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ON n-$*$-PARANORMAL OPERATORS

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Abstract. A Hilbert space operator $T \in \mathscr{B}(\mathscr{H})$ is said to be n-$*$-paranormal, $T \in \mathfrak{C}(n)$ for short, if $\|T^*x\|^n \leq \|T^n x\| \|x\|^{n-1}$ for all $x \in \mathscr{H}$. We proved some properties of class $\mathfrak{C}(n)$ and we proved an asymmetric Putnam-Fuglede theorem for n-$*$-paranormal. Also, we study some invariants of Weyl type theorems. Moreover, we will prove that a class n-$*$ paranormal operator is finite and it remains invariant under compact perturbation and some orthogonality results will be given.

1. Introduction

Throughout this paper let $\mathscr{H}$ be a separable complex Hilbert space with inner product $\langle\cdot,\cdot\rangle$. Let $\mathscr{B}(\mathscr{H})$ denote the C^*-algebra of all bounded linear operators on $\mathscr{H}$. If $T \in \mathscr{B}(\mathscr{H})$ we shall write $\ker(T)$ and $T\mathscr{H}$ for the null space and range of T. Also, let $\sigma(T), \sigma_a(T)$, $\sigma_r(T)$ and $\sigma_p(T)$ denote the spectrum, approximate point spectrum, the residual spectrum and the point spectrum of T.

An operator is called $*$-paranormal if $\|T^*x\|^2 \leq \|T^2x\|$ for all unit vector $x \in \mathscr{H}$. In order to discuss the relations between $*$-paranormal and p-hyponormal and log-hyponormal operators, Duggal, Jean and Kim [17], introduced $*$-class A operators defined by $|T^2| \geq |T^*|$, and they showed that $*$-class A is a subclass of $*$-paranormal and contains p-hyponormal and log-hyponormal operators. Another generalization of hyponormal operators are n-$*$-paranormal operator. An operator $T \in \mathscr{B}(\mathscr{H})$ is n-$*$-paranormal if $\|T^*x\|^n \leq \|T^n x\|$ for each unit vector x. Moreover, by n-paranormal operator we mean an operator $T \in \mathscr{B}(\mathscr{H})$ which satisfies $\|Tx\|^n \leq \|T^n x\|$ for each unit vector x. For $n = 2$, n-paranormal and n-$*$-paranormal operators are simply called paranormal and $*$-paranormal operators, respectively.

The inclusion relations between the above mentioned classes of operators are shown below (see [7, 13, 14, 29]).

hyponormal $\subset$ $*$-class $A \subset$ $*$-paranormal $\subset$ n-$*$-paranormal $\subset$ $n+1$-paranormal

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and hyponormal $\subset$ paranormal $\subset n+1$-paranormal.

The organization of the paper is as follows. We introduce our notation and terminology in Section 2. In Section 3, we consider some properties of class n-$*$-paranormal which will be used in the sequel. In Section 4, some variants of Weyl type theorems such as property (w), property (gw), property (t), property (gt) and other properties will be studied. Section 5 is devoted to extend the asymmetric Putnam-Fuglede theorem to the class of n-$*$-paranormal. Finally, in Section 6, we will show that the class n-$*$-paranormal operator is finite and it remains invariant under compact perturbation.

2. Notation and terminology

If the range $T\mathscr{H}$ of $T \in \mathscr{B}(\mathscr{H})$ is closed and $\alpha(T)$=dimker$(T) < \infty$ (resp., $\beta(T)$=co-dim$T\mathscr{H} < \infty$), then T is an upper semi-Fredholm (resp., lower semi-Fredholm) operator. Let $SF_+(\mathscr{H})$ (resp., $SF_-(\mathscr{H})$) denote the semigroup of upper semi Fredholm (resp., lower semi Fredholm) operator on $\mathscr{H}$. An operator $T \in \mathscr{B}(\mathscr{H})$ is said to be semi-Fredholm, $T \in SF$, if $T \in SF_+(\mathscr{H}) \cup SF_-(\mathscr{H})$ and Fredholm if $T \in SF_+(\mathscr{H}) \cap SF_-(\mathscr{H})$. If T is semi-Fredholm, then the index of T is defined by $\text{ind}(T) = \alpha(T) - \beta(T)$. The classes of upper semi-Weyl operators $W_+(\mathscr{H})$ and lower semi-Weyl operators $W_-(\mathscr{H})$ are defined by

$$W_+(\mathscr{H}) = \{T \in \mathscr{B}(\mathscr{H}) : T \,\text{is upper semi Fredholm and}\, \text{ind}(T) \leq 0\},$$

$$W_-(\mathscr{H}) = \{T \in \mathscr{B}(\mathscr{H}) : T \,\text{is lower semi Fredholm and}\, \text{ind}(T) \geq 0\}.$$

