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ON 2–HYPONORMAL TOEPLITZ OPERATORS WITH FINITE RANK SELF–COMMUTATORS

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Abstract. Suppose T_φ is a 2–hyponormal Toeplitz operator whose self–commutator has rank $n \geq 1$. If $H_{\bar{\varphi}}\left(\ker\,[T_\varphi^*, T_\varphi]\right)$ contains a vector e_n in a canonical orthonormal basis $\{e_k\}_{k\in\mathbb{Z}_+}$ of $H^2(\mathbb{T})$, then φ should be an analytic function of the form $\varphi = qh$, where q is a finite Blaschke product of degree at most n and h is an outer function.

1. Introduction

Let $L^2 \equiv L^2(\mathbb{T})$ be the set of square integrable functions on the unit circle $\mathbb{T}$. It is well known that $\{e_n(z) \equiv z^n : n = 0, \pm 1, \pm 2, \ldots\}$ forms a canonical orthonormal basis for L^2. The Hardy space $H^2 \equiv H^2(\mathbb{T})$ is the closed linear span of $\{e_n : n = 0, 1, \ldots\}$. If $f \in H^2$, then f will be called *analytic* and if $f \in L^2 \ominus H^2$, then f will be called *co-analytic*. Let $L^\infty \equiv L^\infty(\mathbb{T})$ be the set of all bounded measurable functions on $\mathbb{T}$ and define $H^\infty := L^\infty \cap H^2$.

Now if $\varphi \in L^\infty$, we define $T_\varphi : H^2 \to H^2$ by

$$T_\varphi g := P(\varphi g) \quad (g \in H^2),$$

where P is the orthogonal projection from L^2 to H^2. The operator T_φ is called a *Toeplitz operator with symbol* φ. If $\varphi \in L^\infty$, we define $H_\varphi : H^2 \to H^2$ by

$$H_\varphi(g) := J(I-P)(\varphi g) \quad (g \in H^2),$$

where J is the unitary operator defined by $J(z^{-n}) = z^{n-1}$ $(n = 1, 2, \ldots)$. The operator H_φ is called a *Hankel operator with symbol* φ.

The following properties follow from the definition:

$$T_\varphi^* = T_{\bar{\varphi}}, \quad H_\varphi^* = H_{\tilde{\varphi}} \quad (\text{where } \tilde{\varphi}(z) = \overline{\varphi(\bar{z})}); \tag{1}$$

$$H_\varphi T_h = H_{\varphi h} = T_{\tilde{h}}^* H_\varphi \quad (\varphi \in L^\infty, h \in H^\infty). \tag{2}$$

Let $\mathcal{H}$ and $\mathcal{K}$ be infinite dimensional complex Hilbert spaces and let $\mathcal{B}(\mathcal{H}, \mathcal{K})$ be the set of all bounded linear operators from $\mathcal{H}$ to $\mathcal{K}$. We abbreviate $\mathcal{B}(\mathcal{H}) :=$

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$\mathcal{B}(\mathcal{H},\mathcal{H})$. On the other hand, an operator $T \in \mathcal{B}(\mathcal{H})$ is called normal if the self-commutator $[T^*,T] := T^*T - TT^*$ is zero and is called hyponormal if $[T^*,T] \geq 0$, i.e., the self-commutator is positive semi-definite. An operator $T \in \mathcal{B}(\mathcal{H})$ is called *subnormal* if there exists a Hilbert space $\mathcal{K}$ containing $\mathcal{H}$ such that

$$N = \begin{pmatrix} T & * \\ 0 & * \end{pmatrix} \quad \text{on } \mathcal{K} \equiv \mathcal{H} \oplus \mathcal{H}'$$

is normal, i.e., T has a normal extension. In general, it looks so very hard to determine the subnormality of the operator. The Bram-Halmos criterion for subnormality states that an operator T is subnormal if and only if

$$\sum_{i,j} (T^i x_j, T^j x_i) \geq 0 \tag{3}$$

for all finite collections $x_0, x_1, \ldots, x_k \in \mathcal{H}$ ([4], [6, II.1.9]). We can easily show that (3) is equivalent to the following positivity test:

$$\begin{pmatrix} I & T^* & \ldots & T^{*k} \\ T & T^*T & \ldots & T^{*k}T \\ \vdots & \vdots & \ddots & \vdots \\ T^k & T^*T^k & \ldots & T^{*k}T^k \end{pmatrix} \geq 0 \quad (\text{all } k \geq 1). \tag{4}$$

The condition (4) depends on the size of the matrix. If $k = 1$, then it is equivalent to the hyponormality of T. If the condition (4) holds for a fixed $k \in \mathbb{Z}_+$, then we say that T is k-hyponormal. For example, the operator $T \in \mathcal{B}(\mathcal{H})$ is 2-hyponormal if

$$\begin{pmatrix} I & T^* & T^{*2} \\ T & T^*T & T^{*2}T \\ T^2 & T^*T^2 & T^{*2}T^2 \end{pmatrix} \geq 0.$$

Many authors have studied the hyponormality and the subnormality of the Toeplitz operators on the Hardy space of the unit circle (cf. [1], [3], [7]-[10], [12], [13]-[15], [16], [19], [22], [23]).

