^{n+1}_1 \mid x^{n+1} = 0\}$ *with* $A = O$ *and* $\varepsilon = 1$.

(ii) $\mathbb{R}^n = \{x \in \mathbb{R}^{n+1}_1 \mid x^1 = 0\}$ *with* $A = O$ *and* $\varepsilon = -1$.

Case 2: $a^2 + 4b > 0$:

(1) *If* $b \neq 0$,

(i) $S^n_1(\lambda_i^2) = \{x \in \mathbb{R}^{n+1}_1 \mid -x_1^2 + \sum_{i=2}^{n+1} x_i^2 = \frac{1}{\lambda_i^2}\}$ *with* $A = \lambda_i I$, $i = 1, 2$ *and* $\varepsilon = 1$.

(ii) $H^n(-\lambda_i^2) = \{x \in \mathbb{R}^{n+1}_1 \mid -x_1^2 + \sum_{i=2}^{n+1} x_i^2 = -\frac{1}{\lambda_i^2}\}$ *with* $A = \lambda_i I$, $i = 1, 2$ *and* $\varepsilon = -1$.

(2) *If* $b = 0$,

(i) $\mathbb{R}^n_1 = \{x \in \mathbb{R}^{n+1}_1 \mid x^{n+1} = 0\}$ *with* $A = O$ *and* $\varepsilon = 1$.

(ii) $\mathbb{R}^n = \{x \in \mathbb{R}^{n+1}_1 \mid x^1 = 0\}$ *with* $A = O$ *and* $\varepsilon = -1$.

(iii) $S^n_1(a^2) = \{x \in \mathbb{R}^{n+1}_1 \mid -x_1^2 + \sum_{i=2}^{n+1} x_i^2 = \frac{1}{a^2}\}$ *with* $A = aI$ *and* $\varepsilon = 1$.

(iv) $H^n(-a^2) = \{x \in \mathbb{R}^{n+1}_1 \mid -x_1^2 + \sum_{i=2}^{n+1} x_i^2 = -\frac{1}{a^2}\}$ *with* $A = aI$ *and* $\varepsilon = -1$.

(v) $\mathbb{R}^r \times S^{n-r}_1(a^2) = \{x \in \mathbb{R}^{n+1}_1 \mid -x_1^2 + \sum_{i=r+2}^{n+1} x_i^2 = \frac{1}{a^2}\}$ *with* $A = O_r \oplus aI_{n-r}$ *and* $\varepsilon = 1$.

(vi) $\mathbb{R}^r \times H^{n-r}(-a^2) = \{x \in \mathbb{R}^{n+1}_1 \mid -x_1^2 + \sum_{i=r+2}^{n+1} x_i^2 = -\frac{1}{a^2}\}$ *with* $A = O_r \oplus aI_{n-r}$ *and* $\varepsilon = -1$.

Proof. Case 1: Suppose $a^2 + 4b = 0$, $\lambda_1 = \lambda_2 = \frac{a}{2}$, we distinguish the following two cases:

(1) If $a \neq 0$, $\lambda_1 = \lambda_2 \neq 0$, we get $\mu_1 = \mu_2 = \cdots = \mu_n = \frac{a}{2}$ and $c = \varepsilon\frac{a^2}{4} \neq 0$ for $\varepsilon = \pm 1$, then M is totally umbilical and is as follows:

(i) $M = S^n_1(\frac{a^2}{4}) = \{x \in \mathbb{R}^{n+1}_1 \mid -x_1^2 + \sum_{i=2}^{n+1} x_i^2 = \frac{4}{a^2}\}$ with $A = \frac{a}{2}I$ and $\varepsilon = 1$.

(ii) $M = H^n(-\frac{a^2}{4}) = \{x \in \mathbb{R}^{n+1}_1 \mid -x_1^2 + \sum_{i=2}^{n+1} x_i^2 = -\frac{4}{a^2}\}$ with $A = \frac{a}{2}I$ and $\varepsilon = -1$.

(2) If $a = 0$, $\lambda_1 = \lambda_2 = 0$, we get $\mu_1 = \mu_2 = \cdots = \mu_n = 0$ and $c = 0$, then M is as follows:

(i) $M = \mathbb{R}^n_1 = \{x \in \mathbb{R}^{n+1}_1 \mid x^{n+1} = 0\}$ with $A = O$ and $\varepsilon = 1$.

(ii) $M = \mathbb{R}^n = \{x \in \mathbb{R}^{n+1}_1 \mid x^1 = 0\}$ with $A = O$ and $\varepsilon = -1$.

Case 2: Suppose $a^2 + 4b > 0$:

(1) If $b \neq 0$, $\mu_1 = \mu_2 = \cdots = \mu_n = \lambda_i \neq 0$, we get $c = \varepsilon\lambda_i^2 \neq 0$ for $\varepsilon = \pm 1$ and $i = \pm 1$, then M is as follows:

(i) $M = S_1^n(\lambda_i^2) = \{x \in \mathbb{R}_1^{n+1} \mid -x_1^2 + \sum_{i=2}^{n+1} x_i^2 = \frac{1}{\lambda_i^2}\}$ with $A = \lambda_i I$, $i = 1, 2$ and $\varepsilon = 1$.

