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SOME DOUBLY-WARPED PRODUCT GRADIENT RICCI SOLITONS

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Abstract. In this paper, we study certain doubly-warped products which admit gradient Ricci solitons with harmonic Weyl curvature and non-constant soliton function. The metric is of the form $g = dx_1^2 + p(x_1)^2 dx_2^2 + h(x_1)^2\ \tilde{g}$ on $\mathbb{R}^2 \times N$, where x_1, x_2 are the local coordinates on $\mathbb{R}^2$ and $\tilde{g}$ is an Einstein metric on the manifold N. We obtained a full description of all the possible *local* gradient Ricci solitons.

1. Introduction

A gradient Ricci soliton consists of a Riemannian manifold (M, g) and a smooth function f satisfying $\nabla df = -Rc + \lambda g$, where Rc denotes the Ricci tensor of g and λ is a constant. They are important as singularity models of the Ricci flow in Hamilton's theory.

It is very interesting to find non-trivial gradient Ricci solitons. Some collection of explicit examples have been found and studied; see Chapters 1 and 2 of [1] and its references.

The goal of this paper is modest: we shall analyze gradient Ricci solitons of restricted feature. Here we consider $(\mathbb{R}^2 \times N, g)$ where g has harmonic Weyl curvature and is of the form $g = dx_1^2 + p(x_1)^2 dx_2^2 + h(x_1)^2 \tilde{g}$ in the local coordinates x_1, x_2 on $\mathbb{R}^2$, with two functions p, h and $\tilde{g}$ is an Einstein metric on the manifold N. The motivation for this study is explained in the next paragraphs.

Ivey in [3] has considered Riemannian metrics of the form $g = dx_1^2 + p(x_1)^2 g_{S^k} + h(x_1)^2 \tilde{g}$, where g_{S^k} is the standard metric on S^k, $k \geq 1$ and $\tilde{g}$ is an Einstein metric with positive scalar curvature. By studying phase-plane trajectories of the soliton ordinary differential equations, he showed that there exists a complete gradient Ricci soliton metrics of the above form with $\lambda = 0$.

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Recently, gradient Ricci solitons with harmonic Weyl curvature became an interesting theme to study [2, 6]. Now one may need to understand noncompact solitons with λ of any sign. So, the author wanted to understand Ivey's examples in the context of harmonic Weyl curvature.

Although we work only in $k = 1$ case of Ivey's examples, we treat λ and $\tilde{g}$ with any signs. We note that four dimensional case is already studied in [5], but our argument here is all-dimension inclusive.

It turns out that, with the condition of harmonic Weyl curvature, a good analysis is possible. Indeed, after analyzing several cases we could make a full description of all the possible *local* gradient Ricci solitons in the main theorem, Theorem 5.4.

This paper is organized as follows. In Section 2 we mainly compute the curvature components of the soliton metric with harmonic Weyl curvature. In Section 3 we characterize four possible simple cases arising from the soliton equation. They are expressed as a linear relation in terms of $\frac{p'}{p}$ and $\frac{h'}{h}$. In Section 4 we reduce the general case to three quadratic cases in Lemma 4.2. In the last Section 5, we analyze the remaining three cases and summarize them in Theorem 5.4.

2. Analysis of the soliton metric with harmonic Weyl curvature

We recall one formula for a gradient Ricci soliton with harmonic Weyl curvature from [2]. For the Riemannian curvature tensor $R(X,Y,Z,W) = \langle \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z, W \rangle$;

Lemma 2.1. *For a gradient Ricci soliton* (M, g, f) *with* $\delta W = 0$*, i.e., harmonic Weyl curvature, we have*;

$$\begin{aligned} R(X,Y,Z,\nabla f) &= \frac{1}{n-1} R(X,\nabla f) g(Y,Z) - \frac{1}{n-1} R(Y,\nabla f) g(X,Z) \\ &= \frac{1}{2(n-1)} dR(X) g(Y,Z) - \frac{1}{2(n-1)} dR(Y) g(X,Z). \end{aligned}$$

We shall consider a gradient Ricci soliton with $\delta W = 0$ such that its metric g on $\mathbb{R}^2 \times N^{n-2}$, $n \geq 4$, can be written in local coordinates $x_1, \ldots, x_n$ as

$$g = dx_1{}^2 + p(x_1)^2 dx_2^2 + h(x_1)^2 \tilde{g} \qquad \text{with} \quad |\nabla f| \neq 0, \tag{1}$$

where x_1, x_2 are the coordinates for $\mathbb{R}^2$ and $x_3, \ldots, x_n$ are local coordinates for N and $\tilde{g}$ is an Einstein metric on N. Moreover, we shall consider the soliton function f to be a function of x_1 only.

