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S-ITERATION PROCESS FOR ASYMPTOTIC POINTWISE NONEXPANSIVE MAPPINGS IN COMPLETE HYPERBOLIC METRIC SPACES

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Abstract. In this paper, we study the modified S-iteration process for asymptotic pointwise nonexpansive mappings in a uniformly convex hyperbolic metric space. We then prove the convergence of the sequence generated by the modified S-iteration process.

1. Introduction

Let C be a nonempty subset of a metric space (X, d). We say that $x \in C$ is a fixed point of T if

$$Tx = x.$$

The fixed point set of T is denoted by $Fix(T)$. A mapping $T : C \to C$ is said to be asymptotic pointwise nonexpantive if for any $x, y \in C$, there exists a sequence of non-negative number $\{k_n(x)\}$ such that, for all $x, y \in C$,

$$d(T^n(x), T^n(y)) \leq k_n(x)d(x, y)$$

and $\lim_{n\to\infty} k_n(x) = 1$.

Example 1.1 ([2])**.** Let B denote the unit ball in the Hilbert space l^2 and let T be defined as follows

$$T : (x_1, x_2, x_3, \ldots) \to (0, x_1^2, A_2x_2, A_3x_3, \ldots),$$

where A_i is a sequence of number such that $0 < A_i < 1$ and $\prod_{i=2}^{\infty} A_i = \frac{1}{2}$. Then T is asymptotically nonexpansive, but not nonexpansive.

In 2013, Idn Dehaish et al. [3] studied the following modified Mann iteration process defined by: $x_1 \in C$ and

$$x_{n+1} = t_nT^n(x_n) \oplus (1 - t_n)x_n, \quad n \in \mathbb{N} \tag{1}$$

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where $\{t_n\} \subset (0,1)$. This iteration process was introduced by Schu [7]. They proved that the modified Mann iteration process defined by (1) converges in a weaker sense to a fixed point of T in a complete uniformly convex metric space.

Agrawal et al. [1] introduced the S-iteration process defined by: $x_1 \in C$ and

$$\begin{aligned} x_{n+1} &= \alpha_n T^n y_n + (1-\alpha_n)T^n x_n \\ y_n &= \beta_n T^n x_n + (1-\beta_n)x_n, \quad n \in \mathbb{N}, \end{aligned}$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in $(0,1)$.

In this paper, motivated by [1] and [3], we consider the modified S-iteration process defined by: $x_1 \in C$ and

$$\begin{aligned} x_{n+1} &= \alpha_n T^n y_n \oplus (1-\alpha_n)T^n x_n \\ y_n &= \beta_n T^n x_n \oplus (1-\beta_n)x_n, \quad n \in \mathbb{N}, \end{aligned}$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in $(0,1)$. We prove the existence of the fixed point of an asymptotic pointwise mapping defined on uniformly convex hyperbolic metric spaces. Moreover, we study the convergence of the modified S-iteration process associated with asymptotic pointwise mappings.

2. Preliminaries and lemmas

In this section, we provide some basic concepts, definitions and lemmas which will be used in the sequel. Throughout this paper, (M,d) will denote a metric space. Suppose that there exists a family $\mathcal{F}$ of metric segments such that any two point x, y in M are endpoint of a unique metric segment $[x,y] \in \mathcal{F}$ ($[x,y]$ is an isometric image of the real line interval $[0, d(x,y)]$). We shall denote by $\beta x \oplus (1-\beta)y$ the unique point z of $[x,y]$ which satisfies

$$d(x,z) = (1-\beta)d(x,y) \text{ and } d(z,y) = \beta d(x,y),$$

where $\beta \in [0,1]$. Such metric spaces are usually called convex metric spaces [5]. Moreover, if we have

$$d(\alpha p \oplus (1-\alpha)x, \alpha q \oplus (1-\alpha)y) \le \alpha d(p,q) + (1-\alpha)d(x,y)$$

for all p, q, x, y in M, and $\alpha \in [0,1]$, then M is said to be a hyperbolic metric space (see [6]).