Let $a := a(T)$ be the ascent of an operator T; i.e., the smallest nonnegative integer p such that $\ker(T^p) = \ker(T^{p+1})$. If such integer does not exist we put $a(T) = \infty$. Analogously, let $d := d(T)$ be descent of an operator T; i.e., the smallest nonnegative integer s such that $T^s\mathscr{H} = T^{s+1}\mathscr{H}$, and if such integer does not exist we put $d(T) = \infty$. It is well known that if $a(T)$ and $d(T)$ are both finite, then $a(T) = d(T)$ [19, Proposition 38.3]. Moreover, $0 < a(T - \lambda I) = d(T - \lambda I) < \infty$ precisely when λ is a pole of the resolvent of T, see Heuser [19, Proposition 50.2].

A bounded linear operator T acting on a Hilbert space $\mathscr{H}$ is Weyl, $T \in W$, if $T \in W_+(\mathscr{H}) \cap W_-(\mathscr{H})$ and Browder, $T \in \mathcal{B}$, if T is Fredholm of finite ascent and descent. Let $\mathbb{C}$ denote the set of complex numbers. The Weyl spectrum $\sigma_w(T)$ and Browder spectrum $\sigma_b(T)$ of T are defined by

$$\sigma_w(T) = \{\lambda \in \mathbb{C} : T - \lambda \notin W\}$$

and

$$\sigma_b(T) = \{\lambda \in \mathbb{C} : T - \lambda \notin \mathcal{B}\}.$$

Let $E^0(T) = \{\lambda \in \text{iso}\sigma(T) : 0 < \alpha(T - \lambda) < \infty\}$ and $\pi^0(T)$ denote the set of all normal eigenvalues (Riesz points) of T. According to Coburn [15], Weyl's theorem holds for T if $\Delta(T) = \sigma(T) \setminus \sigma_w(T) = E^0(T)$ and Browder's theorem holds for T if $\Delta(T) = \pi^0(T)$.

Let $SF_+^-(\mathscr{H}) = \{T \in SF_+(\mathscr{H}) : \text{ind}\,(\mathrm{T}) \leq 0\}$. The *upper semi Weyl* *spectrum* is defined by $\sigma_{SF_+^-}(T) = \{\lambda \in \mathbb{C} : T - \lambda \notin SF_+^-(\mathscr{H})\}$. According to Rakočević [24], an operator $T \in \mathscr{B}(\mathscr{X})$ is said to satisfy *a-Weyl's theorem* if $\Delta_a(T) = \sigma_a(T) \setminus \sigma_{SF_+^-}(T) = E_a^0(T)$, where

$$E_a^0(T) = \{\lambda \in \mathrm{iso}\,\sigma_\mathrm{a}(\mathrm{T}) : 0 < \alpha(\mathrm{T} - \lambda\mathrm{I}) < \infty\}.$$

It is known [24] that an operator satisfying a-Weyl's theorem satisfies Weyl's theorem, but the converse does not hold in general.

An operator $T \in \mathscr{B}(\mathscr{H})$ is called B-Fredholm, $T \in \mathcal{BF}$, if there exists a natural number n, for which the induced operator $T_n = T \mid T^n\mathscr{H}$, $T_0 = T$ is Fredholm in the usual sense [10]. The class of B-Weyl operator $T \in \mathscr{B}(\mathscr{H})$ is defined by $\mathcal{BW} = \{T \in \mathcal{BF} : ind(T_n) = 0\}$. The B-Weyl spectrum $\sigma_{BW}(T)$ is defined by $\sigma_{BW}(T) = \{\lambda \in \mathbb{C} : T - \lambda \notin \mathcal{BW}\}$ [10]. As a stronger version of Weyl's theorem, generalized Weyl's theorem was introduced by Berkani [11]. Let $E(T)$ be the set of all eigenvalues of T which are isolated in $\sigma(T)$. We say that T satisfies generalized Weyl's theorem if $\Delta^g(T) = \sigma(T) \setminus \sigma_{BW}(T) = E(T)$.

Following [10], we say that T satisfies *generalized Browders's theorem*, if $\Delta^g(T) = \pi(T)$, where $\pi(T)$ is the set of poles of T.

