Commun. Korean Math. Soc. **31** (2016), No. 3, pp. 591–601
http://dx.doi.org/10.4134/CKMS.c150213
pISSN: 1225-1763 / eISSN: 2234-3024

SOME INTEGRAL TRANSFORMS AND FRACTIONAL INTEGRAL FORMULAS FOR THE EXTENDED HYPERGEOMETRIC FUNCTIONS

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Abstract. Integral transforms and fractional integral formulas involving well-known special functions are interesting in themselves and play important roles in their diverse applications. A large number of integral transforms and fractional integral formulas have been established by many authors. In this paper, we aim at establishing some (presumably) new integral transforms and fractional integral formulas for the generalized hypergeometric type function which has recently been introduced by Luo *et al.* [9]. Some interesting special cases of our main results are also considered.

1. Introduction and preliminaries

The theory of special functions has been one of the most rapidly growing research subjects in mathematical analysis. A lot of special functions in mathematical physics and engineering, such as Jacobi and Laguerre polynomials, can be expressed in terms of the generalized Gauss hypergeometric functions or confluent hypergeometric functions (see, *e.g.*, [17]). Certain extensions of the hypergeometric functions and several other familiar special functions have been presented and investigated (see, *e.g.*, [4], [5], [8], [12] and for a very recent work, see also [2], [16]). Very recently, Luo *et al.* [9] introduced the following extended generalized hypergeometric type function and investigated its various properties. The extended generalized hypergeometric function is defined, for

Received November 9, 2015.
2010 *Mathematics Subject Classification.* Primary 26A33, 33B15, 33C05, 33C15, 33C20, 44A10, 44A20; Secondary 33B99, 33C99.
Key words and phrases. gamma function, beta function, extended beta function, generalized hypergeometric functions, extended generalized hypergeometric functions, decomposition formula, integral transforms, fractional integral operators.

$z \in \mathbb{C}$, by

$$ {}_pF_q^{(\alpha,\beta;\kappa,\mu)}\left[\begin{array}{c} a_1,\ldots,a_p \\ b_1,\ldots,b_q \end{array};z;\gamma\right] := \sum_{n=0}^{\infty} \Theta\left(n/p,q\right)\frac{z^n}{n!} \tag{1.1}$$
$$(\min\left\{\Re(\kappa),\ \Re(\mu)\right\} \geq 0,\ \min\{\Re(\alpha),\ \Re(\beta),\ \Re(\gamma)\} > 0)$$

whose coefficient is given by

$$\Theta\left(n|p,q\right) = \begin{cases} \left(a_1\right)_n \displaystyle\prod_{j=1}^{q} \frac{\mathcal{B}_{\gamma}^{(\alpha,\beta;\kappa,\mu)}\left(a_{j+1}+n, b_j - a_{j+1}\right)}{\mathcal{B}\left(a_{j+1}, b_j - a_{j+1}\right)} \\ \quad \left(p=q+1;\ \Re\left(b_j\right) > \Re\left(a_{j+1}\right) > 0;\ |z|<1\right), \\ \displaystyle\prod_{j=1}^{q} \frac{\mathcal{B}_{\gamma}^{(\alpha,\beta;\kappa,\mu)}\left(a_j+n, b_j-a_j\right)}{\mathcal{B}\left(a_j, b_j-a_j\right)} \\ \quad \left(p=q;\ \Re\left(b_j\right) > \Re\left(a_j\right) > 0\right), \\ \displaystyle\prod_{i=1}^{r} \frac{1}{\left(b_i\right)_n} \prod_{j=1}^{p} \frac{\mathcal{B}_{\gamma}^{(\alpha,\beta;\kappa,\mu)}\left(a_j+n, b_r+j-a_j\right)}{\mathcal{B}\left(a_j, b_r+j-a_j\right)} \\ \quad \left(r=q-p>0;\ \Re\left(b_{r+j}\right) > \Re\left(a_j\right) > 0\right). \end{cases}$$