Definition. Let (M,d) be a hyperbolic metric space. We say that M is uniformly convex if for any $a \in M$, for every $r > 0$, and for each $\epsilon > 0$,

$$\delta(r,\epsilon) = \inf\left\{1 - \frac{1}{r}d\left(\frac{1}{2}x \oplus \frac{1}{2}y, a\right); d(x,a) \le r, d(y,a) \le r, d(x,y) \ge r\epsilon\right\} > 0.$$

It is noted that if (M,d) is uniformly convex, then for every $s \ge 0$, $\epsilon > 0$, there exists $\eta(s,\epsilon) > 0$ depending on s and ϵ such that

$$\delta(r,\epsilon) > \eta(s,\epsilon) > 0 \text{ for any } r > s.$$

Theorem 2.1 ([3])**.** *Assume that (M,d) is complete and uniformly convex. Let $C \subset M$ be nonempty, convex, and closed. Then for any $x \in M$, there exists a unique best approximant of x in C, i.e., a unique $x_0 \in C$ such that*

$$d(x,x_0) = d(x,C).$$

Lemma 2.2 ([4])**.** *Let (M,d) be uniformly convex. Assume that there exists $r \geq 0$ such that*

$$\limsup_{n\to\infty} d(x_n,a) \leq r, \ \limsup_{n\to\infty} d(y_n,a) \leq r \text{ and } \lim_{n\to\infty} d\left(a, \frac{1}{2}x_n \oplus \frac{1}{2}y_n\right) = r.$$

Then

$$\lim_{n\to\infty} d(x_n,y_n) = 0.$$

Theorem 2.3 ([4])**.** *Let (M,d) be uniformly convex. Fix $a \in M$. For each $r > 0$ and for each $\varepsilon > 0$ denote*

$$\Psi(r,\varepsilon) = \inf\left\{\frac{1}{2}d^2(a,x) + \frac{1}{2}d^2(a,y) - d^2\left(a, \frac{1}{2}x \oplus \frac{1}{2}y\right)\right\},$$

where the infimum is taken over all $x,y \in M$ such that $d(a,x) \leq r$, $d(a,y) \leq r$, and $d(x,y) \geq r\varepsilon$. Then $\Psi(r,\varepsilon) > 0$ for any $0 < r$ and for each $\varepsilon > 0$. Moreover, for a fixed $r > 0$, we have

(i) $\Psi(r,0) = 0$;
(ii) $\Psi(r,\varepsilon)$ *is a nondecreasing function of* ε;
(iii) *if* $\lim_{n\to\infty} \Psi(r,t_n) = 0$, *then* $\lim_{n\to\infty} t_n = 0$.

Definition ([4])**.** We say that (M,d) is 2-uniformly convex if

$$c_M = \inf\left\{\frac{\Psi(r,\varepsilon)}{r^2\varepsilon^2}; r > 0, \varepsilon > 0\right\} > 0.$$

Theorem 2.4 ([3])**.** *Assume that (M,d) is 2-uniformly convex. Then for any $\alpha \in (0,1)$, there exists $C_M > 0$ such that*

$$d^2(a,\alpha x \oplus (1-\alpha)y) + C_M \min(\alpha^2,(1-\alpha)^2)d^2(x,y) \leq \alpha d^2(a,x) + (1-\alpha)d^2(a,y)$$

for any $a,x,y \in M$.

Recall that $\tau : M \to \mathbb{R}_+$ is called a type if there exists $\{x_n\}$ in M such that

$$\tau(x) = \limsup_{n\to\infty} d(x,x_n).$$

Theorem 2.5 ([4])**.** *Assume that (M,d) is complete and uniformly convex. Let C be any nonempty, closed, bounded, and convex subset of M. Let τ be a type defined on C. Then any minimizing sequence of τ is convergent. Its limit is independent of the minimizing sequence.*

In fact if M is 2-uniformly convex, and τ is a type defined on a nonempty, closed, bounded, and convex subset C of M, then there exists a unique $x_0 \in C$ such that

$$\tau^2(x_0) + 2c_M d^2(x_0, x) \leq \tau^2(x)$$

for any $x \in C$.

Theorem 2.6 ([3])**.** *Let (M, d) be a complete hyperbolic metric space which is 2-uniformly convex. Let C be a nonempty, closed, convex, and bounded subset of M. Let $T : C \to C$ be asymptotic pointwise nonexpansive. Then T has a fixed point in C. Moreover, the fixed point set $Fix(T)$ is closed and convex.*

3. Main results

In this section, we prove some lemmas which will be used in our theorem.

