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EXPONENTIAL STABILITY OF INFINITE DIMENSIONAL LINEAR SYSTEMS

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Abstract. In this paper, we show that if $\mathcal{A}$ is a differential subalgebra of Banach algebras $\mathcal{B}(\ell^r)$, $1 \le r \le \infty$, then solutions of the infinite dimensional linear system associated with a matrix in $\mathcal{A}$ have its p-exponential stability being equivalent to each other for different $1 \le p \le \infty$.

1. Introduction

In this paper, we consider the following linear system associated with an infinite-dimensional matrix A,

$$\frac{d}{dt}T(t) = AT(t) \quad \text{and} \quad T(0) = I, \tag{1.1}$$

where I is the identity matrix. The above linear system (1.1) is said to be *p-exponentially stable* if there exist constants D and $\alpha > 0$ such that

$$\|T(t)\|_{\mathcal{B}(\ell^p)} \le De^{-\alpha t} \quad \text{for all} \quad t \ge 0, \tag{1.2}$$

where ℓ^p, $1 \le p \le \infty$, is the space of all p-summable sequences with its norm denoted by $\|\cdot\|_p$, and $\mathcal{B}(\ell^p)$ is the space of bounded linear operators on ℓ^p with its norm denoted by $\|\cdot\|_{\mathcal{B}(\ell^p)}$ ([1]). In finite-dimensional setting, the linear system (1.1) has the p-exponential stability with $p = 2$ if and only if all eigenvalues of the matrix A have negative real parts. The above characterization of p-exponential stability plays a crucial role to solve the Lyapunov equation

$$A^T P + PA + Q = 0, \tag{1.3}$$

where Q is a positive definite matrix and all eigenvalues of the matrix A have negative real parts ([5, 8]). In infinite-dimensional setting, it was shown in [7] and [8] that the p-exponential stability with $p = 2$ implies the existence of a unique solution of Lyapunov equation, provided that the matrix A has additional off-diagonal decay.

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Let $0 \le \theta < 1$ and $\mathcal{B}$ be a Banach algebra of matrices. We say that a matrix algebra $\mathcal{A}$ with norm $\|\cdot\|_{\mathcal{A}}$ is *a differential subalgebra* of the Banach algebra $\mathcal{B}$ with order θ if there exist positive constants C and C' such that

$$\|AB\|_{\mathcal{A}} \le C\|A\|_{\mathcal{A}}\|B\|_{\mathcal{A}}\left(\left(\frac{\|A\|_{\mathcal{B}}}{\|A\|_{\mathcal{A}}}\right)^{1-\theta} + \left(\frac{\|B\|_{\mathcal{B}}}{\|B\|_{\mathcal{A}}}\right)^{1-\theta}\right), \tag{1.4}$$

and

$$\|A\|_{\mathcal{B}} \le C'\|A\|_{\mathcal{A}} \tag{1.5}$$

for all $A, B \in \mathcal{A}$. The aim of this paper is to show that if $\mathcal{A}$ is a differential subalgebra of Banach algebras $\mathcal{B}(\ell^r)$, $1 \le r \le \infty$, with order $\theta \in [0, 1)$ and if the linear system (1.1) is p-exponentially stable for some $1 \le p \le \infty$, then the linear system (1.1) is q-exponentailly stable for any $1 \le q \le \infty$.

The paper is organized as follows. In Section 2, we introduce some differential subalgebras of $\mathcal{B}(\ell^p)$, including Gröchenig-Schur class, Gohberg-Baskakov-Sjöstrand class, and Beurling class. In Section 3, we show that if an infinite matrix A satisfies (1.4) and (1.5), then p-exponential stability of the associated linear system (1.1) are equivalent to each other for different $1 \le p \le \infty$.

In this paper, the capital C is an absolute constant which may be different at each occurrence.