The hyponormality of the Toeplitz operator T_φ was characterized in terms of the symbol φ by C. Cowen [8]. This theorem is referred to *Cowen's theorem*: T_φ is hyponormal if and only if there exists a bounded analytic function $k \in H^\infty$ with norm ≤ 1 such that $\varphi - k\overline{\varphi} \in H^\infty$.

On the other hand, we may guess that if an operator T has a finite rank self-commutator, then T enjoys some nice properties (for example, the rank of the self-commutator measures a kind of deviation from the normality). Thus we are tempted to guess that if a "nice" operator has a finite rank self-commutator then it comes to a normal operator. This guess was considered by some authors (cf. [2], [3], [17], [18], [20], [21]). A good candidate of a nice operator may be a Toeplitz operator. Indeed, in [11] and [12], the following question was addressed: If T_φ is a 2-hyponormal Toeplitz operator with nonzero finite rank self-commutator, does it follow that T_φ is analytic?

In this note we consider 2-hyponormal Toeplitz operators with finite rank self-commutators under some constraint on the kernel of the self-commutators.

2. The main result

We recall that a function $\varphi \in L^\infty$ is said to be *bounded type* (or *in the Navanlinna class*) if there are functions $\psi_1, \psi_2 \in H^\infty$ such that

$$\varphi(z) := \frac{\psi_1(z)}{\psi_2(z)} \quad \text{for almost all } z \in \mathbb{T}.$$

For example, rational functions are of bounded type.

To proceed we first recall:

Lemma 2.1 ([12, Corollary 6]). *If T_φ is 2-hyponormal and if φ or $\bar{\varphi}$ is of bounded type, then T_φ is normal or analytic.*

However, the assertion of Lemma 2.1 is not true if the assumption "φ or $\overline{\varphi}$ is of bounded type" is dropped (cf. [11]). Hence we may ask a question: If T_φ has a finite rank self-commutator, what do you say about a relationship between 2-hyponormality and normality. In [11] and [12], the authors have addressed the question: If T_φ is a 2-hyponormal Toeplitz operator with finite rank self-commutator, does it follow that T_φ is normal or analytic? In this note we determine the form of the symbol φ if T_φ is a 2-hyponormal Toeplitz operator with finite rank self-commutator and the kernel of the self-commutator satisfies a property.

A function $\theta \in H^\infty$ is called inner if $|\theta(z)| = 1$ for almost all $z \in \mathbb{T}$. A finite Blaschke product $b \in H^\infty$ is of the form

$$b(z) := e^{i\theta} \prod_{j=1}^{n} \frac{z - \alpha_j}{1 - \overline{\alpha}_j z} \quad (|\alpha_j| < 1).$$

Also a function $f \in H^2$ is called outer if

$$H^2 = \bigvee \{z^n f : n \geq 0\}.$$

Thus f is an outer function if and only if it is a cyclic vector for the shift operator. It is well-known that if f is a nonzero function in H^2, then there exist an inner function θ and an outer function e in H^2 such that

$$f = \theta e,$$

which is called the inner-outer factorization of f.

Our main result now follows:

Theorem 2.2. *Suppose T_φ is a 2–hyponormal Toeplitz operator whose self–commutator has rank $n \geq 1$. If $H_{\bar{\varphi}}\left(\ker [T_\varphi^*, T_\varphi]\right)$ contains a vector e_n in a canonical orthonormal basis $\{e_k\}_{k \in \mathbb{Z}_+}$ of H^2, then φ should be an analytic function of the form $\varphi = qh$, where q is a finite Blaschke product of degree at most n and h is an outer function.*

Proof. We first claim that

(5) $$\varphi \text{ is analytic.}$$

In view of Lemma 2.1, we may assume that both φ and $\overline{\varphi}$ are not of bounded type. To show (5), we assume to the contrary that φ is not analytic. By the same argument as [11, Theorem 8], if we define an operator $A : \operatorname{ran} H_{\bar\varphi} \to \operatorname{ran} H_\varphi$ by