(ii) $M = H^n(-\lambda_i^2) = \{x \in \mathbb{R}_1^{n+1} \mid -x_1^2 + \sum_{i=2}^{n+1} x_i^2 = -\frac{1}{\lambda_i^2}\}$ with $A = \lambda_i I$, $i = 1, 2$ and $\varepsilon = -1$.

(2) If $b = 0$, then $a \neq 0$ and $\lambda_1 = 0, \lambda_2 = a$ or $\lambda_1 = a, \lambda_2 = 0$.

Suppose $\mu_1 = \mu_2 = \cdots = \mu_n = 0$, we get $c = 0$. Then we have

(i) $M = \mathbb{R}_1^n = \{x \mid x^{n+1} = 0\}$ with $A = O$ and $\varepsilon = 1$.

(ii) $M = \mathbb{R}^n = \{x \mid x^1 = 0\}$ with $A = O$ and $\varepsilon = -1$.

Suppose $\mu_1 = \mu_2 = \cdots = \mu_n = a \neq 0$, we get $c = \varepsilon a^2 \neq 0$. Then we have

(iii) $M = S_1^n(a^2) = \{x \in \mathbb{R}_1^{n+1} \mid -x_1^2 + \sum_{i=2}^{n+1} x_i^2 = \frac{1}{a^2}\}$ with $A = aI$ and $\varepsilon = 1$.

(iv) $M = H^n(-a^2) = \{x \in \mathbb{R}_1^{n+1} \mid -x_1^2 + \sum_{i=2}^{n+1} x_i^2 = -\frac{1}{a^2}\}$ with $A = aI$ and $\varepsilon = -1$.

Suppose for $1 < r < n$, $\mu_1 = \cdots = \mu_r = a \neq 0$ and $\mu_{r+1} = \cdots = \mu_n = 0$, then $c_1 = \varepsilon a^2 \neq 0$ and $c_2 = 0$. Then we have

(v) $\mathbb{R}^r \times S_1^{n-r}(a^2) = \{x \in \mathbb{R}_1^{n+1} \mid -x_1^2 + \sum_{i=r+2}^{n+1} x_i^2 = \frac{1}{a^2}\}$ with $A = O_r \oplus aI_{n-r}$ and $\varepsilon = 1$.

(vi) $\mathbb{R}^r \times H^{n-r}(-a^2) = \{x \in \mathbb{R}_1^{n+1} \mid -x_1^2 + \sum_{i=r+2}^{n+1} x_i^2 = -\frac{1}{a^2}\}$ with $A = O_r \oplus aI_{n-r}$ and $\varepsilon = -1$. □

Theorem 2. *The only generalized golden shaped hypersurfaces in de Sitter space* S_1^{n+1} *are as follows*:

Case 1: If $a^2 + 4b = 0$:

(i) $S_1^n(1 + \frac{a^2}{4}) = \{x \in S_1^{n+1} \subset \mathbb{R}_1^{n+2} \mid x_{n+2} = \frac{|a|}{\sqrt{a^2+4}}\}$ *with* $A = \frac{a}{2}I$ *and* $\varepsilon = 1$.

(ii) *If* $a = \pm 2$, $\mathbb{R}^n = \{x \in S_1^{n+1} \subset \mathbb{R}_1^{n+2} \mid x_1 = x_{n+2} + t_0\}(t_0 > 0)$ *with* $A = \pm I$ *and* $\varepsilon = -1$.

(iii) *If* $|a| > 2$, $H^n(1 - \frac{a^2}{4}) = \{x \in S_1^{n+1} \subset \mathbb{R}_1^{n+2} \mid x_{n+2} = \frac{|a|}{\sqrt{a^2-4}}\}$ *with* $A = \frac{a}{2}I$ *and* $\varepsilon = -1$.

(iv) *If* $-2 < a < 2$, $S^n(1 - \frac{a^2}{4}) = \{x \in S_1^{n+1} \subset \mathbb{R}_1^{n+2} \mid x_1 = \frac{|a|}{\sqrt{4-a^2}}\}$ *with* $A = \frac{a}{2}I$ *and* $\varepsilon = -1$.

Case 2: If $a^2 + 4b > 0$:

(1) *Suppose* $\mu_1 = \mu_2 = \cdots = \mu_n = \lambda_1$, *there are the following four cases*:

(i) $S_1^n(1 + \lambda_1^2) = \{x \in S_1^{n+1} \subset \mathbb{R}_1^{n+2} \mid x_{n+2} = \frac{|\lambda_1|}{\sqrt{1+\lambda_1^2}}\}$ *with* $A = \lambda_1 I$ *and* $\varepsilon = 1$.

(ii) *If* $(a, b) \in \{(a, b) \mid a - b + 1 = 0, a < -2\} \cup \{(a, b) \mid a + b - 1 = 0, a < 2\}$, $\mathbb{R}^n = \{x \in S_1^{n+1} \subset \mathbb{R}_1^{n+2} \mid x_1 = x_{n+2} + t_0\}(t_0 > 0)$ *with* $A = \pm I$ *and* $\varepsilon = -1$.