Let $SBF_+^-(\mathscr{H})$ denote the class of all *upper B-Fredholm* operators such that $\mathrm{ind}\,(\mathrm{T}) \leq 0$. The *upper B-Weyl spectrum* $\sigma_{SBF_+^-}(T)$ of T is defined by

$$\sigma_{SBF_+^-}(T) = \{\lambda \in \mathbb{C} : T - \lambda \notin SBF_+^-(\mathscr{X})\}.$$

Following [11], we say that *generalized a-Weyl's theorem* holds for $T \in \mathscr{B}(\mathscr{X})$ if $\Delta_a^g(S) = \sigma_a(T) \setminus \sigma_{SBF_+^-}(T) = E_a(T)$, where $E_a(T) = \{\lambda \in iso\sigma_a(T) : \alpha(T - \lambda) > 0\}$ is the set of all eigenvalues of T which are isolated in $\sigma_a(T)$ and that $T \in \mathscr{B}(\mathscr{X})$ obeys *generalized a-Browder's theorem* if $\Delta_a^g(T) = \pi_a(T)$.

Following [23], we say that $T \in \mathscr{B}(\mathscr{H})$ satisfies *property* (w) if $\Delta_a(T) = E^0(T)$. The property (w) has been studied in [4, 23]. In Theorem 2.8 of [4], it is shown that property (w) implies Weyl's theorem, but the converse is not true in general. We say that $T \in \mathscr{B}(\mathscr{H})$ satisfies *property* (gw) if $\Delta_a^g(T) = E(T)$. Property (gw) has been introduced and studied in [5]. Property (gw) extends property (w) to the context of B-Fredholm theory, and it is proved in [5] that an operator possessing property (gw) satisfies property (w) but the converse is not true in general. According to [12], an operator $T \in \mathscr{B}(\mathscr{H})$ is said to possess *property* (gb) if $\Delta_a^g(T) = \pi(T)$, and is said to possess *property* (b) if $\Delta_a(T) = \pi^0(T)$. It is shown in Theorem 2.3 of [12] that an operator possessing property (gb) satisfies property (b) but the converse is not true in general.

Following [28], we say that $T \in \mathscr{B}(\mathscr{H})$ satisfies property (t) if $\Delta_+(T) = \sigma(T) \setminus \sigma_{SF_+^-}(T) = E^0(T)$. The property (t) has been studied in [28]. In Theorem 2.4 of [28] it is shown that property (t) implies property (w), but the converse is true in general. We say that $T \in \mathscr{B}(\mathscr{H})$ is said to be possesses property (gt) if $\Delta_+^g(T) = \sigma(T) \setminus \sigma_{SBF_+^-}(T) = E(T)$. Property (gt) extends property (t) to the context of B-Fredholm theory and it is proved in [28, Theorem 2.1] that an operator T possessing property (gt) possesses property (t), but the converse is not true.

We say that $T \in \mathscr{B}(\mathscr{H})$ has the *single-valued extension property* (SVEP) at point $\lambda \in \mathbb{C}$ if for every open neighborhood U_λ of λ, the only analytic function $f : U_\lambda \longrightarrow \mathscr{H}$ which satisfies the equation $(T-\mu)f(\mu) = 0$ is the constant function $f \equiv 0$. It is well-known that $T \in \mathscr{B}(\mathscr{H})$ has SVEP at every point of the resolvent $\rho(T) := \mathbb{C} \setminus \sigma(T)$. Moreover, from the identity Theorem for analytic function it easily follows that $T \in \mathscr{B}(\mathscr{H})$ has SVEP at every point of the boundary $\partial\sigma(T)$ of the spectrum. In particular, T has SVEP at every isolated point of $\sigma(T)$. In [20, Proposition 1.8], Laursen proved that if T is of finite ascent, then T has SVEP.

The quasi-nilpotent part of an operator $T \in \mathscr{B}(\mathscr{H})$ is the set

$$H_0(T) = \{x \in \mathscr{H} : \lim_{n \longrightarrow \infty} \|T^n x\|^{\frac{1}{n}} = 0\}.$$

If $T \in \mathscr{B}(\mathscr{H})$, the analytic core $K(T)$ is the set of all $x \in \mathscr{H}$ such that there exist a constant $c > 0$ and a sequence of elements $x_n \in \mathscr{H}$ such that $x_0 = x$, $Tx_n = x_{n-1}$, and $\|x_n\| \leq c^n \|x\|$ for all $n \in \mathbb{N}$.

$H_0(T)$ and $K(T)$ are generally non-closed hyperinvariant subspaces of T such that $\ker(T^n) \subseteq H_0(T)$ for all $n \in \mathbb{N} \cup \{0\}$ and $TK(T) = K(T)$; also if $\lambda \in \mathrm{iso}\sigma(T)$, then $\mathscr{H} = H_0(T-\lambda) \oplus K(T-\lambda)$, where $H_0(T-\lambda)$ and $K(T-\lambda)$ are closed [1].

3. Spectral properties of n-$*$-paranormal

In this section we consider some properties of n-$*$-paranormal which will be used in the sequel. Recall that an operator $B \in \mathscr{B}(\mathscr{H})$ is said to be simply polaroid if the isolated points of the spectrum of the operator are simple poles (i.e., order one poles) of the resolvent of the operator.

