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LIGHTLIKE HYPERSURFACES OF AN INDEFINITE GENERALIZED SASAKIAN SPACE FORM WITH A SYMMETRIC METRIC CONNECTION OF TYPE (ℓ, m)

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Abstract. We define a new connection on a semi-Riemannian manifold. Its notion contains two well known notions; (1) semi-symmetric connection and (2) quarter-symmetric connection. In this paper, we study the geometry of lightlike hypersurfaces of an indefinite generalized Sasakian space form with a symmetric metric connection of type (ℓ, m).

1. Introduction

A linear connection $\bar{\nabla}$ on a semi-Riemannian manifold $(\bar{M}, \bar{g})$ is said to be a *symmetric connection of type* (ℓ, m) if its torsion tensor $\bar{T}$ satisfies

$$\bar{T}(\bar{X}, \bar{Y}) = \ell\{\theta(\bar{Y})\bar{X} - \theta(\bar{X})\bar{Y}\} + m\{\theta(\bar{Y})J\bar{X} - \theta(\bar{X})J\bar{Y}\}, \tag{1.1}$$

where ℓ and m are smooth functions, J is a tensor field of type $(1, 1)$ and θ is a 1-form associated with a unit vector field ζ by $\theta(\bar{X}) = \bar{g}(\bar{X}, \zeta)$. Moreover, if $\bar{\nabla}$ satisfies $\bar{\nabla}\bar{g} = 0$, then it is called a *symmetric metric connection of type* (ℓ, m). In the following, we denote by $\bar{X}$, $\bar{Y}$ and $\bar{Z}$ the vector fields on $\bar{M}$.

In case of $\ell = 1$ and $m = 0$, $\bar{\nabla}$ is called a *semi-symmetric metric connection.* The notion of semi-symmetric metric connection on a Riemannian manifold was introduced by H. A. Hayden [8] and later studied by some authors [18]. In case of $\ell = 0$ and $m = 1$, $\bar{\nabla}$ is called a *quarter-symmetric metric connection.* The notion of quarter-symmetric metric connection was introduced by K. Yano-T. Imai [19], and since then it have been studied by S. C. Rastogi [16, 17], D. Kamilya-U. C. De [11], R. S. Mishra-S. N. Pandey [12], S. Golab [7], N. Pušić [15], J. Nikić-N. Pušić [13] and some others.

The lightlike version of Riemannian manifolds equipped with semi-symmetric or quarter-symmetric metric connections have been studied by several authors. In this paper, we study the geometry of lightlike hypersurface of an indefinite generalized Sasakian space form with a symmetric metric connection of type

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$(\ell,\, m)$, in which the tensor field J, the 1-form θ and the vector field ζ, defined by (1.1), are identical with the tensor field J, the 1-form θ and the vector field ζ of the indefinite almost contact structure $(J,\zeta,\theta,\bar{g})$ on $\bar{M}$.

2. Preliminaries

Let (M,g) be a lightlike hypersurface, with a screen distribution $S(TM)$, of a semi-Riemannian manifold $(\bar{M},\bar{g})$ with a symmetric metric connection $\bar{\nabla}$ of type $(\ell,\, m)$. Then the normal bundle $TM^{\perp}$ of M is a subbundle of the tangent bundle TM of M and satisfies $TM = TM^{\perp}\oplus S(TM)$. Denote by $F(M)$ the algebra of smooth functions on M and by $\Gamma(E)$ the $F(M)$ module of smooth sections of a vector bundle E over M. For any null section ξ of $TM^{\perp}$ on a coordinate neighborhood $\mathcal{U}\subset M$, there exists a unique null section N of a unique vector bundle $tr(TM)$ in $S(TM)^{\perp}$ satisfying

$$\bar{g}(\xi,N)=1,\quad \bar{g}(N,N)=\bar{g}(N,X)=0,\quad \forall\, X\in\Gamma(S(TM)).$$

We call $tr(TM)$ and N the *transversal vector bundle* and the *null transversal vector field* of M with respect to $S(TM)$ respectively. In the following, we denote by X, Y, Z and W the smooth vector fields on M, unless otherwise specified.

As the tangent bundle $T\bar{M}$ of $\bar{M}$ is satisfied $T\bar{M} = TM\oplus tr(TM)$, the Gauss and Weingarten formulas of M are given by

$$\bar{\nabla}_X Y = \nabla_X Y + B(X,Y)N, \tag{2.1}$$

$$\bar{\nabla}_X N = -A_{_N} X + \tau(X)N, \tag{2.2}$$

respectively, where ∇ is the linear connection on M, B is the local second fundamental form on TM, $A_{_N}$ is its shape operator and τ is a 1-form on TM.

We note $TM = TM^{\perp}\oplus S(TM)$ and denote by P the projection morphism of TM on $S(TM)$. Then the Gauss and Weingarten formulas of $S(TM)$ are given by

$$\nabla_X PY = \nabla^*_X PY + C(X,PY)\xi, \tag{2.3}$$

$$\nabla_X \xi = -A^*_{\xi} X - \tau(X)\xi, \tag{2.4}$$

respectively, where ∇^* is the linear connection on $S(TM)$, C is the local screen second fundamental form of $S(TM)$, A^*_{ξ} is its shape operator.