Here the generalized beta function $B_{\gamma}^{(\alpha,\beta;k,\mu)}(x,y)$ is defined by

$$B_{\gamma}^{(\alpha,\beta;\kappa,\mu)}(x,y) := \int_0^1 t^{x-1}(1-t)^{y-1}{}_1F_1\left(\alpha;\beta;-\frac{\gamma}{t^{\kappa}(1-t)^{\mu}}\right)\,dt \tag{1.2}$$
$$(\min\{\Re(x),\ \Re(y),\ \Re(\alpha),\ \Re(\beta),\ \Re(\gamma),\ \Re(\kappa),\ \Re(\mu)\} > 0).$$

Remark 1. The special case of (1.1) when $p = 2$ and $q = 1$ would reduce immediately to the extended Gauss hypergeometric type function defined by

$$ {}_2F_1^{(\alpha,\beta;\kappa,\mu)}\left[\begin{array}{cc} a, & b \\ & c \end{array};z;\gamma\right] = \sum_{n=0}^{\infty}\left(a\right)_n \frac{\mathcal{B}_{\gamma}^{(\alpha,\beta;\kappa,\mu)}\left(b+n,c-b\right)}{\mathcal{B}\left(b,c-b\right)}\frac{z^n}{n!}\quad (|z|<1) \tag{1.3}$$
$$(\min\left\{\kappa,\ \mu\right\} \geq 0;\ \min\{\Re\left(\alpha\right),\Re\left(\beta\right),\ \Re(\gamma)\} > 0;\ \Re\left(c\right) > \Re\left(b\right) > 0).$$

Setting $\gamma = 0$ in (1.1) and (1.2) is easily seen to yield the familiar generalized hypergeometric function ${}_pF_q[z]$ and the classical beta function $B(x,y)$ function, respectively.

The present investigation requires the concept of Hadamard product which can be used to decompose a newly-emerged function into two known functions. Let $f\left(z\right) := \sum_{n=0}^{\infty} a_n z^n$ and $g\left(z\right) := \sum_{n=0}^{\infty} b_n z^n$ be two power series whose radii of convergence are given by R_f and R_g, respectively. Then their Hadamard product (see [13]) is the power series defined by

$$\left(f * g\right)\left(z\right) := \sum_{n=0}^{\infty} a_n b_n z^n.$$

The radius of convergence R of the Hadamard product series $(f * g)(z)$ satisfies $R_f \cdot R_g \leq R$. If, in particular, one of the power series defines an entire function, then the Hadamard product series defines an entire function, too.

Consider the function ${}_sF_{s+r}^{(\alpha,\beta;\kappa,\mu)}[z;r]$ one of whose Hadamard products can, for example, be given as follows:

$$\begin{aligned}&{}_sF_{s+r}^{(\alpha,\beta;\kappa,\mu)}\left[\begin{matrix}x_1,\dots,x_s\\ y_1,\dots,y_{s+r}\end{matrix};z;\gamma\right]\\ &= {}_1F_r\left[\begin{matrix}1;\\ y_1,\dots,y_r;\end{matrix}\ z\right] * {}_sF_s^{(\alpha,\beta;\kappa,\mu)}\left[\begin{matrix}x_1,\dots,x_s\\ y_{1+r},\dots,y_{s+r}\end{matrix};z;\gamma\right]\ (|z|<\infty),\end{aligned}$$

where ${}_1F_r$ is a special case of the generalized hypergeometric functions ${}_pF_q$ (see, *e.g.*, [17, p. 71]).

In diverse areas in engineering and mathematical physics, integral transforms and fractional integral operators play an important role in the view point of application. A remarkably large number of integral transforms and fractional integral formulas involving various special functions have been investigated by many authors. Very recently, certain interesting integral transforms and fractional integral formulas involving the $F_p^{(\alpha,\beta;m)}(\cdot)$ were presented (see [6]). Here, we also aim to establish certain (presumably) new integral transforms and fractional integral formulas involving the generalized Gauss hypergeometric type functions ${}_2F_1^{(\alpha,\beta;\kappa,\mu)}(z;\gamma)$ given by Luo *et al.* [9].