Theorem 3.1. *Let $T \in \mathscr{B}(\mathscr{H})$ be algebraically $\mathfrak{C}(n)$ operator and let λ be an isolated point in $\sigma(T)$. Then λ is a simple pole of the resolvent of T. (That is, operators in $\mathfrak{C}(n)$ are simply polaroid.)*

Proof. Let $\lambda \in \mathrm{iso}\sigma(T)$. Then T has a direct sum decomposition $T = T_1 \oplus T_2$ on $\mathscr{H} = \mathscr{H}_1 \oplus \mathscr{H}_2$ such that $\sigma(T_1) = \{\lambda\}$ and $\sigma(T_2) = \sigma(T) \setminus \{\lambda\}$. Let q be a nonconstant complex polynomial such that $q(T)$ is class $\mathfrak{C}(n)$ operator. Then $\mathscr{H}_1$ is a $q(T)$-invariant subspace, and hence $q(T_1)$ is in $\mathfrak{C}(n)$ (see [13, Theorem 8]) such that $\sigma(q(T_1)) = q(\sigma(T_1)) = q(\{\lambda\})$. But then $q(\lambda) \in \pi^0(T_1)$ and so $\lambda \in \pi^0(T_1)$. Hence, since $\lambda \notin \sigma(T_2)$, we have $\lambda \in \pi^0(T)$. $\square$

Theorem 3.2. *Let $T \in \mathscr{B}(\mathscr{H})$ be such that $T \in \mathfrak{C}(n)$, $0 \neq \sigma_p(T)$ and*

$$T = \begin{pmatrix} \lambda & T_{12} \\ 0 & T_{22} \end{pmatrix} \ \textit{on} \ \ker(T-\lambda) \oplus \ker(T-\lambda)^{\perp}.$$

Then

$$T_{12}\left(\frac{T_{22}}{\lambda} + \cdots + \left(\frac{T_{22}}{\lambda}\right)^{n+1}\right) = (n+1)T_{12}, \tag{3.1}$$

and

$$\left\|T_{22}^{n+2}x\right\|^{2/(n+2)}\|x\|^{2(n+1)/(n+2)} \geq \|T_{12}x\|^2 + \|T_{22}x\|^2 \tag{3.2}$$

for any $x \in \ker(T-\lambda)^{\perp}$. *In particular,* $T_{22} \in \mathfrak{P}(n+1)$ *for every* $n \geq 2$.

Proof. Since $\mathfrak{C}(n) \subseteq \mathfrak{P}(n+1)$ for every $n \geq 2$, the proof follows immediately from Theorem 3.1 of [32]. $\square$

Corollary 3.3. *Let* $T \in \mathscr{B}(\mathscr{H})$ *be such that* $T \in \mathfrak{C}(n)$ *and* $\lambda \neq 0$. *Then* $\ker(T_{22}-\lambda) = \{0\}$, *where* T_{22} *is as in Theorem 3.2.*

Proof. Let $x \in \ker(T_{22}-\lambda)$. Then $\|(T-\lambda)x\|^2 \leq \|T_{12}x\|^2 \leq 0$ by (3.2). Hence $x \in \ker(T-\lambda) \cap \ker(T-\lambda)^{\perp} = \{0\}$ and so $\ker(T_{22}-\lambda) = \{0\}$. $\square$

Corollary 3.4. *If* $T \in \mathfrak{C}(n)$ *and* $\lambda\mu \neq 0$, *then* $\ker(T-\lambda) \perp \ker(T-\mu)$ *for* $\mu \neq \lambda$.

Proof. Let

$$T = \begin{pmatrix} \lambda & T_{12} \\ 0 & T_{22} \end{pmatrix} \text{ on } \ker(T-\lambda) \oplus \ker(T-\lambda)^{\perp}$$

and $x = x_1 \oplus x_2 \in \ker(T-\mu)$. Then

$$0 = (T-\mu)x = [(\lambda-\mu)x_1 + T_{12}x_2] \oplus (T-\mu)x_2.$$

By $(T_{22}-\mu)x_2 = 0$ and (3.2), we have $\|T_{12}x_2\|^2 = 0$. So, $x_1 = 0$ for $\lambda \neq \mu$, which implies $x \in \ker(T-\lambda)^{\perp}$ and hence $\ker(T-\lambda) \perp \ker(T-\mu)$. $\square$

Corollary 3.5. *If* $T \in \mathfrak{C}(n)$, *then* T *has SVEP.*

Proof. Let f be an analytic function such that $(T-\lambda)f(\lambda) = 0$ on an open set U. By assumption, $f(\lambda) \in \ker(T-\lambda)$ for each $\lambda \in D$. Thus $f(\lambda) \perp f(\mu)$ for any two different nonzero numbers λ and μ in D by Corollary 3.4. Therefore, for any sequence $\{\mu_n\}$ of non-zero complex numbers such that $\mu_n \longrightarrow \lambda$, thus $\|f(\lambda)\|^2 = \lim_{\mu_n \to \lambda} \langle f(\lambda), f(\mu_n)\rangle = 0$. That is, T has SVEP. $\square$

Recall that T is said to have totally finite ascent if $T-\lambda$ has finite ascent for every $\lambda \in \mathbb{C}$.