Note that B and C are not symmetric. As $B(X,Y)=\bar{g}(\bar{\nabla}_X Y,\xi)$, we show that B is independent of the choice of $S(TM)$ and satisfies

$$B(X,\xi)=0. \tag{2.5}$$

The above second fundamental forms are related to their shape operators by

$$g(A^*_{\xi}X,Y)=B(X,Y),\qquad \bar{g}(A^*_{\xi}X,N)=0, \tag{2.6}$$

$$g(A_{_N}X,PY)=C(X,PY),\quad \bar{g}(A_{_N}X,N)=0. \tag{2.7}$$

Denote by $(2.6)_i$ the i-th equation of the two equations in (2.6). We use the same notations for any others.

The induced connection ∇ on M is not a metric one and satisfies

$$(\nabla_X g)(Y,Z) = B(X,Y)\eta(Z) + B(X,Z)\eta(Y), \tag{2.8}$$

where η is a 1-form on TM such that

$$\eta(X) = \bar{g}(X,N).$$

3. Symmetric metric connection of type (ℓ, m)

The definition of an indefinite trans-Sasakian manifold, with an indefinite trans-Sasakian structure (J, ζ, θ, $\bar{g}$) of type (α,β), was introduced by Oubina [14]. This definition on indefinite trans-Sasakian manifold was presented in the author's paper [10]. We quote Oubina's definition in itself as follow:

An odd-dimensional semi-Riemannian manifold ($\bar{M}$, $\bar{g}$) is called an *indefinite trans-Sasakian manifold* if there exists a set $\{J, \zeta, \theta, \bar{g}\}$ and two smooth functions α and β, where J is a $(1,1)$-type tensor field, ζ is a vector field which is called the *structure vector field* and θ is a 1-form such that

$$J^2\bar{X} = -\bar{X} + \theta(\bar{X})\zeta, \qquad \theta(\zeta) = 1, \qquad \theta\circ J = 0, \tag{3.1}$$
$$\theta(\bar{X}) = \epsilon\bar{g}(\bar{X},\zeta), \qquad \bar{g}(J\bar{X},J\bar{Y}) = \bar{g}(\bar{X},\bar{Y}) - \epsilon\theta(\bar{X})\theta(\bar{Y}),$$

$$(\bar{\nabla}_{\bar{X}}J)\bar{Y} = \alpha\{\bar{g}(\bar{X},\bar{Y})\zeta - \epsilon\theta(\bar{Y})\bar{X}\} + \beta\{\bar{g}(J\bar{X},\bar{Y})\zeta - \epsilon\theta(\bar{Y})J\bar{X}\}, \tag{3.2}$$

where $\epsilon = 1$ or -1 according as ζ is spacelike or timelike. In this case, the set $\{J, \zeta, \theta, \bar{g}\}$ is called an *indefinite trans-Sasakian structure of type* (α,β).

Note that [10] if $\beta = 0$, then $\bar{M}$ is called an *indefinite α-Sasakian manifold*. Indefinite Sasakian manifold is an example of indefinite α-Sasakian manifold such that $\alpha = 1$. If $\alpha = 0$, then $\bar{M}$ is called an *indefinite β-Kenmotsu manifold*. Indefinite Kenmotsu manifold is an example of indefinite β-Kenmotsu manifold such that $\beta = 1$. Indefinite cosymplectic manifold is an another important kind of indefinite trans-Sasakian manifold such that $\alpha = \beta = 0$.

From (3.1), we see that ζ is a timelike or spacelike unit vector field. In the sequel, we shall assume that ζ is a spacelike vector field, *i.e.*, $\epsilon = 1$, without loss generality. From (3.1) and (3.2), we get

$$\bar{\nabla}_{\bar{X}}\zeta = -\alpha J\bar{X} + \beta(\bar{X} - \theta(\bar{X})\zeta), \qquad d\theta(\bar{X},\bar{Y}) = \alpha g(\bar{X},\, J\bar{Y}). \tag{3.3}$$

It is known [9] that, for any lightlike hypersurface M of an indefinite almost contact metric manifold $\bar{M}$, $J(TM^{\perp})$ and $J(tr(TM))$ are subbundles of $S(TM)$, of rank 1. In the entire discussion of this article, we shall assume that ζ is tangent to M. Călin [3] proved that if ζ is tangent to M, then it belongs to $S(TM)$. Then there exist two non-degenerate almost complex distributions D_o and D with respect to J, *i.e.*, $J(D_o) = D_o$ and $J(D) = D$, such that

$$S(TM) = J(TM^{\perp}) \oplus J(tr(TM)) \oplus_{orth} D_o,$$
$$D = TM^{\perp} \oplus_{orth} J(TM^{\perp}) \oplus_{orth} D_o.$$