2. Differentiable matrix algebras

In this section, we introduce some matrix algebras which are differential subalgebras of $\mathcal{B}(\ell^r)$, $1 \le r \le \infty$.

A weight u in this paper is a matrix on $\mathbb{Z}^d \times \mathbb{Z}^d$ with

$$1 \le u(i,j) = u(j,i) < \infty \tag{2.1}$$

and

$$D(u) := \sup_{i\in\mathbb{Z}^d} u(i,i) < \infty. \tag{2.2}$$

For a matrix a and a weight u, denote their entry multiplication by

$$au := ((au)(i,j))_{i,j\in\mathbb{Z}^d} = (a(i,j)u(i,j))_{i,j\in\mathbb{Z}^d},$$

and entry reciprocal by

$$u^{-1} = ((u(i,j))^{-1})_{i,j\in\mathbb{Z}^d}.$$

Denote by $|\cdot|_\infty$ the infinite norm on the d-dimensional Euclidean space $\mathbf{R}^d$. In the next definition, we introduce some matrix algebras.

Definition 2.1. Let $1 \le p \le \infty$ and u be a weight. The *Gröchenig-Schur class*

$$\mathcal{A}_{p,u} := \{A : \|A\|_{\mathcal{A}_{p,u}} < \infty\} \tag{2.3}$$

contains all matrices $A := (a(i,j))_{i,j\in\mathbb{Z}^d}$ with

$$\|A\|_{\mathcal{A}_{p,u}} := \max\Big\{\sup_{i\in\mathbb{Z}^d}\|((au)(i,j))_{j\in\mathbb{Z}^d}\|_p, \sup_{j\in\mathbb{Z}^d}\|((au)(i,j))_{i\in\mathbb{Z}^d}\|_p\Big\} < \infty; \tag{2.4}$$

the *Gohberg-Baskakov-Sjöstrand class*

$$\mathcal{C}_{p,u} := \{A : \|A\|_{\mathcal{C}_{p,u}} < \infty\} \tag{2.5}$$

includes matrices A with its norm

$$\|A\|_{\mathcal{C}_{p,u}} := \Big\|\Big(\sup_{i-j=k}|(au)(i,j)|\Big)_{k\in\mathbb{Z}^d}\Big\|_p \tag{2.6}$$

being finite; and the *Beurling class* is the set

$$\mathcal{B}_{p,u} := \{A : \|A\|_{\mathcal{B}_{p,u}} < \infty\}, \tag{2.7}$$

where

$$\|A\|_{\mathcal{B}_{p,u}} := \Big\|\Big(\sup_{|i-j|_\infty\ge|k|_\infty}|(au)(i,j)|\Big)_{k\in\mathbb{Z}^d}\Big\|_p. \tag{2.8}$$

From the above definition, we have

$$\mathcal{B}_{p,u} \subset \mathcal{C}_{p,u} \subset \mathcal{A}_{p,u} \tag{2.9}$$

and

$$\|A\|_{\mathcal{A}_{p,u}} \le \|A\|_{\mathcal{C}_{p,u}} \le \|A\|_{\mathcal{B}_{p,u}} \quad A\in\mathcal{B}_{p,u}. \tag{2.10}$$

The reader may refer to [2, 3, 4, 6, 10, 11, 12, 13] for historical remarks and more properties of the above three classes of matrices.

For $1\le p\le\infty$, a weight u is called a *p-submultiplicative weight* if there exists another weight v satisfying

$$u(i,j) \le u(i,k)v(k,j) + v(i,k)u(k,j) \quad \text{for all} \quad i,j,k\in\mathbb{Z}^d, \tag{2.11}$$

and one of the following three conditions:

$$\sup_{i\in\mathbb{Z}^d}\big\|\big((vu^{-1})(i,j)\big)_{j\in\mathbb{Z}^d}\big\|_{p/(p-1)} + \sup_{j\in\mathbb{Z}^d}\big\|\big((vu^{-1})(i,j)\big)_{i\in\mathbb{Z}^d}\big\|_{p/(p-1)} < \infty \tag{2.12}$$