Lemma 3.1. *Let C be a nonempty, closed, convex, and bounded subset of a complete hyperbolic 2-uniformly convex metric space (M, d). Let $T : C \to C$ be asymptotic pointwise nonexpansive such that $Fix(T) \neq \emptyset$. Assume that $\sum_{n=1}^{\infty}(k_n(x) - 1) < \infty$ for any $x \in C$. Let $\{\alpha_n\}$ and $\{\beta_n\}$ be sequences in $(0, 1)$ such that $0 < a \leq \alpha_n, \beta_n \leq b < 1$ for some a, b. The modified S-iteration process is defined by*

$$\begin{aligned} x_{n+1} &= \alpha_n T^n y_n \oplus (1 - \alpha_n) T^n x_n \\ y_n &= \beta_n T^n x_n \oplus (1 - \beta_n) x_n, \quad n \in \mathbb{N} \end{aligned} \tag{1}$$

where $x_1 \in C$ is a fixed arbitrary point. Then for any $z \in Fix(T)$, $\lim_{n\to\infty} d(x_n, z)$ and $\lim_{n\to\infty} d(y_n, z)$ exist.

Proof. Let $z \in Fix(T)$. Then

$$\begin{aligned} d(y_n, z) &\leq \beta_n d(T^n x_n, z) + (1 - \beta_n) d(x_n, z) \\ &= \beta_n d(T^n x_n, T^n z) + (1 - \beta_n) d(x_n, z) \\ &\leq \beta_n k_n(z) d(x_n, z) + (1 - \beta_n) d(x_n, z) \\ &= d(x_n, z)[\beta_n(k_n(z) - 1) + 1] \end{aligned} \tag{2}$$

and

$$\begin{aligned} d(x_{n+1}, z) &\leq \alpha_n d(T^n y_n, z) + (1 - \alpha_n) d(T^n x_n, z) \\ &= \alpha_n d(T^n y_n, T^n z) + (1 - \alpha_n) d(T^n x_n, T^n z) \\ &\leq \alpha_n k_n(z) d(y_n, z) + (1 - \alpha_n) k_n(z) d(x_n, z) \\ &\leq \alpha_n k_n(z)[\beta_n(k_n(z) - 1) + 1] d(x_n, z) + (1 - \alpha_n) k_n(z) d(x_n, z) \\ &= d(x_n, z) k_n(z)[\alpha_n \beta_n(k_n(z) - 1) + 1]. \end{aligned}$$

So, it follows that

$$\begin{aligned} d(x_{n+1}, z) - d(x_n, z) &\leq d(x_n, z) k_n(z)[\alpha_n \beta_n(k_n(z) - 1) + 1] - d(x_n, z) \\ &= d(x_n, z)[k_n(z)(\alpha_n \beta_n(k_n(z) - 1) + 1) - 1] \end{aligned}$$

for any $n \geq 1$. Put $\delta(C) = \sup\{d(m_1, m_2) : m_1, m_2 \in C\}$. Hence, we obtain

$$d(x_{n+r}, z) - d(x_n, z) \leq \delta(C) \sum_{i=0}^{r-1} [k_{n+i}(z)(\alpha_{n+i}\beta_{n+i}(k_{n+i}(z) - 1) + 1) - 1].$$

Put $a_{n+i}(z) = k_{n+i}(z)\alpha_{n+i}\beta_{n+i}(k_{n+i}(z) - 1) + 1$. Then we have

$$d(x_{n+r}, z) - d(x_n, z) \leq \delta(C) \sum_{i=0}^{r-1} (a_{n+i}(z) - 1)$$

for any $n, r \geq 1$. Letting $r \to \infty$, we see that

$$\limsup_{r\to\infty} d(x_r, z) \leq d(x_n, z) + \delta(C) \sum_{i=n}^{\infty} (a_{n+i}(z) - 1)$$

for any $n \geq 1$. Next we let $n \to \infty$, since C is bounded,

$$\begin{aligned} \limsup_{r\to\infty} d(x_r, z) &\leq \liminf_{n\to\infty} d(x_n, z) + \delta(C) \liminf_{n\to\infty} \sum_{i=n}^{\infty} (a_{n+i}(z) - 1) \\ &= \liminf_{n\to\infty} d(x_n, z). \end{aligned}$$

This shows that $\limsup_{r\to\infty} d(x_r, z) = \liminf_{n\to\infty} d(x_n, z)$ and thus $\lim_{n\to\infty} d(x_n, z) = p$ exists.