2. Integral transforms and generalized Gauss hypergeometric functions

We present three transforms, which exhibit the connection between the Euler, Varma, Laplace and Whittaker integral transforms and generalized Gauss hypergeometric type functions ${}_2F_1^{(\alpha,\beta;\kappa,\mu)}\left[\begin{smallmatrix}l+m,b\\ c\end{smallmatrix};yz;\gamma\right]$ defined by (1.3). To do this, we begin by recalling the following beta transform of a function $f(z)$ (see [14]):

$$B\{f(z):a,b\} = \int_0^1 z^{a-1}(1-z)^{b-1}f(z)\,dz. \tag{2.1}$$

Theorem 2. *Let* $\min\{\Re(l), \Re(m), \Re(\alpha), \Re(\beta), \Re(\gamma), \Re(\kappa), \Re(\mu)\} > 0$ *and* $\Re(c) > \Re(b) > 0$. *Then the following beta transform formula holds true*:

$$\begin{aligned}&B\left\{{}_2F_1^{(\alpha,\beta;\kappa,\mu)}\left[\begin{matrix}l+m,b\\ c\end{matrix};yz;\gamma\right]:l,m\right\}\\ &= B(l,m)\,{}_2F_1^{(\alpha,\beta;\kappa,\mu)}\left[\begin{matrix}l,b\\ c\end{matrix};y;\gamma\right]\ (|y|<1),\end{aligned} \tag{2.2}$$

where B is the beta transform in (2.1) *and the beta transform of* ${}_2F_1^{(\alpha,\beta;\kappa,\mu)}(\cdot)$ *is assumed to exist.*

Proof. Let $\mathcal{L}$ be the left-hand side of (2.2). Applying the beta transform (2.1) to the function (1.3), we get

$$\mathcal{L} = \int_0^1 z^{l-1}(1-z)^{m-1} \sum_{n=0}^{\infty} (l+m)_n \frac{B_\gamma^{(\alpha,\beta;\kappa,\mu)}(b+n, c-b)}{B(b,c-b)} \frac{(yz)^n}{n!}\, dz. \tag{2.3}$$

By changing the order of integration and summation which may be verified under the conditions, and using the classical beta function $B(\alpha, \beta)$ (see, *e.g.*, [17, p. 8]), we obtain

$$\mathcal{L} = \sum_{n=0}^{\infty} (l+m)_n \frac{B_\gamma^{(\alpha,\beta;\kappa,\mu)}(b+n, c-b)}{B(b,c-b)} \frac{\Gamma(l+n)\Gamma(m)}{\Gamma(l+m+n)} \frac{y^n}{n!}, \tag{2.4}$$

which, in view of (1.3), is seen to lead to the right-hand side of (2.2). □

The Varma transform of a function $f(z)$ is defined by the following integral equation (see Mathai *et al.* [11, p. 55]):

$$V(f,k,m;s) = \int_0^\infty (sz)^{m-\frac{1}{2}} \exp\left(-\frac{1}{2}sz\right) W_{k,m}(sz) f(z)\, dz \quad (\Re(s) > 0), \tag{2.5}$$

where $W_{k,m}$ is the Whittaker function defined by (Mathai *et al.* [11, p. 55])

$$W_{k,m}(z) = \sum_{m,-m} \frac{\Gamma(-2m)}{\Gamma(\frac{1}{2}-k-m)} M_{k,m}(z) \tag{2.6}$$

where the summation symbol indicates that the expression following it, a similar expression with m replaced by $-m$ is to be added and

$$M_{k,m}(z) = z^{m+\frac{1}{2}} e^{-\frac{z}{2}} {}_1F_1\left(\frac{1}{2} - k + m; 2m+1; z\right). \tag{2.7}$$

The following formula (see Mathai *et al.* [11, p. 56]) will be used:

$$\begin{aligned} &\int_0^\infty z^{\rho-1} \exp\left(-\frac{1}{2}sz\right) W_{k,\nu}(sz)\, dz = s^{-\rho} \frac{\Gamma(\rho+\nu+\frac{1}{2})\Gamma(\rho-\nu+\frac{1}{2})}{\Gamma(1-k+\rho)} \\ &\qquad (\Re(s) > 0,\ \Re(\rho \pm \nu) > -1/2). \end{aligned} \tag{2.8}$$