(iii) *If* $(a,b) \in \{(a,b)\,|\,a+b-1<0, -2\le a<2\} \cup \{(a,b)\,|\,a-b+1<0, a+b-1<0, a<-2\}$,
$S^n(1-\lambda_1^2) = \{x \in S_1^{n+1} \subset \mathbb{R}_1^{n+2} \mid x_1 = \frac{|\lambda_1|}{\sqrt{1-\lambda_1^2}}\}$ *with* $A=\lambda_1 I$ *and* $\varepsilon=-1$.

(iv) *If* $(a,b) \in \{(a,b)\,|\,a+b-1>0, a<2\} \cup \{(a,b)\,|\,a\ge 2\} \cup \{(a,b)\,|\,a<-2, a-b+1>0\}$,
$H^n(1-\lambda_1^2) = \{x \in S_1^{n+1} \subset \mathbb{R}_1^{n+2} \mid x_{n+2} = \frac{|\lambda_1|}{\sqrt{\lambda_1^2-1}}\}$ *with* $A=\lambda_1 I$ *and* $\varepsilon=-1$.

(2) *Suppose* $\mu_1=\mu_2=\cdots=\mu_n=\lambda_2$,

(i) $S_1^n(1+\lambda_2^2) = \{x \in S_1^{n+1} \subset \mathbb{R}_1^{n+2} \mid x_{n+2} = \frac{|\lambda_2|}{\sqrt{1+\lambda_2^2}}\}$ *with* $A=\lambda_2 I$ *and* $\varepsilon=1$.

(ii) *If* $(a,b) \in \{(a,b)\,|\,a-b+1=0, a>-2\} \cup \{(a,b)\,|\,a+b-1=0, a>2\}$,
$\mathbb{R}^n = \{x \in S_1^{n+1} \subset \mathbb{R}_1^{n+2} \mid x_1 = x_{n+2}+t_0\}(t_0>0)$ *with* $A=\pm I$ *and* $\varepsilon=-1$.

(iii) *If* $(a,b) \in \{(a,b)\,|\,a-b+1>0, -2<a\le 2\} \cup \{(a,b)\,|\,a-b+1>0, a+b-1>0, a>2\}$,
$S^n(1-\lambda_2^2) = \{x \in S_1^{n+1} \subset \mathbb{R}_1^{n+2} \mid x_1 = \frac{|\lambda_2|}{\sqrt{1-\lambda_2^2}}\}$ *with* $A=\lambda_2 I$ *and* $\varepsilon=-1$.

(iv) *If* $(a,b) \in \{(a,b)\,|\,a-b+1<0, a>-2\} \cup \{(a,b)\,|\,a\le -2\} \cup \{(a,b)\,|\,a>2, a+b-1<0\}$,
$H^n(1-\lambda_2^2) = \{x \in S_1^{n+1} \subset \mathbb{R}_1^{n+2} \mid x_{n+2} = \frac{|\lambda_2|}{\sqrt{\lambda_2^2-1}}\}$ *with* $A=\lambda_2 I$ *and* $\varepsilon=-1$.

(3) *Suppose* $\mu_1=\mu_2=\cdots=\mu_r=\lambda_1$, *and* $\mu_{r+1}=\cdots=\mu_n=\lambda_2$, *for* $1\le r\le n$:

(i) *If* $b=-1$ *and* $a>2$,
$S^r(1-\lambda_2^2)\times H^{n-r}(1-\lambda_1^2) = \{x \in S_1^{n+1} \subset \mathbb{R}_1^{n+2} \mid \sum_{i=2}^{2+r} x_i^2 = \frac{1}{1-\lambda_2^2}, -x_1^2+\sum_{i=r+3}^{n+2} x_i^2 = \frac{1}{1-\lambda_1^2}\}$ *with* $A=\lambda_2 I_r \oplus \lambda_1 I_{n-r}$ *and* $\varepsilon=-1$.

(ii) *If* $b=-1$ *and* $a<-2$,
$S^r(1-\lambda_1^2)\times H^{n-r}(1-\lambda_2^2) = \{x \in S_1^{n+1} \subset \mathbb{R}_1^{n+2} \mid \sum_{i=2}^{2+r} x_i^2 = \frac{1}{1-\lambda_1^2}, -x_1^2+\sum_{i=r+3}^{n+2} x_i^2 = \frac{1}{1-\lambda_2^2}\}$ *with* $A=\lambda_1 I_r \oplus \lambda_2 I_{n-r}$ *and* $\varepsilon=-1$.

(iii) *If* $b=1$,
$S^r(1+\lambda_i^2)\times S_1^{n-r}(1+\lambda_j^2) = \{x \in S_1^{n+1} \subset \mathbb{R}_1^{n+2} \mid \sum_{i=2}^{2+r} x_i^2 = \frac{1}{1+\lambda_i^2}, -x_1^2+\sum_{i=r+3}^{n+2} x_i^2 = \frac{1}{1+\lambda_j^2}\}$ *with* $A=\lambda_i I_r \oplus \lambda_j I_{n-r}$, $i,j=1,2$, $i\ne j$ *and* $\varepsilon=-1$.