This assumption of $f = f(x_1)$ may seem too restricted. However, we believe that f should be a function of x_1 only, if $\delta W = 0$.

We first compute the Levi-Civita connection of g in the coordinates. We write $\partial_i := \frac{\partial}{\partial x_i}$ and denote the derivative in x_1 by prime, e.g. $p' := \frac{dp}{dx_1}$. For

$i, j \in \{3, \ldots, n\}$,

$$\tag{2} \begin{aligned} &\nabla_{\partial_1}\partial_1 = 0, \qquad \nabla_{\partial_1}\partial_2 = \frac{p'}{p}\partial_2, \qquad \nabla_{\partial_1}\partial_j = \frac{h'}{h}\partial_j, \\ &\nabla_{\partial_2}\partial_2 = -pp'\partial_1, \qquad \nabla_{\partial_j}\partial_2 = \nabla_{\partial_2}\partial_j = 0, \\ &\nabla_{\partial_i}\partial_j = \nabla^{\tilde{g}}_{\partial_i}\partial_j - hh'\tilde{g}(\partial_i, \partial_j)\partial_1. \end{aligned}$$

From (2), one can compute the curvature components of g in terms of those of $\tilde{g}$ and (derivatives of) p and h. We prefer to write them in terms of a local orthonormal frame field E_i such that $E_1 = \partial_1$, $E_2 = \frac{1}{p}\partial_2$ and for $j \geq 3$, $E_j = \frac{1}{h}e_j$, where e_j's form a local orthonormal frame field for $\tilde{g}$. As $\tilde{g}$ is Einstein, the Ricci tensor of $\tilde{g}$ satisfies $Ric^{\tilde{g}} = \frac{R^{\tilde{g}}}{n-2}\tilde{g}$, where $R^{\tilde{g}}$ is the scalar curvature of $\tilde{g}$. We shall see in (5) that E_i's are Ricci-eigen vector fields.

We shall write the Ricci components $R_{ij} := R(E_i, E_j)$, and curvature components $R_{ijkl} := R(E_i, E_j, E_k, E_l)$. For later convenience we set $a = \frac{p'}{p}$ and $b = \frac{h'}{h}$.

We have

$$R_{1221} = -\frac{p''}{p}, \qquad R_{1jj1} = -\frac{h''}{h} \ \text{ for } j \in \{3, \ldots, n\}.$$

By Lemma 2.1, for $j = 2, \ldots, n$,

$$\tag{3} R_{1jj1}|\nabla f| = \frac{1}{n-1}R_{11}|\nabla f| = \frac{1}{2(n-1)}R'.$$

This implies that

$$\tag{4} R_{1221} = \cdots = R_{1nn1} \quad \text{and} \quad \frac{p''}{p} = \frac{h''}{h} = a' + a^2 = b' + b^2.$$

Ricci components are as follows. For $i \in \{3, \ldots, n\}$,

$$\tag{5} \begin{aligned} R_{11} &= -(n-1)\frac{h''}{h} = -(n-1)(a' + a^2), \\ R_{22} &= -\frac{p''}{p} - (n-2)\frac{p'}{p}\frac{h'}{h} = -a' - a^2 - (n-2)ab, \\ R_{ii} &= -\frac{h''}{h} - \frac{p'}{p}\frac{h'}{h} - (n-3)\frac{h'^2}{h^2} + \frac{R^{\tilde{g}}}{(n-2)h^2} \\ &= -b' - (n-2)b^2 - ab + \frac{R^{\tilde{g}}}{(n-2)h^2}, \\ R_{ij} &= 0, \quad i \neq j \end{aligned}$$

The scalar curvature equals

$$\tag{6} R = -(2n-2)(a' + a^2) - (n-2)\{2ab + (n-3)b^2\} + \frac{R^{\tilde{g}}}{h^2}.$$

3. Four special cases

From (2), we can deduce

$$\nabla_{E_1}E_1 = 0,\ \nabla_{E_2}E_1 = \frac{p'}{p}E_2,\ \nabla_{E_2}E_2 = -\frac{p'}{p}E_1 \quad \text{and,}$$

for $i = 2,\dots,n$, $\nabla_{E_1}E_i = 0$, $\nabla_{E_i}E_1 = \frac{h'}{h}E_i$, $\nabla_{E_2}E_i = \nabla_{E_i}E_2 = 0$. And $\langle\nabla_{E_j}E_j, E_1\rangle = -\frac{h'}{h}$, $j \geq 3$, $\langle\nabla_{E_i}E_j, E_1\rangle = 0$ for distinct $i, j \geq 3$.