Using these distributions, TM is decomposed as follow:

$$TM = D \oplus J(tr(TM)).$$

Consider two null vector fields U and V and their 1-forms u and v such that

$$U = -JN, \quad V = -J\xi, \quad u(X) = g(X, V), \quad v(X) = g(X, U). \tag{3.4}$$

Denote by S the projection morphism of TM on D. Any vector field X of M is expressed as $X = SX + u(X)U$. Applying J to this form, we have

$$JX = FX + u(X)N, \tag{3.5}$$

where F is a tensor field of type $(1, 1)$ globally defined on M by $FX = JSX$. Applying J to (3.5) and using (3.1) and (3.4), we have

$$F^2X = -X + u(X)U + \theta(X)\zeta. \tag{3.6}$$

The vector field U is called the *structure vector field* of M. Applying $\bar{\nabla}_X$ to (3.4) and (3.5) and using (2.1)∼(2.7) and (3.1)∼(3.5), we get

$$B(X, U) = C(X, V) \equiv \sigma(X), \tag{3.7}$$

$$\nabla_X U = F(A_N X) + \tau(X)U - \{\alpha\eta(X) + \beta v(X)\}\zeta, \tag{3.8}$$

$$\nabla_X V = F(A^*_\xi X) - \tau(X)V - \beta u(X)\zeta, \tag{3.9}$$

$$\begin{aligned}(\nabla_X F)(Y) = \; & u(Y)A_N X - B(X, Y)U \\ & + \alpha\{g(X, Y)\zeta - \theta(Y)X\} \\ & + \beta\{\bar{g}(JX, Y)\zeta - \theta(Y)FX\},\end{aligned} \tag{3.10}$$

$$(\nabla_X u)(Y) = -u(Y)\tau(X) - B(X, FY) - \beta\theta(Y)u(X), \tag{3.11}$$

$$\begin{aligned}(\nabla_X v)(Y) = \; & v(Y)\tau(X) - g(A_N X, FY) \\ & - \theta(Y)\{\alpha\eta(X) + \beta v(X)\}.\end{aligned} \tag{3.12}$$

Let $\bar{M}$ be an indefinite trans-Sasakian manifold with a symmetric metric connection of type (ℓ, m). Substituting (2.1) and (3.5) into (1.1) and then, comparing the tangent and transversal components, we get

$$T(X, Y) = \ell\{\theta(Y)X - \theta(X)Y\} + m\{\theta(Y)FX - \theta(X)FY\}, \tag{3.13}$$

$$B(X, Y) - B(Y, X) = m\{\theta(Y)u(X) - \theta(X)u(Y)\}, \tag{3.14}$$

where T is the torsion tensor with respect to ∇. From (2.8) and (3.13), we see that ∇ is a symmetric non-metric connection of type (ℓ, m) in M. From (3.14), we also see that *the local second fundamental form B of M is symmetric, if and only if,* $m = 0$. Replacing X by ξ to (3.14) and then, using (2.5), we have

$$B(\xi, X) = 0, \qquad A^*_\xi\xi = 0. \tag{3.15}$$

Applying $\bar{\nabla}_X$ to $g(\zeta, \xi) = 0$ and $\bar{g}(\zeta, N) = 0$ by turns, we have

$$B(X, \zeta) = -\alpha u(X), \quad C(X, \zeta) = -\alpha v(X) + \beta\eta(X). \tag{3.16}$$

Substituting (3.5) into $(3.3)_1$ and using (2.1), we have

$$\nabla_X\zeta = -\alpha FX + \beta(X - \theta(X)\zeta). \tag{3.17}$$

Denote by $\bar{R}$, R and R^* the curvature tensors of the symmetric metric connection $\bar{\nabla}$ of type (ℓ, m) on $\bar{M}$, and the induced linear connections ∇ and

∇^* on M and $S(TM)$, respectively. Using (3.13) and the Gauss-Weingarten formulas, we obtain the Gauss-Codazzi equations for M and $S(TM)$:

$$\begin{aligned} \bar{R}(X,Y)Z = {} & R(X,Y)Z + B(X,Z)A_{N}Y - B(Y,Z)A_{N}X \\ & + \{(\nabla_X B)(Y,Z) - (\nabla_Y B)(X,Z) \\ & + \tau(X)B(Y,Z) - \tau(Y)B(X,Z) \\ & - \ell[\theta(X)B(Y,Z) - \theta(Y)B(X,Z)] \\ & - m[\theta(X)B(FY,Z) - \theta(Y)B(FX,Z)]\}N, \end{aligned} \tag{3.18}$$

$$\begin{aligned} \bar{R}(X,Y)N = {} & -\nabla_X(A_{N}Y) + \nabla_Y(A_{N}X) + A_{N}[X,Y] \\ & + \tau(X)A_{N}Y - \tau(Y)A_{N}X \\ & + \{B(Y,A_{N}X) - B(X,A_{N}Y) + 2d\tau(X,Y)\}N, \end{aligned} \tag{3.19}$$