Theorem 3. *Let* $y \geq 0$, $\Re(s) \geq 0$, $\min\{\Re(\alpha), \Re(\beta), \Re(\gamma), \Re(\kappa), \Re(\mu)\} > 0$, $\Re(c) > \Re(b) > 0$, *and* $|\frac{y}{s}| < 1$. *Then the following Varma transform formula holds true*:

$$\begin{aligned} &V\left\{z^{l-1}\, {}_2F_1^{(\alpha,\beta;\kappa,\mu)}\left[\begin{matrix} a, b \\ c \end{matrix}; yz; \gamma\right]\right\} = \frac{1}{s^l} \frac{\Gamma(l)\Gamma(2m+l)}{\Gamma(m+l-k+\frac{1}{2})} \\ &\quad \times {}_2F_1^{(\alpha,\beta;\kappa,\mu)}\left[\begin{matrix} a, b \\ c \end{matrix}; \frac{y}{s}; \gamma\right] *_2 F_1\left[\begin{matrix} 2m+l, l; \\ m+l-k+\frac{1}{2}; \end{matrix}\ \frac{y}{s}\right], \end{aligned} \tag{2.9}$$

where V *is the Varma transform in* (2.5) *and both sides of* (2.9) *are assumed to exist.*

Proof. Let $\mathcal{L}$ be the left-hand side of (2.9). Then, a similar argument as in the proof of Theorem 2 is seen to give the following result:

$$\mathcal{L} = \frac{1}{s^l}\sum_{n=0}^{\infty}(a)_n \frac{B_{\gamma}^{(\alpha,\beta;\kappa,\mu)}(b+n,c-b)}{B(b,c-b)} \frac{\Gamma(l+n)\Gamma(2m+l+n)}{\Gamma(m+l-k+\frac{1}{2}+n)} \frac{(\frac{y}{s})^n}{n!}, \tag{2.10}$$

which, upon using Hadamard product series and (1.3), leads to the right-hand side of (2.9). □

It is interesting to observe that, for $k=-m+\frac{1}{2}$ in (2.9), the Varma transform defined by (2.5) reduces to the well-known Laplace transform of a function $f(z)$ (see, *e.g.*, [14]):

$$L\{f(z):s\} = \int_0^\infty e^{-sz} f(z)\,dz. \tag{2.11}$$

In fact, we have an interesting Laplace transform asserted by the following corollary.

Corollary 1. *Let* $y \geq 0$, $\Re(s) \geq 0$, $\min\{\Re(\alpha), \Re(\beta), \Re(\gamma), \Re(\kappa), \Re(\mu)\} > 0$, $\Re(c) > \Re(b) > 0$, *and* $|\frac{y}{s}| < 1$. *Then the following Laplace transform formula holds true*:

$$\begin{aligned} &L\left\{z^{l-1}\, {}_2F_1^{(\alpha,\beta;\kappa,\mu)}\left[\begin{matrix} a,b \\ c \end{matrix}; yz;\gamma\right]\right\} \\ &= \frac{\Gamma(l)}{s^l}\, {}_2F_1^{(\alpha,\beta;\kappa,\mu)}\left[\begin{matrix} a,b \\ c \end{matrix}; \frac{y}{s};\gamma\right] * {}_1F_0\left[\begin{matrix} l; \\ -; \end{matrix} \frac{y}{s}\right], \end{aligned} \tag{2.12}$$

where L *is the Laplace transform in* (2.11) *and both sides of* (2.12) *are assumed to exist.*

Theorem 4. *Suppose that* $w \geq 0$, $\Re(\gamma) \geq 0$, $\min\{\Re(\alpha), \Re(\beta), \Re(k), \Re(\mu)\} > 0$, $\Re(c) > \Re(b) > 0$, *and* ρ, $\delta \in \mathbb{C}$ *are parameters. Then the following Whittaker transform formula holds true*:

$$\begin{aligned} &\int_0^\infty t^{\rho-1} e^{\frac{-\delta t}{2}} W_{\lambda,\mu}(\delta t)\, {}_2F_1^{(\alpha,\beta;\kappa,\mu)}\left[\begin{matrix} a,b \\ c \end{matrix}; wt;\gamma\right] dt \\ &= \delta^{-\rho} \frac{\Gamma\left(\frac{1}{2}+\mu+\rho\right)\Gamma\left(\frac{1}{2}-\mu+\rho\right)}{\Gamma\left(\frac{1}{2}-\lambda+\rho\right)} \\ &\times {}_2F_1^{(\alpha,\beta;\kappa,\mu)}\left[\begin{matrix} a,b \\ c \end{matrix}; \frac{w}{\delta};\gamma\right] * {}_2F_1\left[\begin{matrix} \frac{1}{2}+\mu+\rho, \frac{1}{2}-\mu+\rho; \\ \frac{1}{2}-\lambda+\rho; \end{matrix} \frac{w}{\delta}\right], \end{aligned} \tag{2.13}$$

provided that the integral transform converges.

Proof. Let $\mathcal{L}$ be the left-hand side of (2.13). Then, by applying (1.3) and setting $\delta t = \nu$, and changing the order of integration and summation, we obtain

$$\begin{aligned}\mathcal{L} = \delta^{-\rho} \sum_{n=0}^{\infty} (a)_n \frac{B_{\gamma}^{(\alpha,\beta;\kappa,\mu)}(b+n, c-b)}{B(b, c-b)} \frac{(w)^n}{\delta^n n!} d\nu \\ \times \int_0^{\infty} \nu^{\rho+n-1} e^{\frac{-\nu}{2}} W_{\lambda,\mu}(\nu) d\nu.\end{aligned} \tag{2.14}$$

Here we use the following integral formula involving the Whittaker function (see Mathai *et al.* [11, p. 56])

$$\begin{aligned}\int_0^{\infty} t^{\nu-1} e^{-\frac{t}{2}} W_{\lambda,\mu}(t)\, dt = \frac{\Gamma\left(\frac{1}{2}+\mu+\nu\right)\Gamma\left(\frac{1}{2}-\mu+\nu\right)}{\Gamma\left(\frac{1}{2}-\lambda+\nu\right)} \\ (\Re(\nu \pm \mu) > -1/2)\,.\end{aligned} \tag{2.15}$$

Then, after a little simplification, we get

$$\begin{aligned}\mathcal{L} = \delta^{-\rho} \sum_{n=0}^{\infty} (a)_n \frac{B_{\gamma}^{(\alpha,\beta;\kappa,\mu)}(b+n, c-b)}{B(b, c-b)} \frac{(w)^n}{\delta^n n!} \\ \times \frac{\Gamma\left(\frac{1}{2}+\mu+\rho+n\right)\Gamma\left(\frac{1}{2}-\mu+\rho+n\right)}{\Gamma\left(\frac{1}{2}-\lambda+\rho+n\right)},\end{aligned} \tag{2.16}$$

which, upon using Hadamard product series and (1.3), leads to the right-hand side of (2.13). □

It is noted in passing that the case $\kappa = \mu = m$ in Theorems 2 and 4, and Corollary 1 is seen to yield the known results in [6].

3. Fractional calculus of the generalized Gauss hypergeometric functions

Recently, fractional integral operators involving the various special functions have been actively investigated (see, *e.g.*, [1], [3], [7] and [15]). Here we establish some fractional integral formulas for the generalized Gauss hypergeometric type functions $F_p^{(\alpha,\beta;k,\mu)}(a, b; c; z)$. To do this, we recall the following pair of Saigo hypergeometric fractional integral operators (see Mathai *et al.* [11, p. 104]): For $\Re(\mu) > 0$,

$$(I_{0,x}^{\mu,\nu,\eta} f(t))(x) = \frac{x^{-\mu-\nu}}{\Gamma(\mu)} \int_0^x (x-t)^{\mu-1} {}_2F_1\left(\mu+\nu, -\eta; \mu; 1-\frac{t}{x}\right) f(t)\, dt \tag{3.1}$$

and

$$\begin{aligned}&(J_{x,\infty}^{\mu,\nu,\eta} f(t))(x) \\ &= \frac{1}{\Gamma(\mu)} \int_x^{\infty} (t-x)^{\mu-1} t^{-\mu-\nu} {}_2F_1\left(\mu+\nu, -\eta; \mu; 1-\frac{x}{t}\right) f(t)\, dt,\end{aligned} \tag{3.2}$$

where the function $f(t)$ is so constrained that the defining integrals in (3.1) and (3.2) exist.