From these, we can write the soliton equation

$$\nabla df(E_i, E_j) = -(Rc - \lambda g)(E_i, E_j)$$

as below, using (5) and the definition $\nabla df(E_i, E_j) = E_iE_j(f) - (\nabla_{E_i}E_j)f$.

For $j \in \{3,\dots,n\}$,

$$f'' = \frac{p''}{p} + (n-2)\frac{h''}{h} + \lambda = (n-1)(a' + a^2) + \lambda. \tag{7}$$

$$f'a = f'\frac{p'}{p} = a' + a^2 + (n-2)ab + \lambda. \tag{8}$$

$$f'b = f'\frac{h'}{h} = b' + (n-2)b^2 + ab - \frac{R^{\tilde{g}}}{(n-2)h^2} + \lambda. \tag{9}$$

Remark 3.1. At this point we note the real analyticity of most functions and tensors involved. In fact, as g and f are real analytic (in harmonic coordinates) [4], so is $|\nabla f|$ where $\nabla f \neq 0$. Since $R' = dR(E_1) = dR(\frac{\nabla f}{|\nabla f|})$ is real analytic, so is $R(E_1, E_j, E_j, E_1)$ from (3). As $-b' - b^2 = R(E_1, E_3, E_3, E_1)$ is real analytic, so is $b = \frac{h'}{h}$, as well as $a = \frac{p'}{p}$. And h and p are real analytic.

This will help our argument; when we analyze an equation of the type $P_1 \cdot P_2 = 0$ (identically) on a domain where P_i is each a polynomial in a, b, then $P_1 = 0$ or $P_2 = 0$.

To study the soliton metric g of (1) with $\delta W = 0$, we first consider four special cases: when $a = 0$, $b = 0$, $a = b$ or $a = -b$ on a domain.

Lemma 3.2. *For the soliton metric g of* (1) *with $\delta W = 0$, if it satisfies $a = 0$* (*identically*) *on a domain, then we have $\lambda = 0$ and g is locally isometric to one of the following.*

(i) *a domain in $\mathbb{R}^2 \times (N, \tilde{g})$ with $g = dx_1^2 + dx_2^2 + \tilde{g}$, where $\tilde{g}$ is a Ricci flat metric. And f is linear.*

(ii) *a domain in $\mathbb{R}^2 \times (N, \tilde{g})$ with $g = dx_1^2 + dx_2^2 + x_1^2\tilde{g}$ where $\tilde{g}$ is a positive Einstein metric and f constant.*

Proof. As a vanishes, $\lambda = 0$ from (8). As p is constant, say $p = p_0 > 0$, we have $\frac{p''}{p} = \frac{h''}{h} = 0$. So, h is a linear function. Shifting by a constant if necessary ($x_1 \mapsto x_1 +$constant), we may set $h(x_1) = a_0$ or $h(x_1) = cx_1$ for some non-zero constant a_0, c. If $h(x_1) = a_0$, then from (9) we get $R^{\tilde{g}} = 0$. Then $g = dx_1^2 + dx_2^2 + \tilde{g}$, where $\tilde{g}$ is Ricci flat. And $f^{''} = 0$ from (7).

If $h(x_1) = cx_1$, $c \neq 0$, then from (7), $f^{''} = 0$. From (9) we get $f^{'}(x_1) = \frac{1}{x_1}(n-3-\frac{R^{\tilde{g}}}{(n-2)c^2})$. So, $f^{'} = 0$ and $c^2 = \frac{R^{\tilde{g}}}{(n-2)(n-3)} > 0$.

So, $g = dx_1^2 + dx_2^2 + x_1^2\tilde{g}$ with $\tilde{g}$ positive Einstein metric and f is a constant. □

Remark 3.3. In the statement of Lemma 3.2, x_1 may be a shift of the original x_1 in (1) by a constant. Likewise, $\tilde{g}$ may be a constant multiple of the original $\tilde{g}$ in (1). In other words, x_1 and $\tilde{g}$ in the statement already absorbed some constant. In later lemmas, these will constantly happen. And x_2 can also be a constant multiple of the original x_2.

The next is when $b = 0$.

Lemma 3.4. *For the soliton metric g of* (1) *with $\delta W = 0$, if $b = 0$, but a is never zero on a domain, then g is locally isometric to a domain in $\mathbb{R}^2 \times (N^{n-2}, \tilde{g})$ with $g = dx_1^2 + x_1^2 dx_2^2 + \tilde{g}$, where $\tilde{g}$ is an Einstein metric with $\frac{R^{\tilde{g}}}{n-2} = \lambda$ on N. And $f = \frac{\lambda}{2}x_1^2 + C_1$ for a constant C_1.*

Proof. As h is constant, say $h = h_0 > 0$, we have $\frac{p^{''}}{p} = \frac{h^{''}}{h} = 0$. So, p is a linear function. As a is never zero, by shifting x_1 by a constant, we may set $p(x_1) = cx_1$, for some non-zero constant c.