Lemma 3.6. *If* $T \in \mathfrak{C}(n)$, *then* $\ker(T-\lambda) = \ker(T-\lambda)^2$ *for each* $\lambda \in \mathbb{C}$. *In particular,* T *has totally finite ascent.*

Proof. By definition, $\ker(T^{n+1}) = \ker(T)$, so that $\ker(T^2) = \ker(T)$. Assume $0 \neq \lambda \in \sigma_p(T)$ because the case $\lambda \notin \sigma_p(T)$ is obvious. Let $0 \neq x \in \ker(T-\lambda)^2$ and $x = x_1 \oplus x_2 \in \ker(T-\lambda) \oplus \ker(T-\lambda)^{\perp}$. Then

$$0 = \ker(T-\lambda)^2 x = \begin{pmatrix} 0 & T_{12}(T_{22}-\lambda) \\ 0 & (T_{22}-\lambda)^2 \end{pmatrix} x = T_{12}(T_{22}-\lambda)x_2 \oplus (T_{22}-\lambda)^2 x_2.$$

Since $\ker(T_{22}-\lambda) = \{0\}$ by Theorem 3.2, it follows that $x_2 = 0$ and $x = x_1 \in \ker(T-\lambda)$. $\square$

Recall that $T \in \mathscr{B}(\mathscr{H})$ is said to be normaloid if $r(T) = \|T\|$, where $r(T)$ is the spectral radius of T.

Definition 3.7. An operator T is said to be $\widehat{n}$-$*$-paranormal (abbreviation $T \in \mathfrak{C}(\widehat{n})$) if

$$\left\|T^{1+i}x\right\|^{1/(1+i)} \|x\|^{i/(i+1)} \geq \|T^*x\| \text{ for all } x \in \mathscr{H} \text{ and } i \geq n.$$

Operators which are $\widehat{n}$-$*$-paranormal is a generalization of $*$-paranormal operators and a subclass of n-$*$-paranormal operators. It is easy to see that $\widehat{1}$-$*$-paranormal equals $*$-paranormality. Recall that $T \in \mathscr{B}(\mathscr{H})$ is called $\widehat{n}$-paranormal (abbreviation $T \in \mathfrak{P}(\widehat{n})$) if $\left\|T^{1+i}x\right\|^{1/(1+i)} \|x\|^{i/(i+1)} \geq \|Tx\|$ for all $x \in \mathscr{H}$ and $i \geq n$.

Theorem 3.8. *In $\mathscr{B}(\mathscr{H})$, it holds $\mathfrak{C}(\widehat{n}) \subseteq \mathfrak{P}(\widehat{n+1})$ for all $n \geq 1$.*

Proof. By the definition it holds $\|T^*x\|^{i+1} \leq \left\|T^{i+1}x\right\| \|x\|^i$. Therefore

$$\|T^*Tx\|^{i+1} \leq \left\|T^{i+2}x\right\| \|Tx\|^i$$

and

$$\|Tx\|^{2(i+1)} \leq \|T^*Tx\|^{i+1} \|x\|^{i+1} \leq \left\|T^{i+2}x\right\| \|Tx\|^i \|x\|^{i+1}.$$

Hence

$$\|Tx\|^{i+2} \leq \left\|T^{i+2}x\right\| \|x\|^{i+1}.$$

That is, $T \in \mathfrak{P}(\widehat{n+1})$. □

Lemma 3.9. *Let $T \in \mathscr{B}(\mathscr{H})$. If $T \in \mathfrak{C}(\widehat{n})$, then T is normaloid.*

Proof. Since $T \in \mathfrak{C}(\widehat{n})$, we have $T \in \mathfrak{P}(\widehat{n+1})$ by Theorem 3.8, and so the result follows by Lemma 2.6 of [31]. □

Recall that T is convexoid if $conv\sigma(T) = \overline{W(T)}$, where $W(T)$ is convex with convex hull $conv\sigma(T) \subseteq \overline{W(T)}$.

Lemma 3.10. *Let $T \in \mathscr{B}(\mathscr{H})$ be such that $T \in \mathfrak{C}(n)$. Assume that $\sigma(T) = \{\lambda\}$. Then $T = \lambda$.*

Proof. We have two cases:

Case I: $\lambda = 0$. T being normaloid, and so $T = 0$.