for the Gröchenig-Schur class $\mathcal{A}_{p,u}$;

$$\big\|\big(\sup_{i-j=k}(vu^{-1})(i,j)\big)_{k\in\mathbb{Z}^d}\big\|_{p/(p-1)} < \infty \tag{2.13}$$

for the Gohberg-Baskakov-Sjöstrand class $\mathcal{C}_{p,u}$; and

$$\big\|\big(\sup_{|i-j|_\infty\ge|k|_\infty}(vu^{-1})(i,j)\big)_{k\in\mathbb{Z}^d}\big\|_{p/(p-1)} < \infty \tag{2.14}$$

for the Beurling class $\mathcal{B}_{p,u}$. For a p-submultiplicative weight u, we call the weight v in (2.11) *a companion weight* of u.

Lemma 2.2. *Let $1\le p\le\infty$, u be a weight, and $\mathcal{F}_{p,u}$ be one of three classes $\mathcal{A}_{p,u}$, $\mathcal{C}_{p,u}$ and $\mathcal{B}_{p,u}$. Then the following statements hold.*

(i) *If* $\|u^{-1}\|_{\mathcal{F}_{p/(p-1),u_0}} < \infty$ *for the trivial weight* u_0 *having every entry to be* 1*, then*

$$\|A\|_{\mathcal{B}(\ell^r)} \le \|u^{-1}\|_{\mathcal{F}_{p/(p-1),u_0}} \|A\|_{\mathcal{F}_{p,u}}, \ 1 \le r \le \infty. \tag{2.15}$$

(ii) *If* u *is a* p*-submultiplicative weight and* v *is a companion weight of* u*, then there exists a positive constant* C *such that for all* $A, B \in \mathcal{F}_{p,u}$

$$\|AB\|_{\mathcal{F}_{p,u}} \le C(\|A\|_{\mathcal{F}_{p,u}} \|B\|_{\mathcal{F}_{1,v}} + \|A\|_{\mathcal{F}_{1,v}} \|B\|_{\mathcal{F}_{p,u}}), \tag{2.16}$$

where AB *is the matrix multiplication.*

Proof. (i) It is well-known that

$$\|A\|_{\mathcal{B}(\ell^r)} \le \max\Big(\sup_{j\in\mathbb{Z}} \sum_{i\in\mathbb{Z}^d} |a(i,j)|, \sup_{i\in\mathbb{Z}} \sum_{j\in\mathbb{Z}^d} |a(i,j)|\Big), \ 1 \le r \le \infty.$$

Since for any $A \in \mathcal{A}_{p,u}$,

$$\|A\|_{\mathcal{B}(\ell^r)} \le \|A\|_{\mathcal{A}_{1,u_0}} \le \|u^{-1}\|_{\mathcal{A}_{p/(p-1),u_0}} \|A\|_{\mathcal{A}_{p,u}},$$

this together with (2.10) proves the conclusion (i).

(ii) We prove the conclusion for the Gohberg-Baskakov-Sjöstrand class only. The reader may follow similar argument to prove the conclusion (ii) for the Gröchenig-Schur class and for the Beurling class. Take $A := (a(i,j))_{i,j\in\mathbb{Z}^d}, B := (b(i,j))_{i,j\in\mathbb{Z}^d}$ in $\mathcal{C}_{p,u}$, and write $AB := (c(i,j))_{i,j\in\mathbb{Z}^d}$. Note from (2.11) that

$$\begin{aligned} |(cu)(i,j)| &= \Big|\sum_{\ell\in\mathbb{Z}^d} a(i,\ell)b(\ell,j)u(i,j)\Big| \\ &\le \sum_{\ell\in\mathbb{Z}^d} |(au)(i,\ell)||(bv)(\ell,j)| + \sum_{\ell\in\mathbb{Z}^d} |(av)(i,\ell)||(bu)(\ell,j)|. \end{aligned} \tag{2.17}$$