Proof. Case 1: If $a^2+4b=0$, $\lambda_1=\lambda_2=\frac{a}{2}$, we distinguish the following four cases:

(i) If $\varepsilon=1$, we get $c=1+\frac{a^2}{4}$. Then $M=S_1^n(1+\frac{a^2}{4}) = \{x \in S_1^{n+1} \subset \mathbb{R}_1^{n+2} \mid x_{n+2} = \frac{|a|}{\sqrt{a^2+4}}\}$ with $A=\frac{a}{2}I$.

(ii) If $\varepsilon = -1$ and $a = \pm 2$, we get $\mu_1 = \mu_2 = \cdots = \mu_n = \pm 1$ and $c = 0$. Then $M = \mathbb{R}^n = \{x \in S_1^{n+1} \subset \mathbb{R}_1^{n+2} \mid x_1 = x_{n+2} + t_0\}(t_0 > 0)$ with $A = \pm I$.

(iii) If $\varepsilon = -1$ and $|a| > 2$, we get $c = 1 - \frac{a^2}{4} < 0$. Then we have $M = H^n(1 - \frac{a^2}{4}) = \{x \in S_1^{n+1} \subset \mathbb{R}_1^{n+2} \mid x_{n+2} = \frac{|a|}{\sqrt{a^2-4}}\}$ with $A = \frac{a}{2}I$.

(iv) If $\varepsilon = -1$ and $-2 < a < 2$, we get $c = 1 - \frac{a^2}{4} > 0$. Then we have $M = S^n(1 - \frac{a^2}{4}) = \{x \in S_1^{n+1} \subset \mathbb{R}_1^{n+2} \mid x_1 = \frac{|a|}{\sqrt{4-a^2}}\}$ with $A = \frac{a}{2}I$.

Case 2: If $a^2 + 4b > 0$:

(1) Suppose $\mu_1 = \mu_2 = \cdots = \mu_n = \lambda_1$, there are the following four cases:

(i) If $\varepsilon = 1$, we get $c = 1 + \lambda_1^2 \geq 1$. Then we have $M = S_1^n(1 + \lambda_1^2) = \{x \in S_1^{n+1} \subset \mathbb{R}_1^{n+2} \mid x_{n+2} = \frac{|\lambda_1|}{\sqrt{1+\lambda_1^2}}\}$ with $A = \lambda_1 I$.

(ii) If $\varepsilon = -1$ and $(a, b) \in \{(a, b) \mid a - b + 1 = 0, a < -2\} \cup \{(a, b) \mid a + b - 1 = 0, a < 2\}$, we get $c = 1 - \lambda_1^2 = 0$. Then we have $M = \mathbb{R}^n = \{x \in S_1^{n+1} \subset \mathbb{R}_1^{n+2} \mid x_1 = x_{n+2} + t_0\}(t_0 > 0)$ with $A = \pm I$.

(iii) If $\varepsilon = -1$ and $(a, b) \in \{(a, b) \mid a + b - 1 < 0, -2 \leq a < 2\} \cup \{(a, b) \mid a - b + 1 < 0, a + b - 1 < 0, a < -2\}$, we get $0 < c = 1 - \lambda_1^2 \leq 1$. Then we have $M = S^n(1 - \lambda_1^2) = \{x \in S_1^{n+1} \subset \mathbb{R}_1^{n+2} \mid x_1 = \frac{|\lambda_1|}{\sqrt{1-\lambda_1^2}}\}$ with $A = \lambda_1 I$.

(iv) If $\varepsilon = -1$ and $(a, b) \in \{(a, b) \mid a + b - 1 > 0, a < 2\} \cup \{(a, b) \mid a \geq 2\} \cup \{(a, b) \mid a < -2, a - b + 1 > 0\}$, we get $c = 1 - \lambda_1^2 < 0$. Then we have $M = H^n(1 - \lambda_1^2) = \{x \in S_1^{n+1} \subset \mathbb{R}_1^{n+2} \mid x_{n+2} = \frac{|\lambda_1|}{\sqrt{\lambda_1^2-1}}\}$ with $A = \lambda_1 I$.

(2) Suppose $\mu_1 = \mu_2 = \cdots = \mu_n = \lambda_2$,

(i) If $\varepsilon = 1$, we get $c = 1 + \lambda_2^2 \geq 1$. Then we have $M = S_1^n(1 + \lambda_2^2) = \{x \in S_1^{n+1} \subset \mathbb{R}_1^{n+2} \mid x_{n+2} = \frac{|\lambda_2|}{\sqrt{1+\lambda_2^2}}\}$ with $A = \lambda_2 I$.

(ii) If $\varepsilon = -1$ and $(a, b) \in \{(a, b) \mid a - b + 1 = 0, a > -2\} \cup \{(a, b) \mid a + b - 1 = 0, a > 2\}$, we get $c = 1 - \lambda_2^2 = 0$. Then we have $M = \mathbb{R}^n = \{x \in S_1^{n+1} \subset \mathbb{R}_1^{n+2} \mid x_1 = x_{n+2} + t_0\}(t_0 > 0)$ with $A = \pm I$.