The operator $I_{0,x}^{\mu,\nu,\eta}(\cdot)$ contains both the Riemann-Liouville $R_{0,x}^{\mu}(\cdot)$ and the Erdélyi-Kober $E_{0,x}^{\mu,\eta}(\cdot)$ fractional integral operators by means of the following relationships:

$$(R_{0,x}^{\mu}f(t))(x) = (I_{0,x}^{\mu,-\mu,\eta}f(t))(x) = \frac{1}{\Gamma(\mu)}\int_0^x (x-t)^{\mu-1}f(t)\,dt \tag{3.3}$$

and

$$(E_{0,x}^{\mu,\eta}f(t))(x) = (I_{0,x}^{\mu,0,\eta}f(t))(x) = \frac{x^{-\mu-\eta}}{\Gamma(\mu)}\int_0^x (x-t)^{\mu-1}t^{\eta}f(t)\,dt. \tag{3.4}$$

It is noted that the operator (3.2) unifies the Weyl type and the Erdélyi-Kober fractional operators as follows:

$$(W_{x,\infty}^{\mu}f(t))(x) = (J_{x,\infty}^{\mu,-\mu,\eta}f(t))(x) = \frac{1}{\Gamma(\mu)}\int_x^{\infty} (t-x)^{\mu-1}f(t)\,dt \tag{3.5}$$

and

$$(K_{x,\infty}^{\mu,\eta}f(t))(x) = (J_{x,\infty}^{\mu,0,\eta}f(t))(x) = \frac{x^{\eta}}{\Gamma(\mu)}\int_x^{\infty} (t-x)^{\mu-1}t^{-\mu-\eta}f(t)\,dt. \tag{3.6}$$

We also use the following image formulas which are easy consequences of the operators (3.1) and (3.2) (see Mathai *et al.* [11, p. 107]):

$$\begin{gathered}(I_{0,x}^{\mu,\nu,\eta}t^{\lambda-1})(x) = \frac{\Gamma(\lambda)\Gamma(\lambda-\nu+\eta)}{\Gamma(\lambda-\nu)\Gamma(\lambda+\mu+\eta)}x^{\lambda-\nu-1}\\ (\Re(\lambda)>0,\ \Re(\lambda-\nu+\eta)>0)\end{gathered} \tag{3.7}$$

and

$$\begin{gathered}(J_{x,\infty}^{\mu,\nu,\eta}t^{\lambda-1})(x) = \frac{\Gamma(\nu-\lambda+1)\Gamma(\eta-\lambda+1)}{\Gamma(1-\lambda)\Gamma(\nu+\mu-\lambda+\eta+1)}x^{\lambda-\nu-1}\\ (\Re(\nu-\lambda+1)>0,\ \Re(\eta-\lambda+1)>0).\end{gathered} \tag{3.8}$$

The Saigo fractional integrations of the generalized Gauss hypergeometric type functions $F_p^{(\alpha,\beta;k,\mu)}(a,b;c;z)$ are given in Theorems 5 and 6.

Theorem 5. *Let* $x>0$, $\min\{\Re(\gamma),\Re(\mu),\Re(\rho)\}>0$ *and* $\Re(\rho)>\max\{0,\Re(\nu-\eta)\}$. *Then the following fractional integral formula holds true*:

$$\begin{aligned}&\left(I_{0,x}^{\mu,\nu,\eta}\left[t^{\rho-1}\,{}_2F_1^{(\alpha,\beta;\kappa,\mu)}\left[\begin{matrix}a,b\\c\end{matrix};et;\gamma\right]\right]\right)(x)\\ &= x^{\rho-\nu-1}\frac{\Gamma(\rho)\Gamma(\rho-\nu+\eta)}{\Gamma(\rho+\mu+\nu)\Gamma(\rho-\nu)}\\ &\quad\times\,{}_2F_1^{(\alpha,\beta;\kappa,\mu)}\left[\begin{matrix}a,b\\c\end{matrix};ex;\gamma\right] *\,{}_2F_2\left[\begin{matrix}\rho,\rho-\nu+\eta;\\ \rho-\nu,\rho+\mu+\eta;\end{matrix}\ ex\right].\end{aligned} \tag{3.9}$$