(6) $$A(H_{\bar\varphi}h) = H_\varphi h,$$

then we can see that $||A|| \le 1$, so that A has an extension to H^2 since $\operatorname{ran} H_{\bar\varphi}$ is dense in H^2. In particular, A is one–one since H_φ is one-one. Furthermore it was shown in [9] that if U is the unilateral shift then $UA^* = A^*U$. Thus A^* should be an analytic Toeplitz operator (cf. [5]), and hence A is a coanalytic Toeplitz operator, say $T_{\bar\psi}$, where $\psi(z) := \sum_{n=0}^{\infty} b_n z^n$. Thus (6) can be written as

$$T_{\bar\psi} H_{\bar\psi} = H_\varphi,$$

and hence by (2),

(7) $$H_{\tilde\psi\bar\varphi} = H_\varphi.$$

Write

$$\mathcal{E}(\varphi) = \left\{k \in H^\infty : ||k||_\infty \le 1 \text{ and } \varphi - k\overline{\varphi} \in H^\infty\right\}.$$

Then by (7), $\varphi - \tilde\psi\bar\varphi \in H^\infty$, which implies $\tilde\psi \in \mathcal{E}(\varphi)$ by Cowen's theorem. Since $[T_\varphi^*, T_\varphi]$ is of finite rank, we have that $\ker [T_\varphi^*, T_\varphi]$ has finite co–dimension. Also by the hyponormality of T_φ, we have that

$$||H_{\bar\varphi}h|| = ||H_\varphi h|| \quad \text{for all } h \in \ker [T_\varphi^*, T_\varphi].$$

Thus the restriction of T_ψ to $\operatorname{cl} H_{\bar\varphi}(\ker [T_\varphi^*, T_\varphi])$ is an isometry. Since

$$\operatorname{cl} H_{\bar\varphi}(\ker [T_\varphi^*, T_\varphi])$$

has finite co–dimension, we can choose an orthonormal basis $\{f_k\}_{k\in\mathbb{Z}_+}$ for H^2 such that

$$\operatorname{cl} H_{\bar\varphi}(\ker [T_\varphi^*, T_\varphi]) = \bigvee\{f_k : k \ge r\}.$$

Also we choose a unitary operator $V : H^2 \to H^2$ by

$$Vf_k = e_k \quad (k \in \mathbb{Z}_+),$$

where $\{e_k\}_{k\in\mathbb{Z}_+}$ is a canonical orthonormal basis for H^2. Then $VT_{\bar\psi}V^*$ is also an isometry on $\bigvee\{e_k : k \ge r\}$. By our assumption, there exists $s \in \mathbb{Z}_+$ such that $f_s = e_n$. Thus

$$1 = (VT_{\bar\psi}^* T_{\bar\psi} V^* e_s,\ e_s) = (T_{\bar\psi}^* T_{\bar\psi} e_n,\ e_n) = \sum_{j=0}^{n} |b_j|^2.$$

But since $\tilde\psi \in \mathcal{E}(\varphi)$, and hence

$$||\psi||_2 = ||\tilde\psi||_2 \le ||\tilde\psi||_\infty \le 1,$$

that is, $\sum_{j=0}^{\infty}|b_j|^2 \le 1$, it follows that $b_j = 0$ for $j \ge n+1$. Note that $b_0 \ne 0$ because $A \equiv T_{\bar\psi}$ is one–one. But since $T_{\bar\psi}$ is a contraction, and hence so is $T_{\bar\psi}^* T_{\bar\psi}$. Thus a straightforward calculation shows that

$$1 \ge ||T_{\bar\psi}^* T_{\bar\psi} e_n||^2 = \sum_{m=0}^{n-1} \left| \sum_{k=0}^{m} \bar b_k \, b_{n-m+k} \right|^2 \; + \; 1,$$

which implies that $b_1 = b_2 = \cdots = b_n = 0$ by a telescoping argument. Therefore ψ is a constant of modulus 1. But since $\tilde\psi \in \mathcal{E}(\varphi)$ it follows that φ is of the form $\varphi = \bar f + e^{i\theta} f$ for some $f \in H^\infty$ and $\theta \in [0, 2\pi)$, which implies that T_φ is normal, which is a contradiction. This proves (5).