(iii) If $\varepsilon = -1$ and $(a, b) \in \{(a, b) \mid a - b + 1 > 0, -2 < a \leq 2\} \cup \{(a, b) \mid a - b + 1 > 0, a + b - 1 > 0, a > 2\}$, we get $0 < c = 1 - \lambda_2^2 \leq 1$. Then we have $M = S^n(1 - \lambda_2^2) = \{x \in S_1^{n+1} \subset \mathbb{R}_1^{n+2} \mid x_1 = \frac{|\lambda_2|}{\sqrt{1-\lambda_2^2}}\}$ with $A = \lambda_2 I$.

(iv) If $\varepsilon = -1$ and $(a, b) \in \{(a, b) \mid a - b + 1 < 0, a > -2\} \cup \{(a, b) \mid a \leq -2\} \cup \{(a, b) \mid a > 2, a + b - 1 < 0\}$, we get $c = 1 - \lambda_2^2 < 0$. Then we have $M = H^n(1 - \lambda_2^2) = \{x \in S_1^{n+1} \subset \mathbb{R}_1^{n+2} \mid x_{n+2} = \frac{|\lambda_2|}{\sqrt{\lambda_2^2-1}}\}$ with $A = \lambda_2 I$.

(3) Suppose $\mu_1 = \mu_2 = \cdots = \mu_r = \lambda_1$, and $\mu_{r+1} = \cdots = \mu_n = \lambda_2$, for $1 \leq r \leq n$:

(i) If $\varepsilon = -1$, $b = -1$ and $a > 2$, we get $\lambda_1\lambda_2 = 1$, $c_1 = 1 - \lambda_1^2 > 0$ and $c_2 = 1 - \lambda_2^2 < 0$. Then we have $M = S^r(1 - \lambda_2^2) \times H^{n-r}(1 - \lambda_1^2) = \{x \in S_1^{n+1} \subset \mathbb{R}_1^{n+2} \mid \sum_{i=2}^{2+r} x_i^2 = \frac{1}{1-\lambda_2^2}, -x_1^2 + \sum_{i=r+3}^{n+2} x_i^2 = \frac{1}{1-\lambda_1^2}\}$ with $A = \lambda_2 I_r \oplus \lambda_1 I_{n-r}$.

(ii) If $\varepsilon = -1$, $b = -1$ and $a < -2$, we get $\lambda_1\lambda_2 = 1$, $c_1 = 1 - \lambda_1^2 < 0$ and $c_2 = 1 - \lambda_2^2 > 0$. Then we have $M = S^r(1-\lambda_1^2) \times H^{n-r}(1-\lambda_2^2) = \{x \in S_1^{n+1} \subset \mathbb{R}_1^{n+2} \mid \sum_{i=2}^{2+r} x_i^2 = \frac{1}{1-\lambda_1^2}, -x_1^2 + \sum_{i=r+3}^{n+2} x_i^2 = \frac{1}{1-\lambda_2^2}\}$ with $A = \lambda_1 I_r \oplus \lambda_2 I_{n-r}$.

(iii) If $\varepsilon = 1$ and $b = 1$, we get $\lambda_1\lambda_2 = 1$, $c_i = 1 + \lambda_i^2 > 0$ for $i = 1, 2$. Then we have $M = S^r(1+\lambda_i^2) \times S_1^{n-r}(1+\lambda_j^2) = \{x \in S_1^{n+1} \subset \mathbb{R}_1^{n+2} \mid \sum_{i=2}^{2+r} x_i^2 = \frac{1}{1+\lambda_i^2}, -x_1^2 + \sum_{i=r+3}^{n+2} x_i^2 = \frac{1}{1+\lambda_j^2}\}$ with $A = \lambda_i I_r \oplus \lambda_j I_{n-r}$, $i, j = 1, 2$ and $i \neq j$. □

Theorem 3. *The only generalized golden-shaped hypersurfaces in anti-de Sitter space $H_1^{n+1}(-1)$ are as follows*:

Case 1: *If $a^2 + 4b = 0$*:

(i) $H^n(-1 - \frac{a^2}{4}) = \{x \in H_1^{n+1}(-1) \subset \mathbb{R}_2^{n+2} \mid x_1 = \frac{|a|}{\sqrt{a^2+4}}\}$ *with $A = \frac{a}{2}$ and $\varepsilon = -1$.*

(ii) *If $a = \pm 2$, $\mathbb{R}_1^n = \{x \in H_1^{n+1}(-1) \subset \mathbb{R}_2^{n+2} \mid x_1 = x_{n+2} + t_0\}(t_0 > 0)$ with $A = \pm I$ and $\varepsilon = 1$.*

(iii) *If $a \in \{|a| > 2\}$, $S_1^n(\frac{a^2}{4} - 1) = \{x \in H_1^{n+1}(-1) \subset \mathbb{R}_2^{n+2} \mid x_1 = \frac{|a|}{\sqrt{a^2-4}}\}$ with $A = \frac{a}{2}I$ and $\varepsilon = 1$.*

(iv) *If $a \in \{-2 < a < 2\}$, $H_1^n(\frac{a^2}{4} - 1) = \{x \in S_1^{n+1} \subset \mathbb{R}_1^{n+2} \mid x_{n+2} = \frac{|a|}{\sqrt{4-a^2}}\}$ with $A = \frac{a}{2}I$ and $\varepsilon = 1$.*