$$\begin{aligned} R(X,Y)PZ = {} & R^*(X,Y)PZ + C(X,PZ)A^*_\xi Y - C(Y,PZ)A^*_\xi X \\ & + \{(\nabla_X C)(Y,PZ) - (\nabla_Y C)(X,PZ) \\ & - \tau(X)C(Y,PZ) + \tau(Y)C(X,PZ) \\ & - \ell[\theta(X)C(Y,PZ) - \theta(Y)C(X,PZ)] \\ & - m[\theta(X)C(FY,PZ) - \theta(Y)C(FX,PZ)]\}\xi. \end{aligned} \tag{3.20}$$

$$\begin{aligned} R(X,Y)\xi = {} & -\nabla^*_X(A^*_\xi Y) + \nabla^*_Y(A^*_\xi X) + A^*_\xi[X,Y] \\ & - \tau(X)A^*_\xi Y + \tau(Y)A^*_\xi X \\ & + \{C(Y,A^*_\xi X) - C(X,A^*_\xi Y) - 2d\tau(X,Y)\}\xi. \end{aligned} \tag{3.21}$$

4. Indefinite generalized Sasakian space forms

Definition. An indefinite trans-Sasakian manifold $(\bar{M}, J, \zeta, \theta, \bar{g})$ is called an *indefinite generalized Sasakian space form* [1], denoted by $\bar{M}(f_1, f_2, f_3)$, if there exist three smooth functions f_1, f_2 and f_3 on $\bar{M}$ such that

$$\begin{aligned} \bar{R}(\bar{X},\bar{Y})\bar{Z} = {} & f_1\{\bar{g}(\bar{Y},\bar{Z})\bar{X} - \bar{g}(\bar{X},\bar{Z})\bar{Y}\} \\ & + f_2\{\bar{g}(\bar{X},J\bar{Z})J\bar{Y} - \bar{g}(\bar{Y},J\bar{Z})J\bar{X} + 2\bar{g}(\bar{X},J\bar{Y})J\bar{Z}\} \\ & + f_3\{\theta(\bar{X})\theta(\bar{Z})\bar{Y} - \theta(\bar{Y})\theta(\bar{Z})\bar{X} \\ & + \bar{g}(\bar{X},\bar{Z})\theta(\bar{Y})\zeta - \bar{g}(\bar{Y},\bar{Z})\theta(\bar{X})\zeta\}. \end{aligned} \tag{4.1}$$

Note that indefinite Sasakian, Kenmotsu and cosymplectic space forms are important kinds of indefinite generalized Sasakian space forms such that

$$f_1 = \tfrac{c+3}{4}, f_2 = f_3 = \tfrac{c-1}{4}; \quad f_1 = \tfrac{c-3}{4}, f_2 = f_3 = \tfrac{c+1}{4}; \quad f_1 = f_2 = f_3 = \tfrac{c}{4}$$

respectively, where c is a constant J-sectional curvature of each space forms.

Theorem 4.1. *Let M be a lightlike hypersurface of an indefinite generalized Sasakian space form $\bar{M}(f_1, f_2, f_3)$ with a symmetric metric connection of type (ℓ, m). Then the following properties are satisfied*

(1) *α is a constant,*

(2) $\alpha\beta = 0$ *and* $\alpha\ell = \beta m = 0$,
(3) $f_1 - f_2 = \alpha^2 - \beta^2$ *and* $f_1 - f_3 = (\alpha^2 - \beta^2) + \alpha m + \beta\ell - \zeta\beta$.

Proof. Comparing the tangential and transversal components of (3.18) and (4.1) and using (3.5) and the fact that ζ is tangent to M, we get

$$\begin{aligned} R(X,Y)Z \;=\; & f_1\{g(Y,Z)X - g(X,Z)Y\} \\ & + f_2\{\bar{g}(X,JZ)FY - \bar{g}(Y,JZ)FX + 2\bar{g}(X,JY)FZ\} \\ & + f_3\{\theta(X)\theta(Z)Y - \theta(Y)\theta(Z)X + \bar{g}(X,Z)\theta(Y)\zeta \\ & - \bar{g}(Y,Z)\theta(X)\zeta\} \;+\; B(Y,Z)A_{N}X - B(X,Z)A_{N}Y, \end{aligned} \tag{4.2}$$

$$\begin{aligned} & (\nabla_X B)(Y,Z) - (\nabla_Y B)(X,Z) + \tau(X)B(Y,Z) - \tau(Y)B(X,Z) \\ & - \ell\{\theta(X)B(Y,Z) - \theta(Y)B(X,Z)\} \\ & - m\{\theta(X)B(FY,Z) - \theta(Y)B(FX,Z)\} \\ = \; & f_2\{u(Y)\bar{g}(X,JZ) - u(X)\bar{g}(Y,JZ) + 2u(Z)\bar{g}(X,JY)\}. \end{aligned} \tag{4.3}$$