We write $\hat{a}(k) = \sup_{j\in\mathbb{Z}^d} |(au)(j+k,j)|$. Since

$$\sum_{k\in\mathbb{Z}^d} \sup_{j\in\mathbb{Z}^d} |(au)(j+k,j+k')|^p = \sum_{k\in\mathbb{Z}^d} \hat{a}(k-k')^p = \|A\|^p_{\mathcal{C}_{p,u}}$$

for $k' \in \mathbb{Z}^d$, we have that

$$\begin{aligned} &\sum_{k\in\mathbb{Z}^d} \sup_{i-j=k} \Big(\sum_{\ell\in\mathbb{Z}^d} |(au)(i,\ell)||bv(\ell,j)|\Big)^p \\ &\le \sum_{k\in\mathbb{Z}^d} \sup_{i-j=k} \Big(\sum_{\ell\in\mathbb{Z}^d} |(au)(i,\ell)|^p |(bv)(\ell,j)|\Big) \times \Big(\sum_{\ell\in\mathbb{Z}^d} |(bv)(\ell,j)|\Big)^{p-1} \\ &\le \|B\|^{p-1}_{\mathcal{C}_{1,v}} \sum_{k\in\mathbb{Z}^d} \sup_{j\in\mathbb{Z}^d} \Big(\sum_{\ell\in\mathbb{Z}^d} |(au)(j+k,\ell)|^p |(bv)(\ell,j)|\Big) \\ &\le \|B\|^{p-1}_{\mathcal{C}_{1,v}} \sum_{k\in\mathbb{Z}^d} \sum_{k'\in\mathbb{Z}^d} \Big(\sup_{\ell-j=k'} |(au)(j+k,\ell)|^p\Big)\Big(\sup_{\ell-j=k'} |(bv)(\ell,j)|\Big) \\ &\le \|A\|^p_{\mathcal{C}_{p,u}} \|B\|^p_{\mathcal{C}_{1,v}}. \end{aligned}$$

Similarly,

$$\sum_{k\in\mathbb{Z}^d}\sup_{i-j=k}\Big(\sum_{\ell\in\mathbb{Z}^d}|(av)(i,\ell)||(bu)(\ell,j)|\Big)^p\le\|A\|_{\mathcal{C}_{1,v}}^p\|B\|_{\mathcal{C}_{p,u}}^p.$$

The above two estimates together with (2.17) yield

$$\Big(\sum_{k\in\mathbb{Z}^d}\sup_{i-j=k}|(cu)(i,j)|^p\Big)^{1/p}\le 2\big(\|A\|_{\mathcal{C}_{p,u}}\|B\|_{\mathcal{C}_{1,v}}+\|A\|_{\mathcal{C}_{1,v}}\|B\|_{\mathcal{C}_{p,u}}\big).$$

□

For $1\le p\le\infty$, $\tau\ge 0$, a p-submultiplicative weight u and a companion weight v of u, we define $\Delta(\tau)$ and $\Omega_{p/(p-1)}(\tau)$ as follows:

$$\Delta(\tau)=\sup_{i\in\mathbb{Z}^d}\sum_{\substack{j\in\mathbb{Z}^d\text{ with}\\|i-j|_\infty\le\tau}}v(i,j)+\sup_{j\in\mathbb{Z}^d}\sum_{\substack{i\in\mathbb{Z}^d\text{ with}\\|i-j|_\infty\le\tau}}v(i,j),\tag{2.18}$$

