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# SOLUTION FORMULAS FOR SOME NONLOCAL HEAT EQUATIONS

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Abstract. We obtain solution formulas for some nonlocal heat equations. As a simple application, we prove finite time blow-up of the solution.

## 1. Introduction

We are interested in the following heat equations with nonlocal terms

$$\partial_t u - \Delta u - u \left( \int_\Omega |\nabla u|^p dx \right)^q + f(x,t)u = 0, \quad t > 0, \quad x \in \Omega, \tag{1.1}$$

or

$$\partial_t u - \Delta u - u \int_0^t \left( \int_\Omega |\nabla u|^p dx \right)^q ds + f(x,t)u = 0, \quad t > 0, \quad x \in \Omega, \tag{1.2}$$

with initial and boundary conditions

$$u(x, 0) = g(x), \quad x \in \Omega,$$
$$u(x, t) = 0, \quad t > 0, \quad x \in \partial\Omega.$$

Here $p$, $q \geq 1$ are constants and $f$ is a given smooth function. $\Omega$ is a smoothly bounded domain of $\mathbb{R}^n$ or whole space $\mathbb{R}^n$. In the case of $\mathbb{R}^n$, the boundary condition is understood as $\lim_{|x|\to\infty} u(x, t) = 0$.

The following equation has been studied in [2]

$$\partial_t u - \Delta u = u^m(x,t) \left( \int_\Omega |\nabla u(y,t)|^2 dy \right)^r, \quad t > 0, \quad x \in \Omega,$$
$$u(x, 0) = u_0(x), \quad x \in \Omega,$$
$$u(x, t) = 0, \quad t > 0, \quad x \in \partial\Omega,$$

where $m \geq 1$ and $r > 0$. It was mentioned in Remark 4.2 of [2] that in the case $m = 1$, $u$ can be written as

$$u(x,t) = e^{G(t)} v(x,t),$$

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where $G(t) = \int_0^t \left(\int_\Omega |\nabla u(y,s)|^2\, dy\right)^r ds$ and $v$ is the solution of $v_t - \Delta v = 0$ and $v(x,\,0) = u_0(x)$. If $u$ blows up in finite time $T^*$, then we must have $G(T^*) = \infty$, and $u(x,t)$ blows up at every point of $\Omega$ at $t \to T^*$. The model (1.2) is a space-time generalization of (1.1).

We will derive more precise formulas for the nonlocal terms of (1.1) and (1.2), as long as solutions exist. To state our results, let us consider an equation

$$
\begin{aligned}
&\partial_t v - \Delta v + f(x,t)v = 0, \quad t > 0, \quad x \in \Omega,\\
&v(x,\,0) = g(x), \quad x \in \Omega, \qquad\qquad (1.3)\\
&v(x,\,t) = 0, \quad t > 0, \quad x \in \partial\Omega.
\end{aligned}
$$

From now on, we assume that the equations (1.1) and (1.2) admit a unique, classical solution $u$ in time $[0,\,T]$. Our first result is concerned with the equation (1.1).

**Theorem 1.1.** *As long as it exists, the solution $u$ of* (1.1) *is given by*

$$u(x,\,t) = \frac{v(x,\,t)}{\left(1 - pq\int_0^t \left(\int_\Omega |\nabla v(y,s)|^p\, dy\right)^q ds\right)^{\frac{1}{pq}}},$$

*where $v$ is a solution of* (1.3).

Our second result is concerned with the equation (1.2).

**Theorem 1.2.** *As long as it exists, the solution $u$ of* (1.2) *is given by*

$$u(x,t) = v(x,t)\exp\left(\frac{1}{pq}h(t)\right).$$

*Here $h(t)$ is a solution of the following ODE*

$$
\begin{aligned}
&\frac{d^2h}{dt^2} = pq\mu e^h,\\
&h(0) = 0,\ h'(0) = 0,
\end{aligned}
$$

*where $\mu(t) = \left(\int_\Omega |\nabla v(x,t)|^p\, dx\right)^q$ and $v$ is a solution of* (1.3).

As a simple application of Theorems 1.1 and 1.2, we have Corollaries 2.1 and 3.1 which show finite time blow-up of solutions.

We prove Theorem 1.1 in Section 2. Theorem 1.2 is proved in Section 3. We close this study giving a few remarks in Section 4.

## 2. Proof of Theorem 1.1

First of all, we reduce the equation (1.1) to heat equation without nonlocal term. Define $\lambda(t)$ as

$$\lambda(t) = \left(\int_\Omega |\nabla u|^p\, dx\right)^q.$$

Considering integration factor with respect to time variable, it seems natural to transform

$$u(x,t) = v(x,t)e^{\int_0^t \lambda(s)ds}. \tag{2.1}$$

Then the function $v$ satisfies a heat equation

$$\begin{aligned} &\partial_t v - \Delta v + f(x,t)v = 0, \quad t > 0, \quad x \in \Omega, \\ &v(x,\, 0) = g(x), \quad x \in \Omega, \\ &v(x,\, t) = 0, \quad t > 0, \quad x \in \partial\Omega. \end{aligned}$$

Note that $u$ and $v$ have the same initial data $g$.