We next prove that $\lim_{n\to\infty} d(y_n, z)$ exists. Consider

$$\begin{aligned} d(x_{n+1}, z) &\leq \alpha_n d(T^n y_n, z) + (1 - \alpha_n) d(T^n x_n, z) \\ &\leq \alpha_n k_n(z) d(y_n, z) + k_n(z) d(x_n, z) - \alpha_n k_n(z) d(x_n, z), \end{aligned}$$

which implies that

$$\alpha_n k_n(z) d(x_n, z) \leq \alpha_n k_n(z) d(y_n, z) + k_n(z) d(x_n, z) - d(x_{n+1}, z).$$

Then

$$\begin{aligned} k_n(z) d(x_n, z) &\leq k_n(z) d(y_n, z) + \frac{1}{\alpha_n} k_n(z) d(x_n, z) - \frac{1}{\alpha_n} d(x_{n+1}, z) \\ &= k_n(z) d(y_n, z) + \frac{1}{\alpha_n} [k_n(z) d(x_n, z) - d(x_{n+1}, z)]. \end{aligned}$$

This gives

$$\liminf_{n\to\infty} d(x_n, z) \leq \liminf_{n\to\infty} d(y_n, z).$$

It follows that $p \leq \liminf_{n\to\infty} d(y_n, z)$ and hence, by (2), we have

$$\begin{aligned} \limsup_{n\to\infty} d(y_n, z) &\leq \limsup_{n\to\infty} d(x_n, z)[\beta_n(k_n(z) - 1) + 1] \\ &= \limsup_{n\to\infty} d(x_n, z). \end{aligned}$$

We have $\limsup_{n\to\infty} d(y_n, z) \leq p$ and thus $\limsup_{n\to\infty} d(y_n, z) \leq p \leq \liminf_{n\to\infty} d(y_n, z)$. This shows that $\lim_{n\to\infty} d(y_n, z)$ exists. $\square$

Lemma 3.2. *Let C be a nonempty, closed, convex, and bounded subset of a complete hyperbolic 2-uniformly convex metric space (M,d). Let $T : C \to C$ be asymptotic pointwise nonexpansive such that $Fix(T) \neq \emptyset$. Let $\{x_n\}$ be defined by (1). Then*

$$\lim_{n\to\infty} d(x_n, T^n x_n) = 0 \text{ and } \lim_{n\to\infty} d(x_n, T^m x_n) = 0 \quad \text{for all } m \geq 1,$$

provided that $L = \sup_{n\in\mathbb{N}} \sup_{x\in C} k_n(x) < \infty$, i.e., T is a uniformly Lipschitzian mapping on C.

Proof. Let $z \in Fix(T)$. First we will prove that $\lim_{n\to\infty} d(x_n, T^n x_n) = 0$. By Theorem 2.4, we have

$$\begin{aligned} & d^2(z, y_n) + C_M \min(\beta_n^2, (1-\beta_n)^2) d^2(x_n, T^n x_n) \\ & \leq \beta_n d^2(z, x_n) + (1-\beta_n) d^2(z, T^n x_n), \end{aligned}$$

which implies that

$$\begin{aligned} & C_M \min(\beta_n^2, (1-\beta_n^2)) d^2(x_n, T^n x_n) \\ & \leq \beta_n d^2(z, x_n) + (1-\beta_n) d^2(z, T^n x_n) - d^2(z, y_n) \\ & = \beta_n d^2(z, x_n) + (1-\beta_n) d^2(T^n z, T^n x_n) - d^2(z, y_n) \\ & \leq \beta_n d^2(z, x_n) + (1-\beta_n) k_n(z) d^2(z, x_n) - d^2(z, y_n) \\ & = \beta_n d^2(z, x_n) + k_n(z) d^2(z, x_n) - \beta_n k_n(z) d^2(z, x_n) - d^2(z, y_n) \\ & = d^2(z, x_n)[\beta_n + k_n(z) - \beta_n k_n(z)] - d^2(z, y_n) \\ & = d^2(z, x_n)[k_n(z)(1-\beta_n) + \beta_n] - d^2(z, y_n) + d^2(x_n, z) - d^2(x_n, z) \\ & = d^2(z, x_n)[(k_n(z)(1-\beta_n) + \beta_n) - 1] - d^2(z, y_n) + d^2(x_n, z) \\ & = d^2(z, x_n)(k_n(z) - 1)(1-\beta_n) - d^2(z, y_n) + d^2(x_n, z). \end{aligned}$$

Thus $\lim_{n\to\infty} d(x_n, T^n(x_n)) = 0$, by Lemma 3.1 and $k_n(z) \to 1$.