Case 2: *If $a^2 + 4b > 0$*:

(i) *Suppose $\mu_1 = \mu_2 = \cdots = \mu_n = \lambda_1$, there are the following four cases*:

(1) $H^n(-1 - \lambda_1^2) = \{x \in H_1^{n+1}(-1) \subset \mathbb{R}_2^{n+2} \mid x_1 = \frac{|\lambda_1|}{\sqrt{1+\lambda_1^2}}\}$ *with $A = \lambda_1 I$ and $\varepsilon = -1$.*

(2) *If $(a, b) \in \{(a, b) \mid a - b + 1 = 0, a < -2\} \cup \{(a, b) \mid a + b - 1 = 0, a < 2\}$, $\mathbb{R}_1^n = \{x \in H_1^{n+1}(-1) \subset \mathbb{R}_2^{n+2} \mid x_1 = x_{n+2} + t_0\}(t_0 > 0)$ with $A = \pm I$ and $\varepsilon = 1$.*

(3) *If $(a, b) \in \{(a, b) \mid a + b - 1 < 0, -2 \leq a < 2\} \cup \{(a, b) \mid a - b + 1 < 0, a + b - 1 < 0, a < -2\}$, $H_1^n(\lambda_1^2 - 1) = \{x \in H_1^{n+1}(-1) \subset \mathbb{R}_2^{n+2} \mid x_{n+2} = \frac{|\lambda_1|}{\sqrt{1-\lambda_1^2}}\}$ with $A = \lambda_1 I$ and $\varepsilon = 1$.*

(4) *If $(a, b) \in \{(a, b) \mid a + b - 1 > 0, a < 2\} \cup \{(a, b) \mid a \geq 2\} \cup \{(a, b) \mid a < -2, a - b + 1 > 0\}$, $S_1^n(\lambda_1^2 - 1) = \{x \in H_1^{n+1}(-1) \subset \mathbb{R}_2^{n+2} \mid x_1 = \frac{|\lambda_1|}{\sqrt{\lambda_1^2-1}}\}$ with $A = \lambda_1 I$ and $\varepsilon = 1$.*

(ii) *Suppose $\mu_1 = \mu_2 = \cdots = \mu_n = \lambda_2$, there are the following four cases*:

(1) $H^n(-1 - \lambda_2^2) = \{x \in H_1^{n+1}(-1) \subset \mathbb{R}_2^{n+2} \mid x_1 = \frac{|\lambda_2|}{\sqrt{1+\lambda_2^2}}\}$ *with $A = \lambda_2 I$ and $\varepsilon = -1$.*

(2) *If $(a, b) \in \{(a, b) \mid a - b + 1 = 0, a > -2\} \cup \{(a, b) \mid a + b - 1 = 0, a > 2\}$, $\mathbb{R}_1^n = \{x \in H_1^{n+1}(-1) \subset \mathbb{R}_2^{n+2} \mid x_1 = x_{n+2} + t_0\}(t_0 > 0)$ with $A = \pm I$ and $\varepsilon = 1$.*

(3) *If* $(a,b) \in \{(a,b)\,|\,a-b+1>0, -2<a\le 2\} \cup \{(a,b)\,|\,a-b+1>0, a+b-1>0, a>2\}$, $H_1^n(\lambda_2^2-1)=\{x\in H_1^{n+1}(-1)\subset \mathbb{R}_2^{n+2}\mid x_{n+2}=\frac{|\lambda_2|}{\sqrt{1-\lambda_2^2}}\}$ *with* $A=\lambda_2 I$ *and* $\varepsilon=1$.

(4) *If* $(a,b) \in \{(a,b)\,|\,a-b+1<0, a>-2\} \cup \{(a,b)\,|\,a\le -2\} \cup \{(a,b)\,|\,a>2, a+b-1<0\}$, $H_1^n(\lambda_2^2-1)=\{x\in H_1^{n+1}(-1)\subset \mathbb{R}_2^{n+2}\mid x_1=\frac{|\lambda_2|}{\sqrt{\lambda_2^2-1}}\}$ *with* $A=\lambda_2 I$ *and* $\varepsilon=1$.

(iii) *Suppose* $\mu_1=\mu_2=\cdots=\mu_r=\lambda_1$, *and* $\mu_{r+1}=\cdots=\mu_n=\lambda_2$, *for* $1\le r\le n$:

(1) *If* $b=-1$ *and* $a>2$, $S_1^r(\lambda_1^2-1)\times H^{n-r}(\lambda_2^2-1)=\{x\in H_1^{n+1}(-1)\subset \mathbb{R}_2^{n+2}\mid -x_1^2+\sum_{i=3}^{2+r}x_i^2=\frac{1}{\lambda_1^2-1}, -x_2^2+\sum_{i=r+3}^{n+2}x_i^2=\frac{1}{\lambda_2^2-1}\}$ *with* $A=\lambda_1 I_r\oplus\lambda_2 I_{n-r}$ *and* $\varepsilon=1$.