We will derive more precise formula for the nonlocal term $\int_0^t \lambda(s)ds$, as long as it exists. From (2.1), we have

$$|\nabla u(x,t)|^p = |\nabla v(x,t)|^p e^{p\int_0^t \lambda(s)ds}.$$

Integrating on $\Omega$, we obtain

$$\lambda(t) = \mu(t)e^{pq\int_0^t \lambda(s)ds},$$

where we use the notation $\mu(t) = \left(\int_\Omega |\nabla v|^p\, dx\right)^q$. Taking logarithmic function and derivative, we derive a Riccati equation for $\lambda$

$$\lambda' = \frac{\mu'}{\mu}\lambda + pq\lambda^2. \tag{2.2}$$

Let us define $\beta(t) = e^{-pq\int_0^t \lambda(s)ds}$ which implies

$$\lambda = -\frac{1}{pq}\frac{\beta'}{\beta},$$

where we can check $\beta(0) = 1$ and $\lambda(0) = -\frac{1}{pq}\beta'(0)$. Then we obtain the equation for $\beta$, from (2.2),

$$\beta'' = \frac{\mu'}{\mu}\,\beta',$$

from which we derive $\beta'(t) = c\mu(t)$. Considering $-pq\lambda(0) = \beta'(0) = c\mu(0) = c\lambda(0)$, we have $c = -pq$. Integrating and considering $\beta(0) = 1$, we derive

$$\beta(t) = 1 - pq\int_0^t \mu(s)\, ds.$$

Then we arrive at

$$\lambda(t) = -\frac{1}{pq}\frac{\beta'}{\beta} = \frac{\mu(t)}{1 - pq\int_0^t \mu(s)\, ds},$$

which implies

$$\int_0^t \lambda(s)\, ds = -\frac{1}{pq}\log\left(1 - pq\int_0^t \mu(s)ds\right).$$

Then we derive, from (2.1), the desired formula

$$u(x,\,t) = \frac{v(x,\,t)}{\left(1 - pq\int_0^t \left(\int_\Omega |\nabla v(y,s)|^p\,dy\right)^q ds\right)^{\frac{1}{pq}}}.$$

As a simple application of the above formula, we can show finite time blow-up.

**Corollary 2.1.** *Let $T^* > 0$ be finite. For a solution $v$ such that*

$$\lim_{t\to T^*}\int_0^t \left(\int_\Omega |\nabla v(y,s)|^p\,dy\right)^q ds = \frac{1}{pq},$$

*we have* $\lim_{t\to T^*}|u(x,\,t)| = \infty$ *for* $x \in S := \{x\in\Omega \,|\, v(x,\,T^*) \neq 0\}$.

## 3. Proof of Theorem 1.2

Define $\Lambda(t)$ as

$$\Lambda(t) = \int_0^t \left(\int_\Omega |\nabla u|^p\,dx\right)^q ds = \int_0^t \lambda(s)\,ds.$$

As previous section, it seems natural to transform

$$u(x,t) = v(x,t)e^{\int_0^t \Lambda(s)ds}. \tag{3.1}$$

Then the function $v$ is a solution of a heat equation

$$\begin{aligned}
&\partial_t v - \Delta v + f(x,t)v = 0, \quad t>0, \quad x\in\Omega,\\
&v(x,\,0) = g(x), \quad x\in\Omega,\\
&v(x,\,t) = 0, \qquad t>0, \quad x\in\partial\Omega.
\end{aligned}$$

We will derive more precise formula for the nonlocal term $\int_0^t \Lambda(s)ds$, as long as it exists. We have, from (3.1),

$$\left(\int_\Omega |\nabla u(x,t)|^p\right)^q = e^{pq\int_0^t \Lambda(s)ds}\left(\int_\Omega |\nabla v(x,t)|^p\right)^q.$$

Integrating on $[0,\,t]$, we obtain

$$\Lambda(t) = \int_0^t \mu(s)\,e^{pq\int_0^s \Lambda(\tau)d\tau}\,ds,$$

where we use the notation $\mu(t) = \left(\int_\Omega |\nabla v|^p\,dx\right)^q$. Differentiating both sides, we have

$$\Lambda' = \mu(t)\,e^{pq\int_0^t \Lambda(\tau)d\tau}.$$

Taking logarithmic function and derivative, we derive

$$\frac{\lambda'}{\lambda} = \frac{\Lambda''}{\Lambda'} = \frac{\mu'}{\mu} + pq\int_0^t \lambda(s)\,ds \tag{3.2}$$

which leads to

$$\left(\log \frac{\lambda}{\mu}\right)' = pq \int_0^t \lambda(s)\, ds.$$

Taking derivative on both sides, we have

$$\left(\log \frac{\lambda}{\mu}\right)'' = pq\lambda. \tag{3.3}$$

Let $\log \frac{\lambda}{\mu} = h$. Considering the definition of $\lambda$, $\mu$ and (3.2), we can check

$$h(0) = \log \frac{\lambda(0)}{\mu(0)} = 0 \quad \text{and} \quad h'(0) = \frac{\lambda'(0)}{\lambda(0)} - \frac{\mu'(0)}{\mu(0)} = 0.$$