$$\begin{aligned}\Omega_{p/(p-1)}(\tau)=&\sup_{i\in\mathbb{Z}^d}\Big(\sum_{\substack{j\in\mathbb{Z}^d\text{ with}\\|i-j|_\infty>\tau}}((vu^{-1})(i,j))^{p/(p-1)}\Big)^{(p-1)/p}\\&+\sup_{j\in\mathbb{Z}^d}\Big(\sum_{\substack{i\in\mathbb{Z}^d\text{ with}\\|i-j|_\infty>\tau}}((vu^{-1})(i,j))^{p/(p-1)}\Big)^{(p-1)/p}\end{aligned}\tag{2.19}$$

for the Gröchenig-Schur class $\mathcal{A}_{p,u}$;

$$\Delta(\tau)=\sum_{|k|_\infty\le\tau}\sup_{i-j=k}v(i,j),\tag{2.20}$$

$$\Omega_{p/(p-1)}(\tau)=\Big(\sum_{|k|_\infty>\tau}\big(\sup_{i-j=k}(vu^{-1})(i,j)\big)^{p/(p-1)}\Big)^{(p-1)/p}\tag{2.21}$$

for the Gohberg-Baskakov-Sjöstrand class $\mathcal{C}_{p,u}$; and

$$\Delta(\tau)=\sum_{|k|_\infty\le\tau}\sup_{|k|_\infty\le|i-j|_\infty\le\tau}v(i,j),\tag{2.22}$$

$$\Omega_{p/(p-1)}(\tau)=\Big(\sum_{|k|_\infty>\tau/2}\big(\sup_{|i-j|_\infty\ge|k|_\infty}(vu^{-1})(i,j)\big)^{p/(p-1)}\Big)^{(p-1)/p}\tag{2.23}$$

for the Beurling class $\mathcal{B}_{p,u}$.

Polynomial weights $\big((1+|i-j|_\infty)^\alpha\big)_{i,j\in\mathbb{Z}^d}$ and subexponential weights $\big(\exp(D|i-j|_\infty^\delta)\big)_{i,j\in\mathbb{Z}^d}$ are p-submultiplicative and they satisfy (2.24) ([11], [12], [13]), while subpolynomial weights $\big(\exp\big(D(\ln(1+|i-j|_\infty))^\delta\big)\big)_{i,j\in\mathbb{Z}^d}$ are 1-submultiplicative but they do not satisfy (2.24) ([9]), where $1\le p\le\infty$, $\alpha>d/p$, $D>0$ and $\delta\in(0,1)$.

In the following theorem, a sufficient condition is given for a subalgebra to be a differential subalgebra.

Proposition 2.3. *Let $1 \le p \le \infty$ and $\mathcal{F}_{p,u}$ be one of three classes $\mathcal{A}_{p,u}$, $\mathcal{C}_{p,u}$ and $\mathcal{B}_{p,u}$. If u is a p-submultiplicative weight with companion weight v, and there exist $D > 0$ and $0 \le \theta < 1$ such that*

$$\inf_{\tau \ge 0} \left[\Delta(\tau) + t \cdot \Omega_{p/(p-1)}(\tau)\right] \le Dt^{\theta} \quad \text{for all } t \ge 1, \tag{2.24}$$

then $\mathcal{F}_{p,u}$ is a differential subalgebra of $\mathcal{B}(\ell^r)$, $1 \le r \le \infty$, with order θ, that is, there exists $C > 0$ such that for any $A, B \in \mathcal{F}_{p,u}$

$$\|AB\|_{\mathcal{F}_{p,u}} \le C\|A\|_{\mathcal{F}_{p,u}}\|B\|_{\mathcal{F}_{p,u}} \left(\left(\frac{\|A\|_{\mathcal{B}(\ell^r)}}{\|A\|_{\mathcal{F}_{p,u}}}\right)^{1-\theta} + \left(\frac{\|B\|_{\mathcal{B}(\ell^r)}}{\|B\|_{\mathcal{F}_{p,u}}}\right)^{1-\theta} \right). \tag{2.25}$$

Proof. We prove the inequality (2.25) when $\mathcal{F}_{p,u}$ is the Gohberg-Baskakov-Sjöstrand class, so that $\Delta(\tau)$ and $\Omega_{p/(p-1)}(\tau)$ are given by (2.20) and (2.21), respectively. We may follow similar argument to prove the conclusion for the Gröchenig-Schur class and the Beurling class, and we leave the details for the interested reader.