Case II: $\lambda \neq 0$. Here T is invertible, and since $T \in \mathfrak{C}(n)$, we see that T, T^{-1} are normaloid. On the other hand $\sigma(T^{-1}) = \{\frac{1}{\lambda}\}$, so $\|T\| \left\|T^{-1}\right\| = 1$. This implies that $\frac{1}{\lambda}T$ is unitary with its spectrum $\sigma(\frac{1}{\lambda}T) = \{1\}$. It follows that T is convexoid, so $W(T) = \{\lambda\}$. Therefore $T = \lambda$. □

Lemma 3.11. *Let T be a quasinilpotent algebraically n-$*$-paranormal operator. Then T is nilpotent.*

Proof. Suppose that $q(T)$ is n-$*$-paranormal for some non-constant polynomial q. Since $\sigma(q(T)) = q(\sigma(T))$, the operator $q(T) - q(0)$ is quasinilpotent. Since $q(T) \in \mathfrak{C}(n)$, it follows from Lemma 3.10 that $cT^m(T-\lambda_1)(T-\lambda_2)\cdots(T-\lambda_n) = q(T) - q(0) = 0$, where $(m \geq 1)$. Since $T - \lambda_j$ is invertible for every $0 \neq \lambda_j$, $1 \leq j \leq n$, we must have $T^m = 0$. □

Theorem 3.12. *Let T be algebraically n-$*$-paranormal operator. Then T is isoloid.*

Proof. Let $\lambda \in \text{iso}\sigma(T)$ and let $E := \frac{1}{2\pi i}\int_{\partial D}(\lambda - T)^{-1}\, d\lambda$ be the associated Riesz idempotent, where D is a closed disc centred at λ which contains no other points of $\sigma(T)$. We can represent T as the direct sum $T = T_1 \oplus T_2$, where $\sigma(T_1) = \{\lambda\}$ and $\sigma(T_2) = \sigma(T) \setminus \{\lambda\}$. Since T is algebraically n-$*$-paranormal, $q(T)$ is in $\mathfrak{C}(n)$ for some non-constant polynomial q. Since $\sigma(T_1) = \{\lambda\}$, we must have $\sigma(q(T_1)) = q(\sigma(T_1)) = q(\{\lambda\}) = \{q(\lambda)\}$. Since $q(T_1)$ is n-$*$-paranormal, it follows from Lemma 3.11 that $q(T_1) - q(\lambda) = 0$. Put $Q(z) := q(z) - q(\lambda)$. Then $Q(T_1) = 0$, and hence T_1 is algebraically n-$*$-paranormal operator. Since $T_1 - \lambda$ is quasinilpotent and algebraically n-$*$-paranormal operator, it follows from Lemma 3.11 that $T_1 - \lambda$ is nilpotent, and so $\lambda \in \sigma_p(T_1)$. Hence $\lambda \in \sigma_p(T)$. This shows that T is isoloid. □

An operator $T \in \mathscr{B}(\mathscr{H})$ is said to be semi-regular if $T\mathscr{H}$ is closed and $\ker(T) \subset \bigcap_{n\in\mathbb{N}} T^n\mathscr{H}$; T admits a generalized Kato decomposition, GKD for short, if there exists a pair of T-invariant closed subspaces (M, N) such that $\mathscr{H} = M \oplus N$, the restriction $T|_M$ is quasinilpotent and $T|_N$ is semi-regular. We say that T is of Kato type at a point λ if $(T - \lambda)|_M$ is nilpotent in the GKD for $T - \lambda$. Fredholm operators are Kato type.

Theorem 3.13. *Let T be an algebraically n-$*$-paranormal operator. Then T is of Kato type at each $\lambda \in iso\sigma(T)$.*

Proof. Let T be an algebraically n-$*$-paranormal operator and $\lambda \in \text{iso}\sigma(T)$. Then $\mathscr{H} = H_0(T-\lambda) \oplus K(T-\lambda)$, where $T_1 = T|_{H_0(T-\lambda)}$ satisfies $\sigma(T_1) = \{\lambda\}$ and $T|_{K(T-\lambda)}$ is semi-regular. Since T_1 is algebraically n-$*$-paranormal, then there exists a non-constant polynomial $q(\cdot)$ such that $q(T_1)$ is in $\mathfrak{C}(n)$. Then $\sigma(q(T_1)) = q(\sigma(T_1)) = q(\{\lambda\}) = \{q(\lambda)\}$. Applying Lemma 3.10, it follows that $H_0(q(T) - q(\lambda)) = \ker(q(T) - q(\lambda))$. So,

$$0 = q(T_1) - q(\lambda) = c(T_1 - \lambda)^m \prod_{j=1}^{n}(T - \lambda_j)$$

for some complex numbers $c, \lambda_1, \ldots, \lambda_n$, then for each $j = 1, \ldots, n$, $T - \lambda_j$ is invertible, which implies $T_1 - \lambda$ is nilpotent and hence $T - \lambda$ is of Kato type. □