(2) *If* $b=-1$ *and* $a<-2$, $S_1^r(\lambda_2^2-1)\times H^{n-r}(\lambda_1^2-1)=\{x\in H_1^{n+1}(-1)\subset \mathbb{R}_2^{n+2}\mid -x_1^2+\sum_{i=3}^{2+r}x_i^2=\frac{1}{\lambda_2^2-1}, -x_2^2+\sum_{i=r+3}^{n+2}x_i^2=\frac{1}{\lambda_1^2-1}\}$ *with* $A=\lambda_2 I_r\oplus\lambda_1 I_{n-r}$ *and* $\varepsilon=1$.

(3) *If* $b=1$, $H^r(-1-\lambda_i^2)\times H_1^{n-r}(-1-\lambda_j^2)=\{x\in H_1^{n+1}(-1)\subset \mathbb{R}_2^{n+2}\mid -x_1^2+\sum_{i=3}^{2+r}x_i^2=\frac{1}{-1-\lambda_i^2}, -x_2^2+\sum_{i=r+3}^{n+2}x_i^2=\frac{1}{-1-\lambda_j^2}\}$ *with* $A=\lambda_i I_r\oplus\lambda_j I_{n-r}))$, $i,j=1,2$, $i\ne j$ *and* $\varepsilon=-1$.

Proof. Case 1: If $a^2+4b=0$, then $\lambda_1=\lambda_2=\frac{a}{2}$.

(i) If $\varepsilon=-1$, we get $c=-\frac{a^2}{4}-1<0$. Then $M=H^n(-1-\frac{a^2}{4})=\{x\in H_1^{n+1}(-1)\subset R_2^{n+2}\mid x_1=\frac{|a|}{\sqrt{a^2+4}}\}$ with $A=\frac{a}{2}I$.

(ii) If $\varepsilon=1$ and $a=\pm 2$, we get $c=-1+\frac{a^2}{4}=0$. Then $M=\mathbb{R}_1^n=\{x\in H_1^{n+1}(-1)\subset \mathbb{R}_2^{n+2}\mid x_1=x_{n+2}+t_0\}(t_0>0)$ with $A=\pm I$.

(iii) If $\varepsilon=1$ and $|a|>2$, we get $c=-1+\frac{a^2}{4}>0$. Then $M=S_1^n(\frac{a^2}{4}-1)=\{x\in H_1^{n+1}(-1)\subset \mathbb{R}_2^{n+2}\mid x_1=\frac{|a|}{\sqrt{a^2-4}}\}$ with $A=\frac{a}{2}I$.

(iv) If $\varepsilon=1$ and $-2<a<2$, we get $c=-1+\frac{a^2}{4}<0$. Then $M=H_1^n(\frac{a^2}{4}-1)=\{x\in S_1^{n+1}\subset \mathbb{R}_1^{n+2}\mid x_{n+2}=\frac{|a|}{\sqrt{4-a^2}}\}$ with $A=\frac{a}{2}I$.

Case 2: If $a^2+4b>0$:

(i) Suppose $\mu_1=\mu_2=\cdots=\mu_n=\lambda_1$, there are the following four cases:

(1) If $\varepsilon=-1$, we get $c=-\lambda_1^2-1<0$. Then $M=H^n(-1-\lambda_1^2)=\{x\in H_1^{n+1}(-1)\subset \mathbb{R}_2^{n+2}\mid x_1=\frac{|\lambda_1|}{\sqrt{1+\lambda_1^2}}\}$ with $A=\lambda_1 I$.

(2) If $\varepsilon=1$ and $(a,b)\in\{(a,b)\,|\,a-b+1=0, a<-2\}\cup\{(a,b)\,|\,a+b-1=0, a<2\}$, we get $c=\lambda_1^2-1=0$. Then $M=\mathbb{R}_1^n=\{x\in H_1^{n+1}(-1)\subset \mathbb{R}_2^{n+2}\mid x_1=x_{n+2}+t_0\}(t_0>0)$ with $A=\pm I$.

(3) If $\varepsilon=1$ and $(a,b)\in\{(a,b)\,|\,a+b-1<0, -2\le a<2\}\cup\{(a,b)\,|\,a-b+1<0, a+b-1<0, a<-2\}$, we get $-1\le c=\lambda_1^2-1<0$. Then $M=H_1^n(\lambda_1^2-1)=\{x\in H_1^{n+1}(-1)\subset \mathbb{R}_2^{n+2}\mid x_{n+2}=\frac{|\lambda_1|}{\sqrt{1-\lambda_1^2}}\}$ with $A=\lambda_1 I$.

(4) If $\varepsilon = 1$ and $(a,b) \in \{(a,b)\,|\, a+b-1>0, a<2\} \cup \{(a,b)\,|\, a \geq 2\} \cup \{(a,b)\,|\, a<-2, a-b+1>0\}$, we get $c = \lambda_1^2 - 1 > 0$. Then $M = S_1^n(\lambda_1^2-1) = \{x \in H_1^{n+1}(-1) \subset \mathbb{R}_2^{n+2} \mid x_1 = \frac{|\lambda_1|}{\sqrt{\lambda_1^2-1}}\}$ with $A = \lambda_1 I$.