Let $1 \le p \le \infty$, $A, B \in \mathcal{C}_{p,u}$, where $A = (a(i,j))_{i,j \in \mathbb{Z}^d}$ and $B = (b(i,j))_{i,j \in \mathbb{Z}^d}$. Since for any $i, j \in \mathbb{Z}^d$,

$$|b(i,j)| \le \|B\|_{\mathcal{B}(\ell^r)},$$

we have that for any $\tau > 0$

$$\begin{aligned} \|B\|_{\mathcal{C}_{1,v}} &= \sum_{k \in \mathbb{Z}^d} \sup_{i-j=k} |(bv)(i,j)| \\ &\le \|B\|_{\mathcal{B}(\ell^p)} \sum_{|k| \le \tau} \sup_{i-j=k} v(i,j) + \sum_{|k|_\infty > \tau} \sup_{i-j=k} |(bu)(i,j)|(vu^{-1})(i,j) \\ &\le \|B\|_{\mathcal{B}(\ell^p)} \Delta(\tau) + \|B\|_{\mathcal{C}_{p,u}} \Omega_{p/(p-1)}(\tau). \end{aligned}$$

Since by Lemma 2.2(i) there exists $C_1 \ge 1$ such that

$$\|H\|_{\mathcal{B}(\ell^r)} \le C_1 \|H\|_{\mathcal{C}_{p,u}}$$

for all $H \in \mathcal{C}_{p,u}$, we obtain from (2.16) and (2.24) that

$$\begin{aligned} \|AB\|_{\mathcal{C}_{p,u}} &\le C(\|A\|_{\mathcal{C}_{p,u}}\|B\|_{\mathcal{C}_{1,v}} + \|A\|_{\mathcal{C}_{1,v}}\|B\|_{\mathcal{C}_{p,u}}) \\ &\le C\Big(\|A\|_{\mathcal{C}_{p,u}}(\|B\|_{\mathcal{B}(\ell^r)}\Delta(\tau) + C_1\|B\|_{\mathcal{C}_{p,u}}\Omega_{p/(p-1)}(\tau)) \\ &\quad + \|B\|_{\mathcal{C}_{p,u}}(\|A\|_{\mathcal{B}(\ell^r)}\Delta(\tau) + C_1\|A\|_{\mathcal{C}_{p,u}}\Omega_{p/(p-1)}(\tau))\Big) \\ &\le CC_1^{\theta} D \|A\|_{\mathcal{C}_{p,u}}\|B\|_{\mathcal{B}(\ell^r)} \left(\frac{\|B\|_{\mathcal{C}_{p,u}}}{\|B\|_{\mathcal{B}(\ell^r)}}\right)^{\theta} \\ &\quad + CC_1^{\theta} D \|A\|_{\mathcal{B}(\ell^r)}\|B\|_{\mathcal{C}_{p,u}} \left(\frac{\|A\|_{\mathcal{C}_{p,u}}}{\|A\|_{\mathcal{B}(\ell^r)}}\right)^{\theta} \\ &= CC_1^{\theta} D \|A\|_{\mathcal{C}_{p,u}}\|B\|_{\mathcal{C}_{p,u}} \left(\left(\frac{\|A\|_{\mathcal{B}(\ell^r)}}{\|A\|_{\mathcal{C}_{p,u}}}\right)^{1-\theta} + \left(\frac{\|B\|_{\mathcal{B}(\ell^r)}}{\|B\|_{\mathcal{C}_{p,u}}}\right)^{1-\theta} \right). \end{aligned}$$

This completes the proof. □

3. Exponential stability

The following is the main theorem of this paper.