Lemma 3.14. *Let $T \in \mathscr{B}(\mathscr{H})$. If T belongs to class $\mathfrak{C}(n)$, then T is isoloid.*

Proof. Let $\lambda \in \text{iso}\sigma(T)$. Then T and T^* have SVEP at λ. Thus $T - \lambda$ is Kato type, then $a(T - \lambda) = d(T - \lambda) = p$ for some integer $p \geq 1$ and $\mathscr{H} =$

$\ker(T-\lambda)^p \oplus (T-\lambda)^p \mathscr{H}$ ([3, Theorems 2.6 and 2.9] and [21, Proposition 4.10.6]), which implies that λ is an eigenvalue of T [19, Proposition 50.2]. $\square$

4. Variation of Weyl type theorems

We begin this section with the following definition.

Definition 4.1. Let $T \in \mathscr{B}(\mathscr{H})$. Then T is said to be possesses

(1) property (S) if $\Delta^b(T) = \sigma(T) \setminus \sigma_b(T) = E^0(T)$ [27].
(2) property (gS) if $\Delta^d(T) = \sigma(T) \setminus \sigma_D(T) = E(T)$ [27].
(3) property (m) if $\sigma(T) \setminus \sigma_{ub}(T) = E^0(T)$ [26].
(4) property (gm) if $\sigma(T) \setminus \sigma_{LD}(T) = E(T)$ [26].
(5) property (B) if $\sigma(T) \setminus \sigma_{SF_-^+}(T) = \pi^0(T)$ [25].
(6) property (gB) if $\sigma(T) \setminus \sigma_{SBF_-^+}(T) = \pi(T)$ [25].

Theorem 4.2. *Let $T \in \mathscr{B}(\mathscr{H})$. The following assertions hold*:

(i) *If T^* belongs to class $\mathfrak{C}(n)$, then a-Weyl's theorem, or equivalently Weyl's theorem, property (w), property (t), property (m), property (S), property (B), property (b), hold for T.*
(ii) *If T belongs to class $\mathfrak{C}(n)$, then a-Weyl's theorem, or equivalently Weyl's theorem, property (w), property (t), property (m), property (S), property (B), property (b), hold for T^*.*

Proof. (i) The hypothesis T^* belongs to class $\mathfrak{C}(n)$ implies that by Corollary 3.5 that T^* has SVEP and this would be implies that $\sigma(T) = \sigma_a(T)$, $\sigma_w(T) = \sigma_b(T) = \sigma_{SF_+^-}(T) = \sigma_{ub}(T)$ and since T is polaroid if and only if T^* is polaroid and this with T^* has SVEP implies T is a-polaroid and hence $E_a^0(T) = E^0(T) = \pi^0(T) = \pi_a^0(T)$. Therefore

$$E_a^0(T) = \Delta_a(T) = \Delta_+(T) = E^0(T) = \Delta(T) = \pi^0(T) = \sigma(T) \setminus \sigma_{ub}(T).$$

That is, a-Weyl's theorem, or equivalently, property (w), property (t), property (m), property (S), property (B), property (b), hold for T.

(ii) The hypothesis T belongs to class $\mathfrak{C}(n)$ implies that by Corollary 3.5 that T has SVEP and this would be implies that $\sigma(T^*) = \sigma_a(T^*)$, $\sigma_w(T^*) = \sigma_{SF_+^-}(T^*) = \sigma_b(T^*) = \sigma_{ub}(T^*)$, and since T is polaroid if and only if T^* is polaroid and this with T has SVEP implies T^* is a-polaroid and so $E_a^0(T^*) = E^0(T^*) = \pi^0(T^*) = \pi_a^0(T^*)$. Therefore

$$\pi^0(T^*) = \Delta(T^*) = \Delta_a(T^*) = E_a^0(T^*) = \Delta_+(T^*) = E^0(T) = \sigma(T^*) \setminus \sigma_{ub}(T^*).$$

That is, a-Weyl's theorem , or equivalently, property (w), property (t), property (m), property (S), property (B), property (b), hold for T^*. $\square$

For $T \in \mathscr{B}(\mathscr{H})$, let $H_{nc}(\sigma(T))$ denote the set of all analytic functions, defined on an open neighborhood of $\sigma(T)$, such that f is non constant on each of the components of its domain. Define, by the classical functional calculus, $f(T)$ for every $f \in H_{nc}(\sigma(T))$.