Substituting (3.9) into $R(X,Y)V = \nabla_X\nabla_Y V - \nabla_Y\nabla_X V - \nabla_{[X,Y]}V$ and using (2.6), (3.4), (3.5), (3.9)∼(3.14), (3.16), (3.17) and (3.21), we have

$$\begin{aligned} R(X,Y)V \;=\; & B(Y,V)A_{N}X - B(X,V)A_{N}Y - F(R(X,Y)\xi) \\ & + (\alpha^2 - \beta^2)\{u(Y)X - u(X)Y\} \\ & + 2\alpha\beta\{u(Y)FX - u(X)FY\} \\ & + \{-(X\beta)u(Y) + (Y\beta)u(X) \\ & + (\alpha m + \beta\ell)[\theta(X)u(Y) - \theta(Y)u(X)]\}\zeta. \end{aligned}$$

Substituting (4.2) into the left term of this equation, we have

$$\begin{aligned} & F(R(X,Y)\xi) + (f_1 - \alpha^2 + \beta^2)\{u(Y)X - u(X)Y\} \\ & - 2\alpha\beta\{u(Y)FX - u(X)FY\} + 2f_2\bar{g}(X,JY)\xi \\ & + \{(X\beta)u(Y) - (Y\beta)u(X) \\ & + (f_3 + \alpha m + \beta\ell)[u(X)\theta(Y) - u(Y)\theta(X)]\}\zeta = 0. \end{aligned} \tag{4.4}$$

Taking the scalar product with N to (4.4), we obtain

$$\begin{aligned} g(R(X,Y)\xi,\, U) = \; & (f_1 - \alpha^2 + \beta^2)\{u(X)\eta(Y) - u(Y)\eta(X)\} \\ & + \; 2\alpha\beta\{u(Y)v(X) - u(X)v(Y)\} - 2f_2\bar{g}(X,JY). \end{aligned}$$

Taking $X = U$, $Y = \xi$ and $X = U$, $Y = V$ by turns and using (4.2), we have

$$f_1 - f_2 = \alpha^2 - \beta^2, \qquad \alpha\beta = 0, \tag{4.5}$$

due to the facts that $R(U,\xi)\xi = 3f_2V$ and $R(U,V)\xi = -f_2\xi$. Taking the scalar product with ζ to (4.4) and using the fact that $g(FX,\zeta) = 0$, we have

$$\begin{aligned} & (X\beta)u(Y) - (Y\beta)u(X) \\ & + \{f_1 - f_3 - (\alpha^2 - \beta^2) - \alpha m - \beta\ell\}[u(Y)\theta(X) - u(X)\theta(Y)] = 0. \end{aligned}$$

Replacing Y by U to this equation and then, taking $X = \zeta$, we have

$$X\beta + \{f_1 - f_3 - (\alpha^2 - \beta^2) - \alpha m - \beta\ell\}\theta(X) = (U\beta)u(X), \tag{4.6}$$

$$f_1 - f_3 = (\alpha^2 - \beta^2) + \alpha m + \beta\ell - \zeta\beta. \tag{4.7}$$

Substituting (3.17) into $R(X,Y)\zeta = \nabla_X\nabla_Y\zeta - \nabla_Y\nabla_X\zeta - \nabla_{[X,Y]}\zeta$ and using $(3.3)_2$, (3.6), (3.10), (3.13), (3.14), (3.17) and $(4.5)_2$, we have

$$\begin{aligned} R(X,Y)\zeta = &-(X\alpha)FY + (Y\alpha)FX + (X\beta)Y - (Y\beta)X \\ &+ \alpha\{u(X)A_{_N}Y - u(Y)A_{_N}X\} \\ &+ (\alpha^2 - \beta^2 + \alpha m + \beta\ell)\{\theta(Y)X - \theta(X)Y\} \\ &- (\alpha\ell - \beta m)\{\theta(Y)FX - \theta(X)FY\} \\ &- \{(X\beta)\theta(Y) - (Y\beta)\theta(X)\}\zeta. \end{aligned}$$

Substituting (4.2) into this equation and using $(4.5)_1$ and (4.7), we have

$$\begin{aligned} &(X\alpha)FY - (Y\alpha)FX - (X\beta)Y + (Y\beta)X \\ &- (\zeta\beta)\{\theta(Y)X - \theta(X)Y\} + (\alpha\ell - \beta m)\{\theta(Y)FX - \theta(X)FY\} \\ &+ \{(X\beta)\theta(Y) - (Y\beta)\theta(X)\}\zeta = 0. \end{aligned}$$

Taking the scalar product with U to this and using $g(FX,U) = -\eta(X)$, we get

$$\begin{aligned} &\{X\alpha - (\alpha\ell - \beta m)\theta(X)\}\eta(Y) - \{Y\alpha - (\alpha\ell - \beta m)\theta(Y)\}\eta(X) \\ &+ \{X\beta - (\zeta\beta)\theta(X)\}v(Y) - \{Y\beta - (\zeta\beta)\theta(Y)\}v(X) = 0. \end{aligned}$$