Now if T_φ is a hyponormal operator with finite rank self–commutator, then by an argument of [19, Theorem 10], there exists a finite Blaschke product $k \in \mathcal{E}(\varphi)$ such that

$$\deg(k) = \operatorname{rank}[T_\varphi^*, T_\varphi].$$

But since φ is analytic, it follows from the Cowen's theorem that $k\,\bar\varphi =: g \in H^\infty$ with $\deg(k) = n$. Then $\overline{\varphi} = \frac{g}{k}$, so that $\overline{\varphi}$ is of bounded type and we may write

$$\varphi = \theta\overline{a},$$

where θ is an inner function, $a \in H^2$, and θ and a are coprime. Thus $k\overline{\theta}a = g \in H^\infty$, which implies that θ is an inner divisor of k. Let $\varphi := qh$ be the inner-outer factorization of φ, where q is an inner part and h is an outer part. Then we can show that $\deg(q) \le \deg(\theta)$. Therefore we can conclude that q is also a finite Blaschke product of degree at most n. This completes the proof. □

References

[1] M. B. Abrahamse, *Subnormal Toeplitz operators and functions of bounded type*, Duke Math. J. **43** (1976), no. 3, 597–604.

[2] A. Aleman, *Subnormal operators with compact selfcommutator*, Manuscripta Math. **91** (1996), no. 3, 353–367.

[3] I. Amemiya, T. Ito, and T. K. Wong, *On quasinormal Toeplitz operators*, Proc. Amer. Math. Soc. **50** (1975), 254–258.

[4] J. Bram, *Subnormal operators*, Duke Math. J. **22** (1955), 75–94.

[5] A. Brown and P. R. Halmos, *Algebraic properties of Toeplitz operators*, J. Reine Angew. Math. **213** (1963/1964), 89–102.

[6] J. B. Conway, *The Theory of Subnormal Operators*, Math. Surveys and Monographs, **36**, Amer. Math. Soc. Providence, 1991

[7] C. Cowen, *More subnormal Toeplitz operators*, J. Reine Angew. Math. **367** (1986), 215–219.

[8] ———, *Hyponormal and subnormal Toeplitz operators*, Surveys of Some Recent Results in Operator Theory, I (J. B. Conway and B. B. Morrel, eds.), Pitman Research Notes in Mathematics, pp. 155–167, Vol. **171**, Longman, 1988.

[9] ———, *Hyponormality of Toeplitz operators*, Proc. Amer. Math. Soc. **103** (1988), no. 3, 809–812.

[10] C. Cowen and J. Long, *Some subnormal Toeplitz operators*, J. Reine Angew. Math. **351** (1984), 216–220.

[11] R. E. Curto, S. H. Lee, and W. Y. Lee, *Subnormality and 2-hyponormality for Toeplitz operators*, Integral Equations Operator Theory **44** (2002), no. 2, 136–148.

[12] R. E. Curto and W. Y. Lee, *Subnormality and k–hyponormality of Toeplitz operators: A brief survey and open questions*, Proceedings of Le Congres International des Mathematiques de Rabat, 73–81, (M. Mbekhta, ed.), The Theta Foundation, Bucharest, Romania, 2003.

[13] P. Fan, *Remarks on hyponormal trigonometric Toeplitz operators*, Rocky Mountain J. Math. **13** (1983), no. 3, 489–493.

[14] D. R. Farenick and W. Y. Lee, *Hyponormality and spectra of Toeplitz operators*, Trans. Amer. Math. Soc. **348** (1996), no. 10, 4153–4174.

[15] C. Gu, *A generalization of Cowen's characterization of hyponormal Toeplitz operators*, J. Funct. Anal. **124** (1994), no. 1, 135–148.

[16] T. Ito and T. K. Wong, *Subnormality and quasinormality of Toeplitz operators*, Proc. Amer. Math. Soc. **34** (1972), 157–164.

[17] J. E. McCarthy and L. Yang, *Subnormal operators and quadrature domains*, Adv. Math. **127** (1997), no. 1, 52–72.

[18] B. B. Morrel, *A decomposition for some operators*, Indiana Univ. Math. J. **23** (1973), 497–511.

[19] T. Nakazi and K. Takahashi, *Hyponormal Toeplitz operators and extremal problems of Hardy spaces*, Trans. Amer. Math. Soc. **338** (1993), no. 2, 753–767.

[20] D. Xia, *Analytic theory of subnormal operators*, Integral Equations Operator Theory **10** (1987), no. 6, 880–903.

[21] ——, *On pure subnormal operators with finite rank self-commutators and related operator tuples*, Integral Equations Operator Theory **24** (1996), no. 1, 106–125.

[22] D. Yu, *Hyponormal Toeplitz operators on $H^2(\mathbb{T})$ with polynomial symbols*, Nagoya Math. J. **144** (1996), 179–182.

[23] K. Zhu, *Hyponormal Toeplitz operators with polynomial symbols*, Integral Equations Operator Theory **21** (1995), no. 3, 376–381.

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