From (8), $f^{'} = \lambda x_1$. We get $f(x_1) = \frac{1}{2}\lambda x_1^2 + C_1$. From (9), we have $\frac{R^{\tilde{g}}}{(n-2)h_0^2} = \lambda$. And the metric g becomes $g = d{x_1}^2 + x_1^2 dx_2^2 + \tilde{g}$, where $\tilde{g}$ is an Einstein metric with $\frac{R^{\tilde{g}}}{(n-2)} = \lambda$. This proves the lemma. □

Lemma 3.5. *For the soliton metric g of* (1) *with $\delta W = 0$, if the function $a - b = 0$ on a domain, then the metric becomes $g = dx_1^2 + h^2(dx_2^2 + \tilde{g})$ where $\tilde{g}$ is Ricci flat and f, h satisfy*

$$f^{''} = (n-1)\frac{h^{''}}{h} + \lambda,$$
$$f^{'}\frac{h^{'}}{h} = \frac{h^{''}}{h} + (n-2)(\frac{h^{'}}{h})^2 + \lambda.$$

Proof. Suppose $a - b = 0$, i.e., $\frac{p^{'}}{p} = \frac{h^{'}}{h}$. Then $p = ch$ for some constant $c > 0$. Comparing (8) and (9), $R^{\tilde{g}} = 0$. By absorbing the constant c to dx_2, the metric becomes $g = dx_1^2 + h^2(dx_2^2 + \tilde{g})$ where $\tilde{g}$ is Ricci flat. And f, h satisfies the next two equations;

$$f^{''} = (n-1)\frac{h^{''}}{h} + \lambda,$$
$$f^{'}\frac{h^{'}}{h} = \frac{h^{''}}{h} + (n-2)(\frac{h^{'}}{h})^2 + \lambda.$$

□

Lemma 3.6. *For the soliton metric g of* (1) *with $\delta W = 0$, if $a + b = 0$ on a domain, then we have $\lambda = 0$ and g is locally isometric to a domain in $\mathbb{R}^2 \times (N, \tilde{g})$ with $g = dx_1^2 + dx_2^2 + \tilde{g}$, where $\tilde{g}$ is a Ricci flat metric. And f is linear.*

Proof. Suppose $a + b = 0$, i.e., $\frac{p^{'}}{p} = -\frac{h^{'}}{h}$. By differentiating $\frac{p^{''}}{p} - (\frac{p^{'}}{p})^2 = -\frac{h^{''}}{h} + (\frac{h^{'}}{h})^2$. Then by $\frac{p^{''}}{p} = \frac{h^{''}}{h}$, we have $-\frac{h^{''}}{h} + (\frac{h^{'}}{h})^2 = 0$. So, $b = \frac{h^{'}}{h} = C$ for a constant C. So, $h = c_h e^{Cx_1}$ for a constant $c_h > 0$. And $a = -C$.

When $C \neq 0$, from (8) and (9) we have $R^{\tilde{g}} = \lambda = 0$ and $f^{'}$ is a constant. Then (7) becomes $0 = (n-1)C^2$, which is a contradiction.

If $C = 0$, then h and p are both constants. From (7)$\sim$(9), $\lambda = R^{\tilde{g}} = f^{''} = 0$. And $g = dx_1^2 + dx_2^2 + \tilde{g}$ with Ricci-flat metric $\tilde{g}$. □

4. Developing for general cases

In the previous section four simple cases of linear relations are understood. here we develop for all possible cases. It turns out that we only need to study at most quadratic cases.

For simplicity we shall denote $X := \frac{R^{\tilde{g}}}{(n-2)h^2}$. We note that $X^{'} = -2bX$. From (8) and (9) we have $b\{a^{'} + a^2 + (n-2)ab + \lambda\} = a\{b^{'} + (n-2)b^2 + ab - X + \lambda\}$. Rearranging this, using (4), we get;

$$(b-a)a^{'} + a(b^2 - a^2) + (b-a)\lambda = -aX. \tag{10}$$

Here we assume that $a - b$ is never zero on a domain. Taking (8)-(9), we get;

$$\begin{aligned} f^{'}(a-b) &= a^{'} + a^2 + (n-2)ab - b^{'} - (n-2)b^2 - ab + X \\ &= (n-3)ab - (n-3)b^2 + X. \end{aligned} \tag{11}$$