Taking $X = U$, $Y = \xi$ and $X = U$, $Y = V$ to this by turns, we obtain

$$U\alpha = 0, \qquad U\beta = 0. \tag{4.8}$$

From (4.6), (4.7) and $(4.8)_2$, we obtain

$$X\beta = (\zeta\beta)\theta(X). \tag{4.9}$$

Applying ∇_Y to $(3.16)_1$ and using (3.11), (3.16), (3.17) and $(4.5)_2$, we have

$$\begin{aligned} (\nabla_X B)(Y,\zeta) = &-(X\alpha)u(Y) - \beta B(Y,X) \\ &+ \alpha\{u(Y)\tau(X) + B(X,FY) + B(Y,FX)\}. \end{aligned}$$

Substituting this into (4.3) with $Z = \zeta$ and using (3.14) and (3.16), we have

$$\{X\alpha - (\alpha\ell - \beta m)\theta(X)\}u(Y) = \{Y\alpha - (\alpha\ell - \beta m)\theta(Y)\}u(X).$$

Replacing Y by U to this equation and using $(4.8)_1$, we have

$$X\alpha = \{\alpha\ell - \beta m\}\theta(X). \tag{4.10}$$

Substituting (4.9) into $T(X,Y)\beta = X(Y\beta) - Y(X\beta) - [X,Y]\beta$, we have

$$T(X,Y)\beta = X(\zeta\beta)\theta(Y) - Y(\zeta\beta)\theta(X) + 2(\zeta\beta)d\theta(X,Y).$$

Substituting $(3.3)_2$ and (3.13) into this equation and using (4.9), we get

$$X(\zeta\beta)\theta(Y) - Y(\zeta\beta)\theta(X) + 2\alpha(\zeta\beta)\bar{g}(X,JY) = 0,$$

due to $\theta \circ F = 0$. Taking $X = U$ and $Y = \xi$ to this equation, we obtain

$$\alpha(\zeta\beta) = 0. \tag{4.11}$$

Applying ∇_X to $\alpha\beta = 0$ and using (4.9), (4.10) and (4.11), we get $\beta m = 0$. Substituting (4.10) into $T(X,Y)\alpha = X(Y\alpha) - Y(X\alpha) - [X,Y]\alpha$, we get

$$T(X,Y)\alpha = \alpha\{X(\ell)\theta(Y) - Y(\ell)\theta(X)\} + 2\alpha\ell d\theta(X,Y).$$

Substituting $(3.3)_2$ and (3.13) into this equation and using

$$\alpha\{(X\ell)\theta(Y) - (Y\ell)\theta(X)\} + 2\alpha^2\ell\bar{g}(X,JY) = 0.$$

Taking $X = U$ and $Y = \xi$ to this equation, we have $\alpha\ell = 0$. As $\alpha\ell = 0$ and $\beta m = 0$, from (4.10), we see that α is a constant. □

Definition. (1) A screen distribution $S(TM)$ is called *totally umbilical* [4] in M if there exist a smooth function γ such that $A_N X = \gamma PX$, or equivalently,

$$C(X,PY) = \gamma g(X,Y). \tag{4.12}$$

In case $\gamma = 0$, we say that $S(TM)$ is *totally geodesic* in M.

(2) A lightlike hypersurface M is called *screen conformal* [2] if there exist a non-vanishing smooth function φ such that $A_N = \varphi A^*_\xi$, or equivalently,

$$C(X,PY) = \varphi B(X,Y). \tag{4.13}$$

Theorem 4.2. *Let M be a lightlike hypersurfaces of an indefinite generalized Sasakian space form $\bar{M}(f_1, f_2, f_3)$ with a symmetric metric connection of type (ℓ, m). If one of the following four conditions*

(1) *$S(TM)$ is totally umbilical,*
(2) *M is screen conformal,*
(3) *F is parallel with respect to ∇, and*
(4) *U is parallel with respect to ∇*

is satisfied, then $\bar{M}(f_1, f_2, f_3)$ is a flat manifold with an indefinite cosymplectic structure, i.e., $f_1 = f_2 = f_3 = 0$ and $\alpha = \beta = 0$. Moreover, in cases (1), (3), *M is also flat.*

Proof. (1) If $S(TM)$ is totally umbilical, then $(3.16)_2$ is reduced to

$$\gamma\theta(X) = -\alpha v(X) + \beta\eta(X).$$

Taking $X = \zeta$, $X = V$ and $X = \xi$ by turns, we have $\gamma = 0$, $\alpha = 0$ and $\beta = 0$ respectively. As $\gamma = 0$, $S(TM)$ is totally geodesic in M.

As $\alpha = \beta = 0$, $\bar{M}$ is an indefinite cosymplectic manifold and $f_1 = f_2 = f_3$ by Theorem 4.1. As $C = A_N = 0$, using (3.7) and (3.8) we see that

$$B(X,U) = 0, \qquad (\nabla_X B)(Y,U) = 0.$$

Taking $Z = U$ to (4.3) and using the last equations, we have

$$f_2\{u(Y)\eta(X) - u(X)\eta(Y) + \bar{g}(X,JY)\} = 0.$$

Taking $X = \xi$ and $Y = U$ to this equation, we get $f_2 = 0$. Therefore, $f_1 = f_2 = f_3 = 0$ and $\bar{M}(f_1, f_2, f_3)$ is flat. From (4.2) and the facts that $f_1 = f_2 = f_3 = 0$ and $A_{_N} = 0$, we see that $R = 0$. Thus M is also flat.