Theorem 3.1. *Let $0 \le \theta < 1$ and $\mathcal{A}$ be a Banach algebra with norm $\|\cdot\|_{\mathcal{A}}$. Suppose that $\mathcal{A}$ is a differential subalgebra of $\mathcal{B}(\ell^r)$, $1 \le r \le \infty$, with order θ. If the linear system* (1.1) *associated with $A \in \mathcal{A}$ is p-exponentially stable for some $1 \le p \le \infty$, then it is q-exponentially stable for any $1 \le q \le \infty$.*

Proof. Let $0 \le \theta < 1$, $A \in \mathcal{A}$ and $1 \le p \le \infty$. The solution of the system (1.1) is given by

$$\exp(tA) = \sum_{n=0}^{\infty} \frac{(tA)^n}{n!}.$$

Then $\exp(tA) \in \mathcal{A}$ as

$$\|\exp(tA)\|_{\mathcal{A}} \le \sum_{n=0}^{\infty} \frac{|t|^n \|A\|_{\mathcal{A}}^n}{n!} < \infty, \quad t \in \mathbb{R}.$$

By p-exponential stability, there exist constants M and $\alpha > 0$ such that

$$\|\exp(tA)\|_{\mathcal{B}(\ell^p)} \le M e^{-\alpha t} \quad \text{for any } t \ge 0. \tag{3.1}$$

Letting $A = B$ in (1.4) gives

$$\|A^2\|_{\mathcal{A}} \le 2C \|A\|_{\mathcal{A}}^{1+\theta} \|A\|_{\mathcal{B}(\ell^p)}^{1-\theta} \quad \text{for all } \ A \in \mathcal{A}. \tag{3.2}$$

Let $t \ge 1$ and N be the nonnegative integer such that $t/2^N \in [1, 2)$, which implies that

$$\log_2 t - 1 < N \le \log_2 t. \tag{3.3}$$

We put $B = \exp(\frac{t}{2^N} A)$. Observing from (3.1) that for $\beta > 0$ and a nonnegative integer k,

$$\|B^{2^k}\|_{\mathcal{B}(\ell^p)}^{\beta} \le M^{\beta} e^{-\alpha\, t\, 2^{k-N} \beta},$$

it follows from (3.2) that

$$\begin{aligned}
\|B^{2^N}\|_{\mathcal{A}} &\le 2C \|B^{2^{N-1}}\|_{\mathcal{A}}^{1+\theta} \|B^{2^{N-1}}\|_{\mathcal{B}(\ell^p)}^{1-\theta} \\
&\le (2C)^2 \|B^{2^{N-2}}\|_{\mathcal{A}}^{(1+\theta)^2} \|B^{2^{N-2}}\|_{\mathcal{B}(\ell^p)}^{(1+\theta)(1-\theta)} \|B^{2^{N-1}}\|_{\mathcal{B}(\ell^p)}^{1-\theta} \\
&\le \cdots \\
&\le (2C)^N \|B\|_{\mathcal{A}}^{(1+\theta)^N} \|B\|_{\mathcal{B}(\ell^p)}^{(1+\theta)^{N-1}(1-\theta)} \|B^2\|_{\mathcal{B}(\ell^p)}^{(1+\theta)^{N-2}(1-\theta)} \cdots \\
&\quad \cdots \|B^{2^{N-1}}\|_{\mathcal{B}(\ell^p)}^{(1-\theta)} \\
&\le (2C)^N \tilde{C}^{(1+\theta)^N} M^{(1+\theta)^{N-1}(1-\theta)} e^{-\alpha t 2^{-N} (1+\theta)^{N-1}(1-\theta)} \times \\
&\quad \times M^{(1+\theta)^{N-2}(1-\theta)} e^{-\alpha t 2^{-N+1} (1+\theta)^{N-2}(1-\theta)} \cdots M^{(1-\theta)} e^{-\alpha t 2^{-1}(1-\theta)}
\end{aligned}$$