Differentiating (11) and using (4),

$$\begin{aligned} &f^{''}(a-b) + f^{'}(a-b)^{'} \\ &= (n-3)(a^{'}b + ab^{'}) - 2(n-3)bb^{'} - 2bX \\ &= (n-3)a^{'}b + (n-3)(a-2b)(a^{'} - b^2 + a^2) - 2bX \\ &= a^{'}(a-b)(n-3) + (n-3)(a-2b)(-b^2 + a^2) - 2bX. \end{aligned}$$

Differentiating $f^{'}(a-b)$ and using (7), (4) and (11),

$$\begin{aligned} &f^{''}(a-b) + f^{'}(a-b)^{'} \\ &= \{(n-1)(a^{'} + a^2) + \lambda\}(a-b) - (a+b)\{(n-3)ab - (n-3)b^2 + X\}. \end{aligned}$$

Equating the two expressions, we have

$$\begin{aligned} &a^{'}(a-b)(n-3) + (n-3)(a-2b)(-b^2 + a^2) - 2bX \\ &= \{(n-1)(a^{'} + a^2) + \lambda\}(a-b) - (a+b)\{(n-3)ab - (n-3)b^2 + X\}. \end{aligned}$$

The above simplifies to

$$-2a^{'}(a-b)+(b-a)\lambda+2a^2(b-a)+(a-b)b^2(3-n)=(b-a)X.$$

From this last equation, we may state:

Lemma 4.1. *For the soliton metric g of* (1) *with $\delta W=0$, if $a-b$ is never zero on a domain, the following holds.*

$$-2a^{'}-\lambda-2a^2+b^2(3-n)=-X. \tag{12}$$

Removing X from (10) and (12),

$$-(a+b)a^{'}=a^3+(n-2)ab^2+\lambda b. \tag{13}$$

Removing $a^{'}$ from (10) and (12),

$$(a-b)\{2ab-(n-3)b^2+\lambda\}=(a+b)X. \tag{14}$$

Now we can show:

Lemma 4.2. *For the soliton metric g of* (1) *with $\delta W=0$, if $b(a-b)\{2ab-(n-3)b^2+\lambda\}$ is never zero on a domain, the following holds.*

$$\{\lambda+(n-1)ab\}\{\lambda-2a^2+(n-3)ab\}=0. \tag{15}$$

Proof. From the hypothesis that $(a-b)\{2ab-(n-3)b^2+\lambda\}$ is never zero, $(a+b)X$ is also not zero and we may take the natural log of (14) and differentiate it;

$$-a-b+\frac{2a^{'}b+2ab^{'}-2(n-3)bb^{'}}{2ab-(n-3)b^2+\lambda}=\frac{a^{'}+b^{'}}{a+b}-2b.$$

Then,

$$\frac{\{a-(n-4)b\}a^{'}+\{a-(n-3)b\}(a^2-b^2)}{2ab-(n-3)b^2+\lambda}=\frac{a^{'}+a^2-b^2}{a+b}.$$

Arranging terms, we obtain;

$$-a^{'}\{a^2+b^2+(3-n)ab-\lambda\}=(a^2-b^2)\{a^2-(n-2)ab-\lambda\}. \tag{16}$$

We remove $a^{'}$ from (13) and (16) to get;

$$\begin{aligned}&(a+b)(a^2-b^2)\{a^2-(n-2)ab-\lambda\}\\ &=\{a^3+(n-2)ab^2+\lambda b\}\{a^2+b^2+(3-n)ab-\lambda\}.\end{aligned}$$

After simplification, we get (15). □

5. Quadratic cases

From Lemma 4.2 and results of Section 3, we only need to understand three quadratic cases;

$$2ab - (n-3)b^2 + \lambda = 0, \tag{17}$$

$$\lambda + (n-1)ab = 0, \tag{18}$$

$$\lambda - 2a^2 + (n-3)ab = 0. \tag{19}$$

When (17) holds, from (14) we may have either $a + b \equiv 0$ or $X \equiv 0$. Lemma 3.6 covers $a + b \equiv 0$. Now we prove:

Lemma 5.1. *For the soliton metric g of* (1) *with $\delta W = 0$, assume that a, b, $a - b$ and $a + b$ are never zero on a domain. Then $2ab - (n-3)b^2 + \lambda$ and X cannot vanish together.*

Proof. We assume that $2ab - (n-3)b^2 + \lambda = X = 0$. From (6), we get $R = -(2n-2)(a^{'} + a^2) - (n-2)\{4ab + \lambda\}$.