(2) Taking $PY = \zeta$ to (4.13) and using (3.16), we get

$$\alpha v(X) - \beta\eta(X) = \alpha\varphi u(X).$$

Taking $X = V$ and $X = \xi$ by turns, we have $\alpha = 0$ and $\beta = 0$ respectively. Thus $\bar{M}$ is an indefinite cosymplectic manifold and $f_1 = f_2 = f_3$. Taking $X = U$ and $Y = V$ to (3.14) and using $\theta \circ J = 0$, we show that

$$B(U, V) = B(V, U).$$

Taking the scalar product with N to (3.20) and then, substituting (4.2) into the resulting equation, we obtain

$$\begin{aligned}(4.14)\qquad &(\nabla_X C)(Y, PZ) - (\nabla_Y C)(X, PZ)\\ &- \tau(X)C(Y, PZ) + \tau(Y)C(X, PZ)\\ &- \ell\{\theta(X)C(Y, PZ) - \theta(Y)C(X, PZ)\}\\ &- m\{\theta(X)C(FY, PZ) - \theta(Y)C(FX, PZ)\}\\ =\ & f_1\{g(Y, PZ)\eta(X) - g(X, PZ)\eta(Y)\}\\ &+ f_2\{v(Y)\bar{g}(X, JPZ) - v(X)\bar{g}(Y, JPZ) + 2v(PZ)\bar{g}(X, JY)\}\\ &+ f_3\{\theta(X)\eta(Y) - \theta(Y)\eta(X)\}\theta(PZ).\end{aligned}$$

Applying ∇_X to $C(Y, PZ) = \varphi B(Y, PZ)$, we have

$$(\nabla_X C)(Y, PZ) = (X\varphi)B(Y, PZ) + \varphi(\nabla_X B)(Y, PZ).$$

Substituting this into (4.14) and using (4.3), we have

$$\begin{aligned}&\{X\varphi - 2\varphi\tau(X)\}B(Y, PZ) - \{Y\varphi - 2\varphi\tau(Y)\}B(X, PZ)\\ =\ & f_1\{g(Y, PZ)\eta(X) - g(X, PZ)\eta(Y)\}\\ &+ f_2\{[v(Y) - \varphi u(Y)]\bar{g}(X, JPZ) - [v(X) - \varphi u(X)]\bar{g}(Y, JPZ)\\ &+ 2[v(PZ) - \varphi u(PZ)]\bar{g}(X, JY)\}\\ &+ f_3\{\theta(X)\eta(Y) - \theta(Y)\eta(X)\}\theta(PZ).\end{aligned}$$

Replacing Y by ξ to the last equation and using (3.15), we obtain

$$\begin{aligned}&\{\xi\varphi - 2\varphi\tau(\xi)\}B(X, Y)\\ =\ & f_1 g(X, Y) + f_2\{v(X) - \varphi u(X)\}u(Y)\\ &+ 2f_2\{v(Y) - \varphi u(Y)\}u(X) - f_3\theta(X)\theta(Y).\end{aligned}$$

Taking $X = V$, $Y = U$ and then, $X = U$, $Y = V$ by turns, we have

$$\{\xi\varphi - 2\varphi\tau(\xi)\}B(V, U) = f_1 + f_2,$$
$$\{\xi\varphi - 2\varphi\tau(\xi)\}B(U, V) = f_1 + 2f_2,$$

respectively. From these two equations we show that $f_2 = 0$. Thus $f_1 = f_2 = f_3 = 0$ and $\bar{M}(f_1, f_2, f_3)$ is flat.

(3) If F is parallel with respect to ∇, then (3.10) is reduced to

$$u(Y)A_{N}X - B(X,Y)U + \alpha\{g(X,Y)\zeta - \theta(Y)X\} + \beta\{\bar{g}(JX,Y)\zeta - \theta(Y)FX\} = 0. \tag{4.15}$$

Taking the scalar product with N to (4.15), we get $\alpha\eta(X) + \beta v(X) = 0$. From this equation, we obtain $\alpha = 0$ and $\beta = 0$ respectively. Thus $\bar{M}$ is an indefinite cosymplectic manifold and $f_1 = f_2 = f_3$ by Theorem 4.1.