(ii) Suppose $\mu_1 = \mu_2 = \cdots = \mu_n = \lambda_2$, there are the following four cases:

(1) If $\varepsilon = -1$, we get $c = -\lambda_2^2 - 1 < 0$. Then $M = H^n(-1-\lambda_2^2) = \{x \in H_1^{n+1}(-1) \subset \mathbb{R}_2^{n+2} \mid x_1 = \frac{|\lambda_2|}{\sqrt{1+\lambda_2^2}}\}$ with $A = \lambda_2 I$.

(2) If $\varepsilon = 1$ and $(a,b) \in \{(a,b)\,|\, a-b+1=0, a>-2\} \cup \{(a,b)\,|\, a+b-1=0, a>2\}$, we get $c = \lambda_1^2 - 1 = 0$. Then $M = \mathbb{R}_1^n = \{x \in H_1^{n+1}(-1) \subset \mathbb{R}_2^{n+2} \mid x_1 = x_{n+2} + t_0\}(t_0 > 0)$ with $A = \pm I$.

(3) If $\varepsilon = 1$ and $(a,b) \in \{(a,b)\,|\, a-b+1>0, -2<a\leq 2\} \cup \{(a,b)\,|\, a-b+1>0, a+b-1>0, a>2\}$, we get $-1 \leq c = \lambda_1^2 - 1 < 0$. $M = H_1^n(\lambda_2^2-1) = \{x \in H_1^{n+1}(-1) \subset \mathbb{R}_2^{n+2} \mid x_{n+2} = \frac{|\lambda_2|}{\sqrt{1-\lambda_2^2}}\}$ with $A = \lambda_2 I$.

(4) If $\varepsilon = 1$ and $(a,b) \in \{(a,b)\,|\, a-b+1<0, a>-2\} \cup \{(a,b)\,|\, a\leq -2\} \cup \{(a,b)\,|\, a>2, a+b-1<0\}$, we get $c = \lambda_1^2 - 1 > 0$. Then $M = H_1^n(\lambda_2^2-1) = \{x \in H_1^{n+1}(-1) \subset \mathbb{R}_2^{n+2} \mid x_1 = \frac{|\lambda_2|}{\sqrt{\lambda_2^2-1}}\}$ with $A = \lambda_2 I$.

(iii) Suppose $\mu_1 = \mu_2 = \cdots = \mu_r = \lambda_1$, and $\mu_{r+1} = \cdots = \mu_n = \lambda_2$, for $1 \leq r \leq n$:

(1) If $\varepsilon = 1$ and $b = -1$ and $a > 2$, we get $\lambda_1\lambda_2 = 1$ and $c_1 = \lambda_1^2 - 1 > 0$ and $c_2 = \lambda_2^2 - 1 < 0$. Then we have $M = S_1^r(\lambda_1^2-1) \times H^{n-r}(\lambda_2^2-1) = \{x \in H_1^{n+1}(-1) \subset \mathbb{R}_2^{n+2} \mid -x_1^2 + \sum_{i=3}^{2+r} x_i^2 = \frac{1}{\lambda_1^2-1}, -x_2^2 + \sum_{i=r+3}^{n+2} x_i^2 = \frac{1}{\lambda_2^2-1}\}$ with $A = \lambda_1 I_r \oplus \lambda_2 I_{n-r}$.

(2) If $\varepsilon = 1$ and $b = -1$ and $a < -2$, we get $\lambda_1\lambda_2 = 1$ and $c_1 = \lambda_1^2 - 1 < 0$ and $c_2 = \lambda_2^2 - 1 > 0$. Then we have $M = S_1^r(\lambda_2^2-1) \times H^{n-r}(\lambda_1^2-1) = \{x \in H_1^{n+1}(-1) \subset \mathbb{R}_2^{n+2} \mid -x_1^2 + \sum_{i=3}^{2+r} x_i^2 = \frac{1}{\lambda_2^2-1}, -x_2^2 + \sum_{i=r+3}^{n+2} x_i^2 = \frac{1}{\lambda_1^2-1}\}$ with $A = \lambda_2 I_r \oplus \lambda_1 I_{n-r}$.

(3) If $\varepsilon = -1$ and $b = -1$, we get $\lambda_1\lambda_2 = 1$ and $c_i = -1 - \lambda_i^2 < 0$ for $i,j = 1,2$. Then we have $M = H^r(-1-\lambda_i^2) \times H_1^{n-r}(-1-\lambda_j^2) = \{x \in H_1^{n+1}(-1) \subset \mathbb{R}_2^{n+2} \mid -x_1^2 + \sum_{i=3}^{2+r} x_i^2 = \frac{1}{-1-\lambda_i^2}, -x_2^2 + \sum_{i=r+3}^{n+2} x_i^2 = \frac{1}{-1-\lambda_j^2}\}$ with $A = \lambda_i I_r \oplus \lambda_j I_{n-r}$, $i,j = 1,2$ and $i \neq j$. □

Remark 1. Theorem 1, Theorem 2 and Theorem 3 are the generalization of the corresponding results in [9] respectively. That is , when $a = 1$ and $b = 1$, these theorems are the same as theorems in [9].

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