We shall use $\delta W = 0$; $\nabla_k R_{ij} - \nabla_j R_{ik} = -\frac{R_j}{2n-2} g_{ki} + \frac{R_k}{2n-2} g_{ij}$ in $\{E_i\}$. In particular, using (5),

$$\begin{aligned}
0 &= \nabla_1 R_{22} - \nabla_2 R_{12} - \frac{R^{'}}{2n-2} \\
&= \nabla_1(R_{22}) + R(\nabla_{E_2} E_1, E_2) + R(\nabla_{E_2} E_2, E_1) - \frac{R^{'}}{2n-2} \\
&= (R_{22})^{'} + aR_{22} - aR_{11} - \frac{R^{'}}{2n-2}. \\
&= -\{a^{'} + a^2 + (n-2)ab\}^{'} - \frac{R^{'}}{2n-2} - a\{a^{'} + a^2 + (n-2)ab\} \\
&\quad + (n-1)a(a^{'} + a^2). \\
&= (4-n)(ab)^{'} - (n-2)a^2 b + (n-2)a(a^{'} + a^2) \\
&= (4-n)\{a^{'}b + a(a^{'} + a^2 - b^2)\} - (n-2)a^2 b + (n-2)a(a^{'} + a^2) \\
&= \{a^{'} + a(a-b)\}\{2a + (4-n)b\}.
\end{aligned}$$

We should treat two subcases $a^{'} + a(a-b) = 0$ or $2a + (4-n)b = 0$. We shall show these cannot occur.

When $a^{'} + a(a-b) = 0$, we get $p^{''} = \frac{p^{'}h^{'}}{h}$. Then $\frac{p^{'}}{h} = c_1$, a constant. From $\frac{p^{''}}{p} = \frac{h^{''}}{h}$, we also get $\frac{h^{'}}{p} = c_2$, a constant. So, $ab = \frac{p^{'}h^{'}}{ph} = c_1 c_2$. And $2ab - (n-3)b^2 + \lambda = 0$ gives $(n-3)b^2 = 2c_1c_2 + \lambda$. b is a constant and so is a since ab is never zero. Then $a^{'} + a(a-b) = 0$ gives $a(a-b) = 0$, a contradiction to the hypothesis.

When $2a + (4-n)b = 0$, together with $2ab - (n-3)b^2 + \lambda = 0$, we have $b^2 = \lambda \geq 0$. The case of $b = 0$ is violating the hypothesis. So assume $\lambda > 0$. Then $b = \sqrt{\lambda}$ and $a = \frac{n-4}{2}\sqrt{\lambda}$. But this would not satisfy (12). $\square$

Next for the case (18):

Lemma 5.2. *For the soliton metric g of* (1) *with $\delta W = 0$, assume that a, b, $a-b$ and $a+b$ are never zero and that $\lambda + (n-1)ab = 0$ on a domain, then f is constant and g is one of the following.*

(i) $g = dx_1^2 + \cos^2(\sqrt{\frac{\lambda}{n-1}}x_1)dx_2^2 + \sin^2(\sqrt{\frac{\lambda}{n-1}}x_1)\tilde{g}$ *with $\lambda > 0$ and a positive Einstein metric $\tilde{g}$ and $Ric^{\tilde{g}} = \frac{(n-3)\lambda}{n-1}\tilde{g}$.*

(ii) $g = dx_1^2 + \cosh^2(\sqrt{-\frac{\lambda}{n-1}}x_1)dx_2^2 + \sinh^2(\sqrt{-\frac{\lambda}{n-1}}x_1)\tilde{g}$ *with $\lambda < 0$ and a positive Einstein metric $\tilde{g}$ and $Ric^{\tilde{g}} = -\frac{(n-3)\lambda}{n-1}\tilde{g}$.*

Proof. As a, b are never zero, we have $\lambda \neq 0$ from $\lambda + (n-1)ab = 0$. From (12) we have

$$-2\frac{h^{''}}{h} - (n-3)\frac{(h^{'})^2}{h^2} + \frac{R^{\tilde{g}}}{(n-2)h^2} - \lambda = 0. \tag{20}$$

From $\lambda + (n-1)ab = 0$, we have $0 = a^{'}b + ab^{'} = (b^{'} - a^2 + b^2)b + ab^{'} = (a+b)\{b^{'} - b(a-b)\}$. As $a + b \neq 0$, we now have $b^{'} + b^2 = ab = -\frac{\lambda}{(n-1)}$. We get $\frac{h^{''}}{h} = -\frac{\lambda}{n-1}$. Due to (20),

$$-(n-3)\frac{h^{'2}}{h^2} + \frac{R^{\tilde{g}}}{(n-2)h^2} = \frac{(n-3)\lambda}{n-1}. \tag{21}$$

From (9), $f^{'}\frac{h^{'}}{h} = \frac{h^{''}}{h} + \frac{p^{'}}{p}\frac{h^{'}}{h} + (n-3)\frac{h^{'2}}{h^2} - \frac{R^{\tilde{g}}}{(n-2)h^2} + \lambda = -\frac{\lambda}{n-1} - \frac{\lambda}{n-1} - \frac{(n-3)\lambda}{n-1} + \lambda = 0$. As $b \neq 0$, f is a constant.