Then (3.3) can be rewritten as

$$h'' = pq\lambda = pq\mu e^h, \tag{3.4}$$

$$h(0) = 0,\ h'(0) = 0. \tag{3.5}$$

Integrating (3.4) on $[0,\, t]$ and considering the initial data (3.5), we have

$$h'(t) = pq \int_0^t \lambda(s)\, ds = pq\Lambda(t).$$

Integrating again on $[0,\, t]$ and considering the initial data (3.5), we derive

$$h(t) = pq \int_0^t \Lambda(s)\, ds.$$

Then we derive the desired formula

$$u(x,t) = v(x,t) \exp\left(\frac{1}{pq} h(t)\right).$$

As a simple application of Theorem 1.2, we can show finite time blow-up.

**Corollary 3.1.** *Suppose that $\mu(t) \geq m$ for a constant $m > 0$ and $0 \leq t \leq \frac{2}{\sqrt{pqm}}$. Then the solution $h$ of (3.4), (3.5) has the lower bound*

$$e^{\frac{1}{2}h(t)} \geq \frac{2}{2 - \sqrt{pqm}t}. \tag{3.6}$$

*Proof.* First of all, we can assume that $h(t),\ h'(t) \geq 0$. Then we have $h'' \geq pqm$, which implies that $h(t) \geq \frac{pqm}{2}t^2$. Therefore we have $h(t) \geq \log 2$ for the time $t \geq \left(\frac{\log 4}{pqm}\right)^{1/2}$. Note that $\left(\frac{\log 4}{pqm}\right)^{1/2} < \frac{2}{\sqrt{pqm}}$. On the other hand, we have, from (3.4) and (3.5),

$$\left(\frac{1}{2}(h')^2 - pqme^h\right)' \leq 0,$$

which implies

$$\begin{aligned}(h')^2 &\geq 2pqm(e^h - 1)\\ &\geq pqme^h,\end{aligned}$$

where $h(t) \geq \log 2$ is used for the second inequality. Then we $h' \geq \sqrt{pqm}e^{\frac{1}{2}h}$, which leads us to (3.6).

## 4. Remarks

Let us give the following three remarks.

1. Let us consider an equation

$$\partial_t u - \Delta u - u\left(\int_\Omega K(y,t)u(y,t)dy\right) + f(x,t)u = 0, \quad t > 0, \quad x \in \Omega,$$

where $K$ is a given function. Then we can apply the same argument as in section 2 to derive

$$u(x,\, t) = \frac{v(x,\, t)}{1 - \int_0^t \int_\Omega K(y,s)v(y,s)\, dyds}.$$

2. For a special function $f(x,t)$, we have one more reduction as follows. Consider an equation

$$\begin{aligned} \partial_t v - \Delta v &= (E \cdot x)v, \quad t > 0, \quad x \in \mathbb{R}^n, \\ v(x,\, 0) &= v_0(x), \qquad x \in \mathbb{R}^n, \end{aligned} \tag{4.1}$$

where $E = (E_1,\, E_2, \ldots, E_n)$ is constant vector with $|E|^2 = 1$. Applying the modified Avron-Herbst formula [1]

$$v(x,\, t) = \omega(x + t^2E,\, t)\exp\left(tE \cdot x + \frac{t^3}{3}\right),$$

the solution $v$ of (4.1) is related with the solution $\omega$ of

$$\begin{aligned} \partial_t \omega - \Delta\omega &= 0, \\ \omega(x,\, 0) &= v_0(x). \end{aligned}$$

3. For a special function $f(x,\, t) = ct - x$ $(c > 0)$ and $n = 1$, we have traveling waves for (1.1). In fact, assuming the ansatz $u(x,\, t) = \phi(x - ct)$, we have

$$\frac{d^2\phi}{dy^2} + c\frac{d\phi}{dy} + (y + a^2)\phi(y) = 0,$$

where $y = x - ct$ and $a^2 = \left(\int_\mathbb{R} |\phi_x(x - ct)|^p dx\right)^q$. With the notation $\psi(y) = \phi(y - a^2)$, we arrive at

$$\frac{d^2\psi}{dy^2} + c\frac{d\psi}{dy} + y\psi = 0,$$

which is an equation studied in Appendix A of [1]. The solution is given by

$$\psi(y) = e^{-cy/2 + c^3/12}\mathrm{Ai}\left(\frac{c^2}{4} - x\right),$$

where Ai is the Airy function.

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