Proof. Let $\mathcal{L}$ be the left-hand side of (3.9). Then, using (1.3) and changing the order of integration and summation, which is valid under the given conditions, we have

$$\mathcal{L} = \sum_{n=0}^{\infty} (a)_n \frac{B_{\gamma}^{(\alpha,\beta;\kappa,\mu)}(b+n, c-b)}{B(b,c-b)} \frac{(e)^n}{n!} \left(I_{0,x}^{\mu,\nu,\eta}\{t^{\rho+n-1}\}\right)(x). \tag{3.10}$$

Here, making use of the result (3.7), we obtain

$$\begin{aligned} \mathcal{L} = x^{\rho-\nu-1} \sum_{n=0}^{\infty} (a)_n \frac{B_{\gamma}^{(\alpha,\beta;\kappa,\mu)}(b+n, c-b)}{B(b,c-b)} \\ \times \frac{\Gamma(\rho+n)\Gamma(\rho-\nu+\eta+n)}{\Gamma(\rho-\nu+n)\Gamma(\rho+\mu+\eta+n)} \frac{(ex)^n}{n!}, \end{aligned} \tag{3.11}$$

which, in view of Hadamard product series and (1.3), gives the right-hand side of (3.9). □

Theorem 6. *Let* $x > 0$, $\min\{\Re(\gamma), \Re(\mu), \Re(\rho)\} > 0$ *and* $\Re(\rho) < 1 + \min\{\Re(\eta), \Re(\nu)\}$. *Then the following fractional integral formula holds true*:

$$\begin{aligned} &\left(J_{x,\infty}^{\mu,\nu,\eta}\left[t^{\rho-1}\, {}_2F_1^{(\alpha,\beta;\kappa,\mu)}\left[\begin{matrix} a, b \\ c \end{matrix}; \frac{e}{t}; \gamma\right]\right]\right)(x) \\ &= x^{\rho-\nu-1} \frac{\Gamma(1-\rho+\nu)\Gamma(1-\rho+\eta)}{\Gamma(1-\rho)\Gamma(1-\rho-\eta+\nu+\mu)} \\ &\times {}_2F_1^{(\alpha,\beta;\kappa,\mu)}\left[\begin{matrix} a, b \\ c \end{matrix}; \frac{e}{x}; \gamma\right] * {}_2F_2\left[\begin{matrix} 1-\rho+\nu, 1-\rho+\eta; \\ 1-\rho, 1-\rho+\mu+\nu-\eta; \end{matrix}\ \frac{e}{x}\right]. \end{aligned} \tag{3.12}$$

Proof. Similarly as in the proof of Theorem 5, taking the operator (3.2) and the result (3.8) into account will establish the result (3.12). So the details of proof are omitted. □

Setting $\nu = 0$ in Theorems 5 and 6 yields certain interesting results asserted by the following corollaries.

Corollary 2. *Let* $x > 0$, $\min\{\Re(\gamma), \Re(\mu), \Re(\rho)\} > 0$ *and* $\Re(\rho) > \Re(-\eta)$. *Then the right-side Erdélyi-Kober fractional integrals of the generalized Gauss hypergeometric type functions are given as follows*:

$$\begin{aligned} &\left(E_{0,x}^{\mu,\eta}\left[t^{\rho-1}\, {}_2F_1^{(\alpha,\beta;\kappa,\mu)}\left[\begin{matrix} a, b \\ c \end{matrix}; et; \gamma\right]\right]\right)(x) \\ &= x^{\rho-1} \frac{\Gamma(\rho+\eta)}{\Gamma(\rho+\mu)} {}_2F_1^{(\alpha,\beta;\kappa,\mu)}\left[\begin{matrix} a, b \\ c \end{matrix}; ex; \gamma\right] * {}_1F_1\left[\begin{matrix} \rho+\eta; \\ \rho+\mu+\eta; \end{matrix}\ ex\right]. \end{aligned} \tag{3.13}$$