When $\lambda > 0$, the solution of $\frac{h^{''}}{h} = -\frac{\lambda}{n-1}$ is $h = c_h \sin(\sqrt{\frac{\lambda}{n-1}}x_1 + s_0)$ for some constants c_h and s_0. Put it into (21) with setting $x := \sqrt{\frac{\lambda}{n-1}}x_1 + s_0$, we get

$$-\frac{(n-3)\lambda}{n-1}\frac{(1-\sin^2 x)}{\sin^2 x} + \frac{R^{\tilde{g}}}{(n-2)c_h^2 \sin^2 x} = \frac{(n-3)\lambda}{n-1}.$$

This reduces to $\frac{(n-3)\lambda}{n-1} = \frac{R^{\tilde{g}}}{(n-2)c_h^2}$.

From $ab = \frac{p^{'}h^{'}}{ph} = -\frac{\lambda}{(n-1)}$, we get $p = c_p \cos(\sqrt{\frac{\lambda}{n-1}}x_1 + s_0)$. Shifting x_1 by a constant, we can write $g = dx_1^2 + \cos^2(\sqrt{\frac{\lambda}{n-1}}x_1)dx_2^2 + \sin^2(\sqrt{\frac{\lambda}{n-1}}x_1)\tilde{g}$ with a positive Einstein metric $\tilde{g}$ and $Ric^{\tilde{g}} = \frac{(n-3)\lambda}{n-1}\tilde{g}$. It satisfies (7) and (8).

When $\lambda < 0$, $h = c_h \sinh(\sqrt{-\frac{\lambda}{n-1}}x_1 + s_0)$ for some constants c_h and s_0. Put it into (21), we get $-\frac{(n-3)\lambda}{n-1} = \frac{R^{\tilde{g}}}{(n-2)c_h^2}$. And $p = c_p \cosh(\sqrt{-\frac{\lambda}{n-1}}x_1 + s_0)$. Then

$g = dx_1^2 + \cosh^2(\sqrt{-\frac{\lambda}{n-1}}x_1)dx_2^2 + \sinh^2(\sqrt{-\frac{\lambda}{n-1}}x_1)\tilde{g}$ with a positive Einstein metric $\tilde{g}$ and $Ric^{\tilde{g}} = -\frac{(n-3)\lambda}{n-1}\tilde{g}$. It satisfies (7) and (8). $\square$

For the last case (19):

Lemma 5.3. *For the soliton metric g of (1) with $\delta W = 0$, assume that $a, b, a-b$ and $a+b$ are never zero and that $\lambda - 2a^2 + (n-3)ab = 0$ on a domain. Then g is locally isometric to a domain of $\mathbb{R}^n$ with the metric*

$$dx_1^2 + x_1^{\frac{2(n-3)}{n-1}}dx_2^2 + x_1^{\frac{4}{n-1}}\tilde{g},$$

where $\tilde{g}$ is Ricci flat. Also, $\lambda = 0$ and $f = \frac{2(n-3)}{n-1}\ln x_1 + C_2$ for a constant C_2.

Proof. We put $\lambda = 2a^2 - (n-3)ab$ into (13) to get;

$$-a' = \frac{a^3 + (n-2)ab^2 + \lambda b}{a+b} = \frac{a^3 + 2a^2b + ab^2}{a+b} = a(a+b).$$

So, $-a' - a^2 = ab$, i.e., $p''h + p'h' = 0$. Integrating this, we get $p'h = c_1$ for a constant c_1. As $\frac{h''}{h} = \frac{p''}{p}$, we have $h''p + p'h' = 0$, which integrates to $h'p = c_2$. As $a = \frac{p'}{p}$ and $b = \frac{h'}{h}$ are never zero, c_1 and c_2 are not zero. So $\frac{h'}{h} = \frac{c_2}{c_1}\frac{p'}{p}$, i.e., $b = \frac{c_2}{c_1}a$. So, $0 = \lambda - 2a^2 + (n-3)ab = \lambda + \{-2 + \frac{c_2}{c_1}(n-3)\}a^2$.

If $\frac{c_2}{c_1} \neq \frac{2}{n-3}$, then $a = \frac{h'}{h}$ is a constant, say $a = a_0 \neq 0$. Then $h = c_3e^{a_0x_1}, c_3 > 0$. Then $p = c_5e^{-a_0x_1}$, $c_5 > 0$. In this case, we get $a + b = -a_0 + a_0 = 0$, which is contradictory to the hypothesis.