Replacing Y by U to (4.15) such that $\alpha = \beta = 0$ and using (3.7), we get

$$A_{N}X = \sigma(X)U. \tag{4.16}$$

Taking the scalar product with V to (4.15), we get $g(A^*_\xi X, Y) = g(\sigma(X)V, Y)$. As $A^*_\xi X$ and V belong to $S(TM)$, and $S(TM)$ is non-degenerate, we get

$$A^*_\xi X = \sigma(X)V. \tag{4.17}$$

Taking the scalar product with U to (4.16) and using (2.7), we get

$$C(X,U) = 0.$$

Applying ∇_X to $C(Y,U) = 0$ and using (3.8), (4.16) and $FU = 0$, we get

$$(\nabla_X C)(Y,U) = 0.$$

Substituting the last two equation into (4.14) with $PZ = U$, we have

$$(f_1 + f_2)\{v(Y)\eta(X) - v(X)\eta(Y)\} = 0.$$

Taking $X = V$ and $Y = \xi$ to this equation, we obtain $f_1 + f_2 = 0$. Therefore, $f_1 = f_2 = f_3 = 0$ and $\bar{M}(f_1, f_2, f_3)$ is flat. Substituting (4.16) and (4.17) into (4.2) and using the fact that $f_1 = f_2 = f_3 = 0$, we have

$$R(X,Y)Z = \{\sigma(Y)\sigma(X) - \sigma(X)\sigma(Y)\}u(Z)U = 0$$

for all X, Y, $Z \in \Gamma(TM)$. Therefore $R = 0$ and M is also flat.

(4) If U is parallel with respect to ∇, then, from (3.5) and (3.8), we have

$$J(A_{N}X) - u(A_{N}X)N + \tau(X)U - \{\alpha\eta(X) + \beta v(X)\}\zeta. \tag{4.18}$$

Taking the scalar product with ζ and V by turns, we get $\alpha\eta(X) + \beta v(X) = 0$ and $\tau = 0$. Taking $X = \xi$ and $X = V$ to the first result by turns, we have $\alpha = 0$ and $\beta = 0$ respectively. Thus $\bar{M}$ is an indefinite cosymplectic manifold and $f_1 = f_2 = f_3$ by Theorem 4.1.

Applying J to (4.18) and using (2.1), (3.7) and $(3.16)_2$, we obtain

$$A_{N}X = \sigma(X)U. \tag{4.19}$$

Taking the scalar product with U to (4.19), we get

$$C(X,U) = 0.$$

Applying ∇_X to $C(Y,U) = 0$ and using (3.8) and (4.19), we obtain

$$(\nabla_X C)(Y,U) = 0.$$

Substituting the last two equation into (4.14) with $PZ = U$, we have

$$(f_1 + f_2)\{v(Y)\eta(X) - v(X)\eta(Y)\} = 0.$$

Taking $X = V$ and $Y = \xi$ to this equation, we obtain $f_1 + f_2 = 0$. Therefore, $f_1 = f_2 = f_3 = 0$ and $\bar{M}(f_1, f_2, f_3)$ is flat. □

Theorem 4.3. *Let M be a lightlike hypersurfaces of an indefinite generalized Sasakian space form $\bar{M}(f_1, f_2, f_3)$ with a symmetric metric connection of type (ℓ, m). If V is parallel with respect to ∇, then $\bar{M}(f_1, f_2, f_3)$ is a space form with an indefinite α-Sasakian structure such that $\alpha = -m$ and*

$$f_1 = f_3 = \frac{2}{3}\alpha^2, \qquad f_2 = -\frac{1}{3}\alpha^2.$$

Proof. If V is parallel with respect ∇, then, from (3.5) and (3.9), we have

$$J(A^*_\xi X) - u(A^*_\xi X)N - \tau(X)V - \beta u(X)\zeta = 0. \tag{4.20}$$

Taking the scalar product with ζ and U to (4.20) by turns, we have $\beta = 0$ and $\tau = 0$ respectively. Applying J to (4.20) and using (3.1) and $(3.16)_1$, we obtain

$$A^*_\xi X = -\alpha u(X)\zeta + u(A^*_\xi X)U. \tag{4.21}$$

Taking the scalar product with U to this equation, we obtain

$$B(X, U) = 0. \tag{4.22}$$

Replacing Y by U to (3.14) and using the fact that $B(X, U) = 0$, we have

$$B(U, X) = m\theta(X). \tag{4.23}$$

Taking $X = U$ to $(3.16)_1$ and using (4.23), we get

$$\alpha = \alpha u(U) = -B(U, \zeta) = -m\theta(\zeta) = -m.$$

Thus $\bar{M}$ is an indefinite α-Sasakian manifold such that $\alpha = -m$.

Applying ∇_Y to (4.22) and using (3.8), $(3.16)_1$ and (4.21), we have

$$(\nabla_X B)(Y, U) = -\alpha^2\eta(X)u(Y).$$

Substituting the last equation and (4.22) into (4.3) with $Z = U$, we obtain

$$\alpha^2\{u(X)\eta(Y) - u(Y)\eta(X)\} = f_2\{u(Y)\eta(X) - u(X)\eta(Y) + 2\bar{g}(X, JY)\}.$$

Taking $X = \xi$ and $Y = U$, we obtain $3f_2 = -\alpha^2$. From this result and the first two equations of (3) in Theorem 4.1, we get

$$f_1 = f_3 = \frac{2}{3}\alpha^2, \qquad f_2 = -\frac{1}{3}\alpha^2.$$

□

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