Next we prove that $\lim_{n\to\infty} d(x_n, T^m(x_n)) = 0$ for any $m \geq 1$. We see that

$$\begin{aligned} d(x_n, Tx_n) & \leq d(x_n, T^n x_n) + d(T^n x_n, Tx_n) \\ & \leq d(x_n, T^n x_n) + L d(T^{n-1} x_n, x_n) \end{aligned} \tag{1}$$

for any $n \geq 2$. Since

$$\begin{aligned} d(T^{n-1} x_n, x_n) & \leq d(T^{n-1} x_n, T^{n-1} x_{n-1}) + d(T^{n-1} x_{n-1}, x_n) \\ & \leq L d(x_n, x_{n-1}) + d(T^{n-1} x_{n-1}, x_n), \end{aligned} \tag{2}$$

we get that, by (1) and (2),

$$d(x_n, Tx_n) \leq d(x_n, T^n x_n) + L^2 d(x_n, x_{n-1}) + L d(x_n, T^{n-1} x_{n-1}) \tag{3}$$

for any $n \geq 2$. We see that

$$d(x_n, T^{n-1} x_{n-1}) = d((1-\alpha_{n-1}) T^{n-1} x_{n-1} \oplus \alpha_{n-1} T^{n-1} y_{n-1}, T^{n-1} x_{n-1})$$

$$
\begin{aligned}
&= \alpha_{n-1} d(T^{n-1}y_{n-1}, T^{n-1}x_{n-1}) \\
&\leq \alpha_{n-1} L d(y_{n-1}, x_{n-1}) \\
&= \alpha_{n-1} L d((1-\beta_{n-1})x_{n-1} \oplus \beta_{n-1}T^{n-1}x_{n-1}, x_{n-1}) \\
&= \alpha_{n-1} L \beta_{n-1} d(T^{n-1}x_{n-1}, x_{n-1}) \qquad (4)
\end{aligned}
$$

and

$$
\begin{aligned}
d^2(x_n, x_{n-1}) &= d^2((1-\alpha_{n-1})T^{n-1}x_{n-1} \oplus \alpha_{n-1}T^{n-1}y_{n-1}, x_{n-1}) \\
&\leq \alpha_{n-1} d^2(x_{n-1}, T^{n-1}y_{n-1}) + (1-\alpha_{n-1}) d^2(x_{n-1}, T^{n-1}x_{n-1}). \qquad (5)
\end{aligned}
$$

On the other hand, we obtain

$$
\begin{aligned}
d(x_{n-1}, T^{n-1}y_{n-1}) &\leq d(x_{n-1}, T^{n-1}x_{n-1}) + d(T^{n-1}x_{n-1}, T^{n-1}y_{n-1}) \\
&\leq d(x_{n-1}, T^{n-1}x_{n-1}) + L d(x_{n-1}, y_{n-1}) \\
&= d(x_{n-1}, T^{n-1}x_{n-1}) \\
&\quad + L d(x_{n-1}, (1-\beta_{n-1})x_{n-1} \oplus \beta_{n-1}T^{n-1}x_{n-1}) \\
&= d(x_{n-1}, T^{n-1}x_{n-1}) + L\beta_{n-1} d(T^{n-1}x_{n-1}, x_{n-1}) \\
&= (1 + L\beta_{n-1}) d(T^{n-1}x_{n-1}, x_{n-1}). \qquad (6)
\end{aligned}
$$

Substituting (6) in to (5), we obtain

$$
\begin{aligned}
d^2(x_n, x_{n-1}) &\leq \alpha_{n-1}(1 + L\beta_{n-1})^2 d^2(T^{n-1}x_{n-1}, x_{n-1}) \\
&\quad + (1-\alpha_{n-1}) d^2(x_{n-1}, T^{n-1}x_{n-1}) \\
&= [\alpha_{n-1}(1 + L\beta_{n-1})^2 + (1-\alpha_{n-1})] d^2(T^{n-1}x_{n-1}, x_{n-1}). \qquad (7)
\end{aligned}
$$