If $\frac{c_2}{c_1} = \frac{2}{n-3}$, then $\lambda = 0$. From $\frac{h'}{h} = \frac{2}{n-3}\frac{p'}{p}$, we integrates it to $h = c_6p^{\frac{2}{n-3}}$, $c_6 > 0$. Then $c_1 = p'h = c_6p^{\frac{2}{(n-3)}}p'$ and $p = c_7(x_1 + c_8)^{\frac{n-3}{n-1}}$. As x_1 may be defined modulo a constant, we may set $p = c_7x_1^{\frac{n-3}{n-1}}$ and $h = c_9x_1^{\frac{2}{n-1}}$. Put these into (8) and (9), and we get $R^{\tilde{g}} = 0$ and $f' = \frac{2(n-3)}{(n-1)x_1}$. Then $f = \frac{2(n-3)}{n-1}\ln x_1 + C_2$ can be obtained.

So, g is locally isometric to a domain of $\mathbb{R}^n$ with the metric $dx_1^2 + x_1^{\frac{2(n-3)}{n-1}}dx_2^2 + x_1^{\frac{4}{n-1}}\tilde{g}$, where $\tilde{g}$ is Ricci flat. $\square$

We can summarize Lemma 3.2$\sim$Lemma 5.3 with the help of Remark 3.1.

Theorem 5.4. *Let (g, f) be a gradient Ricci soliton on $M = \mathbb{R}^2 \times N^{n-2}$, $n \geq 4$, with $\delta W = 0$ and non-constant f, which can be written in local coordinates $x_1, \ldots, x_n$ as*

$$g = dx_1^2 + p(x_1)^2dx_2^2 + h(x_1)^2\tilde{g}, \tag{22}$$

where $\tilde{g}$ is an Einstein metric of dimension $n-2$ on a manifold N. Then it is locally one of the following:

(i) *$g = dx_1^2 + dx_2^2 + \tilde{g}$, where $\tilde{g}$ is a Ricci flat metric, $\lambda = 0$ and f is linear in x_1.*

(ii) $g = dx_1^2 + dx_2^2 + x_1^2\tilde{g}$ *where* $\tilde{g}$ *is a positive Einstein metric,* $\lambda = 0$ *and* f *is constant.*

(iii) $g = dx_1^2 + x_1^2 dx_2^2 + \tilde{g}$, *where* $\tilde{g}$ *is an Einstein metric with* $\frac{R^{\tilde{g}}}{(n-2)} = \lambda$. *And* $f = \frac{\lambda}{2}x_1^2 + C_1$ *for a constant* C_1.

(iv) $g = dx_1^2 + h^2(dx_2^2 + \tilde{g})$ *where* $\tilde{g}$ *is Ricci flat. And* f, h *satisfy*

$$f'' = (n-1)\frac{h''}{h} + \lambda,$$

$$f'\frac{h'}{h} = \frac{h''}{h} + (n-2)(\frac{h'}{h})^2 + \lambda.$$

(v) $g = dx_1^2 + \cos^2(\sqrt{\frac{\lambda}{n-1}}x_1)dx_2^2 + \sin^2(\sqrt{\frac{\lambda}{n-1}}x_1)\tilde{g}$ *with* $\lambda > 0$ *and a positive Einstein metric* $\tilde{g}$ *and* $Ric^{\tilde{g}} = \frac{(n-3)\lambda}{n-1}\tilde{g}$. f *is constant.*

(vi) $g = dx_1^2 + \cosh^2(\sqrt{-\frac{\lambda}{n-1}}x_1)dx_2^2 + \sinh^2(\sqrt{-\frac{\lambda}{n-1}}x_1)\tilde{g}$ *with* $\lambda < 0$ *and a positive Einstein metric* $\tilde{g}$ *and* $Ric^{\tilde{g}} = -\frac{(n-3)\lambda}{n-1}\tilde{g}$. f *is constant.*

(vii) $g = dx_1^2 + x_1^{\frac{2(n-3)}{n-1}}dx_2^2 + x_1^{\frac{4}{n-1}}\tilde{g}$, *where* $\tilde{g}$ *is Ricci flat with* $\lambda = 0$ *and* $f = \frac{2(n-3)}{n-1}\ln x_1 + C_2$ *for a constant* C_2.

Remark 5.5. The converse of Theorem 5.4 holds, although we omit its detailed computational proof. In fact, it is not hard to check that all the solitons from (i) through (vii) in Theorem 5.4 satisfy $\delta W = 0$, equivalently the equations $\nabla_k R_{ij} - \nabla_j R_{ik} = -\frac{R_j}{2n-2}g_{ki} + \frac{R_k}{2n-2}g_{ij}$.

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