$$= (2C)^N \tilde{C}^{(1+\theta)^N} M^{(1-\theta)\left((1+\theta)^{N-1}+(1+\theta)^{N-2}+\cdots+1\right)} \times \cdots$$

$$\cdots \times e^{-\alpha t(1-\theta)\left(\frac{(1+\theta)^{N-1}}{2^N}+\frac{(1+\theta)^{N-2}}{2^{N-1}}+\cdots+\frac{1}{2}\right)}, \tag{3.4}$$

where $\sup_{s\in[1,2)} \|\exp(sA)\|_{\mathcal{A}} = \tilde{C}$. Observing from (3.3) that

$$e^{-\alpha t(1-\theta)\left(\frac{(1+\theta)^{N-1}}{2^N}+\frac{(1+\theta)^{N-2}}{2^{N-1}}+\cdots+\frac{1}{2}\right)} = e^{-\alpha t} e^{\alpha t\left(\frac{1+\theta}{2}\right)^N}$$

$$\leq e^{-\alpha t} e^{\alpha t (t/2)^{\log_2 \frac{1+\theta}{2}}} \leq e^{-\alpha t} e^{2\alpha t^{\log_2(1+\theta)}}$$

and

$$M^{(1-\theta)\left((1+\theta)^{N-1}+(1+\theta)^{N-2}+\cdots+1\right)} \leq M^{\frac{(1-\theta)}{\theta}(1+\theta)^N} \leq e^{\frac{(1-\theta)}{\theta} t^{\log_2(1+\theta)} \ln M}.$$

Combining the above two estimates with (3.4), we obtain that for any $t \geq 1$

$$\|\exp(tA)\|_{\mathcal{A}} \leq \exp\Big(-\alpha t + t^{\log_2(1+\theta)}(2\alpha + \frac{1-\theta}{\theta}\ln M + \ln \tilde{C}) +$$

$$+ (\ln 2C)(\log_2 t)\Big). \tag{3.5}$$

Since $\max_{t\in[0,1]} \|\exp(tA)\|_{\mathcal{A}} \leq e^{\|A\|_{\mathcal{A}}}$ is bounded, from (3.5) we have that for any $0 < \alpha' < \alpha$, there exists a positive constant M' satisfying

$$\|\exp(tA)\|_{\mathcal{A}} \leq M' e^{-\alpha' t}. \tag{3.6}$$

Combining (1.5) and (3.6) yields that

$$\|\exp(tA)\|_{\mathcal{B}(\ell^q)} \leq C' M' e^{-\alpha' t} \quad \text{for all } t \geq 0, \tag{3.7}$$

where $1 \leq q \leq \infty$. This proves that the linear system (1.1) is q-exponentially stable. □

Let $1 \leq p \leq \infty$. If a Banach algebra $\mathcal{A}$ is one of three classes $\mathcal{A}_{p,u}$, $\mathcal{C}_{p,u}$ and $\mathcal{B}_{p,u}$, and u is a p-submultiplicative weight satisfying (2.24), then (1.5) is satisfied by Lemma 2.2, and $\mathcal{A}$ is a differential subalgebra of $\mathcal{B}(\ell^r)$, $1 \leq r \leq \infty$, by Proposition 2.3. Hence we have the following corollary.

Corollary 3.2. *Let $1 \leq p \leq \infty$, $\mathcal{A}$ be one of three classes $\mathcal{A}_{p,u}, \mathcal{C}_{p,u}$ and $\mathcal{B}_{p,u}$, and u be a p-submultiplicative weight matrix satisfying* (2.24) *and $A \in \mathcal{A}$. Then q-exponential stability of the linear system* (1.1) *associated with A for different $1 \leq q \leq \infty$ is equivalent to each other.*

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