Substituting (4) and (7) in to (3), we have

$$
\begin{aligned}
d(x_n, Tx_n) &\leq d(x_n, T^n x_n) \\
&\quad + L^2[(\sqrt{\alpha_{n-1}(1 + L\beta_{n-1})^2 + (1-\alpha_{n-1})}) d(T^{n-1}x_{n-1}, x_{n-1})] \\
&\quad + L[\alpha_{n-1} L \beta_{n-1} d(T^{n-1}x_{n-1}, x_{n-1})].
\end{aligned}
$$

Hence we get $\lim_{n\to\infty} d(x_n, Tx_n) = 0$. On the other hand, we see that

$$
\begin{aligned}
d(x_n, T^m x_n) &\leq \sum_{k=0}^{m-1} d(T^k x_n, T^{k+1} x_n) \\
&\leq \sum_{k=0}^{m-1} L d(x_n, Tx_n),
\end{aligned}
$$

which implies that $d(x_n, T^m x_n) \leq mL d(x_n, Tx_n)$ for any $m \geq 1$. Thus $\lim_{n\to\infty} d(x_n, T^m x_n) = 0$ for any $m \geq 1$. □

Theorem 3.3. *Let C be a nonempty, closed, convex, and bounded subset of a complete hyperbolic 2-uniformly convex metric space (M, d). Let $T : C \to C$ be asymptotic pointwise nonexpansive such that $Fix(T) \neq \emptyset$. Let $\{x_n\}$ be defined* *by* (1). *Assume that* T *is a uniformly Lipschitzian mapping on* C. *Consider the type* $\tau(x) = \limsup_{n\to\infty} d(x_n, x)$ *on* C. *If* z *is the minimum point of* τ, *i.e.,* $\tau(z) = \inf\{\tau(x) = x \in C\}$. *Then* $Tz = z$.

Proof. We see that

$$d^2\left(x_n, \frac{z + T^m z}{2}\right) + C_M d^2(z, T^m z) \le \frac{1}{2} d^2(x_n, z) + \frac{1}{2} d^2(x_n, T^m z).$$

If we let $n \to \infty$, and $\tau(x) = \limsup_{n\to\infty} d(x_n, x)$, then for any $m \ge 1$

$$\tau^2\left(\frac{z + T^m z}{2}\right) + C_M d^2(z, T^m z) \le \frac{1}{2}\tau^2(z) + \frac{1}{2}\tau^2(T^m z).$$

This shows that

$$\begin{aligned} \tau(T^m z) = \limsup_{n\to\infty} d(x_n, T^m z) &\le \limsup_{n\to\infty} [d(x_n, T^m x_n) + d(T^m x_n, T^m z)] \\ &\le \limsup_{n\to\infty} d(T^m x_n, T^m z). \end{aligned}$$

Therefore, we obtain

$$\begin{aligned} \tau(T^m z) &\le k_m(z) \limsup_{n\to\infty} d(x_n, z) \\ &= k_m(z)\tau(z), \end{aligned}$$

for any $m \ge 1$. Since $\tau(z) \le \tau\left(\frac{z+T^m z}{2}\right)$ and $\tau(T^m z) \le k_m(z)\tau(z)$, we get $\tau^2(z) + C_M d^2(z, T^m z) \le \frac{1}{2}\tau^2(z) + \frac{k_m^2(z)}{2}\tau^2(z)$ for any $m \ge 1$. Hence

$$\begin{aligned} C_M d^2(z, T^m z) &\le \frac{1}{2}\tau^2(z) + \frac{k_m^2(z)}{2}\tau^2(z) - \tau^2(z) \\ &= \tau^2(z)\left[\frac{1}{2} + \frac{k_m^2(z)}{2} - 1\right] \\ &= \tau^2(z)\left[\frac{k_m^2(z) - 1}{2}\right] \end{aligned}$$

for any $m \ge 1$. This shows that $\lim_{m\to\infty} d(z, T^m z) = 0$. We know that

$$\begin{aligned} d(z, Tz) &\le d(z, T^m z) + d(T^m z, Tz) \\ &\le d(z, T^m z) + Ld(T^{m-1} z, z). \end{aligned}$$

Hence $Tz = z$ by Lemma 3.2. This completes the proof. □

Remark 3.4. The S-iteration process studied in this paper is quite different from Mann iteration process defined in [3].

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