Corollary 3. *Let* $x > 0$, $\min\{\Re(\gamma), \Re(\mu), \Re(\rho)\} > 0$ *and* $\Re(\rho) < 1 + \Re(\eta)$. *The following identity holds true*:

$$\begin{aligned}
&\left(K_{x,\infty}^{\mu,\eta}\left[t^{\rho-1}\,{}_2F_1^{(\alpha,\beta;\kappa,\mu)}\left[\begin{matrix} a,b \\ c \end{matrix};\frac{e}{t};\gamma\right]\right]\right)(x) \\
&= x^{\rho-1}\frac{\Gamma(1-\rho+\eta)}{\Gamma(1-\rho-\eta+\mu)} \\
&\quad\times\,{}_2F_1^{(\alpha,\beta;\kappa,\mu)}\left[\begin{matrix} a,b \\ c \end{matrix};\frac{e}{x};\gamma\right] * {}_1F_1\left[\begin{matrix} 1-\rho+\eta; \\ 1-\rho+\mu-\eta; \end{matrix}\ \frac{e}{x}\right].
\end{aligned} \tag{3.14}$$

Further, replacing ν by $-\mu$ in Theorems 5 and 6 and making use of the relations (3.3) and (3.5) gives the other Riemann-Liouville and Weyl fractional integrals of the generalized Gauss hypergeometric type function in (1.3) given by the following corollaries.

Corollary 4. *Let* $x > 0$ *and* $\min\{\Re(\gamma), \Re(\mu), \Re(\rho)\} > 0$. *Then the following formula holds true*:

$$\begin{aligned}
&\left(R_{0,x}^{\mu}\left[t^{\rho-1}\,{}_2F_1^{(\alpha,\beta;\kappa,\mu)}\left[\begin{matrix} a,b \\ c \end{matrix};et;\gamma\right]\right]\right)(x) \\
&= x^{\rho+\mu-1}\frac{\Gamma(\rho+\mu+\eta)}{\Gamma(\rho+\mu)}\,{}_2F_1^{(\alpha,\beta;\kappa,\mu)}\left[\begin{matrix} a,b \\ c \end{matrix};ex;\gamma\right] * {}_1F_1\left[\begin{matrix} \rho; \\ \rho+\mu; \end{matrix}\ ex\right].
\end{aligned} \tag{3.15}$$

Corollary 5. *Let* $x > 0$ *and* $\min\{\Re(\gamma), \Re(\mu), \Re(\rho)\} > 0$. *Then the following formula holds true*:

$$\begin{aligned}
&\left(W_{x,\infty}^{\mu}\left[t^{\rho-1}\,{}_2F_1^{(\alpha,\beta;\kappa,\mu)}\left[\begin{matrix} a,b \\ c \end{matrix};\frac{e}{t};\gamma\right]\right]\right)(x) \\
&= x^{\rho+\mu-1}\frac{\Gamma(1-\rho-\mu)}{\Gamma(1-\rho)}\,\frac{\Gamma(1-\rho+\eta)}{\Gamma(1-\rho-\eta)} \\
&\quad\times\,{}_2F_1^{(\alpha,\beta;\kappa,\mu)}\left[\begin{matrix} a,b \\ c \end{matrix};\frac{e}{x};\gamma\right] * {}_2F_2\left[\begin{matrix} 1-\rho-\mu,\,1-\rho+\eta; \\ 1-\rho,\,1-\rho-\eta; \end{matrix}\ \frac{e}{x}\right].
\end{aligned} \tag{3.16}$$

Concluding remarks

The results presented here are general enough to be suitably specialized to yield a variety of integral transforms and fractional integral formulas for each of the families of the generalized Gauss type hypergeometric functions $F_p^{(\alpha,\beta;\mu)}(a,b;c;z)$, $F_p^{(\alpha,\beta)}(a,b;c;z)$ and $F_p(a,b;c;z)$, which have been investigated by many authors (see, *e.g.*, [4, 5, 8, 10, 13]), and other special functions which are expressible in terms of the Gauss hypergeometric function.

Acknowledgements. The authors should express their deep gratitude for the reviewer's very helpful and detailed comments.

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