Theorem 4.3. *Let $f \in H_{nc}(\sigma(T))$.*

(i) *If T^* belongs to $\mathfrak{C}(n)$, then property (t), or equivalently property (w), property (m), property (S), property (B), property (b), a-Weyl's theorem hold for $f(T)$.*

(ii) *If T belongs to $\mathfrak{C}(n)$, then property (t), or equivalently property (w), property (m), property (S), property (B), property (b), a-Weyl's theorem hold for $f(T^*)$.*

Proof. (i) Since T is polaroid by Theorem 3.1. It follows from Lemma 3.11 of [2] that $f(T)$ is polaroid. By Theorem 2.40 of [1], we have $f(T^*)$ has SVEP since T^* has SVEP by Corollary 3.5. Hence from equivalence 3.5 in [28] we conclude that $f(T)$ is a-polaroid and hence it then follows by [28, Theorem 3.4(i)] that property (t) holds for $f(T)$ and this by (i) of Theorem 4.2 is equivalent to saying that property (w), property (m), property (S), property (B), property (b), a-Weyl's theorem hold for $f(T)$.

(ii) By Theorem 3.1, T is polaroid and so T^* is polaroid by the equivalence 3.1 in [28], and this implies by Lemma 3.11 of [2] that $f(T^*)$ is polaroid. By Theorem 2.40 of [1], we have $f(T)$ has SVEP, hence from equivalence 3.6 in [28] we conclude that $f(T^*)$ is a-polaroid and hence it follows by [28, Theorem 3.4(ii)] that property (t) holds for $f(T^*)$ and this by (ii) of Theorem 4.2 is equivalent to saying that property (w), property (m), property (S), property (B), property (b), a-Weyl's theorem hold for $f(T^*)$. $\square$

Theorem 4.4. *Let $T \in \mathscr{B}(\mathscr{H})$.*

(i) *If T^* belongs to class $\mathfrak{C}(n)$, then property (gt), or equivalently property (gw), property (gm), property (gS), property (gB), property (gb), generalized a-Weyl's theorem hold for T.*

(ii) *If T belongs to class $\mathfrak{C}(n)$, then property (gt), or equivalently property (gw), property (gm), property (gS), property (gB), property (gb), generalized a-Weyl's theorem hold for T^*.*

Proof. (i) Since T^* has SVEP by Corollary 3.5, we have by [1, Corollary 2.45] that $\sigma(T) = \sigma_a(T)$ and from the proof of Theorem 2.14 of [28] we then have $\sigma_{SBF_+^-}(T) = \sigma_{BW}(T) = \sigma_{LD}(T) = \sigma_D(T)$. By Theorem 3.1, T is polaroid, then by equivalence 3.5 in [28], we have T is a-polaroid and so $\pi(T) = \pi_a(T) = E(T) = E_a(T)$ (see the proof of Theorem of [28]). Therefore,

$$\pi(T) = \Delta^g(T) = E(T) = \Delta_+^g(T) = \Delta_a^g(T) = E_a(T).$$

That is, property (gt), or equivalently property (gw), property (gm), property (gS), property (gB), property (gb), generalized a-Weyl's theorem hold for T.

(ii) Since T has SVEP by Corollary 3.5, we have by [1, Corollary 2.45] that $\sigma(T^*) = \sigma(T) = \sigma_a(T^*)$ and from the proof of Theorem 2.15 of [28] we then have $\sigma_{SBF_+^-}(T^*) = \sigma_{BW}(T^*) = \sigma_{LD}(T^*) = \sigma_D(T^*)$. Since T^* is polaroid by Theorem 3.1 (T is polaroid if and only if T^* is polaroid) and T has

SVEP, it then follows by equivalence 3.1 in [28] that T^* is a-polaroid and so $\pi(T^*) = \pi_a(T^*) = E(T^*) = E_a(T^*)$. Consequently

$$\pi(T^*) = \Delta^g(T^*) = E(T^*) = \Delta_+^g(T^*) = \Delta_a^g(T^*) = E_a(T^*).$$

That is, property (gt), or equivalently property (gw), property (gm), property (gS), property (gB), property (gb), generalized a-Weyl's theorem hold for T^*. □

Theorem 4.5. *Let $f \in H_{nc}(\sigma(T))$.*

(i) *If T^* belongs to class $\mathfrak{C}(n)$, then property (gt), or equivalently property (gw), property (gm), property (gS), property (gB), property (gb), generalized a-Weyl's theorem hold for $f(T)$.*

(ii) *If T belongs to class $\mathfrak{C}(n)$, then property (gt), or equivalently property (gw), property (gm), property (gS), property (gB), property (gb), generalized a-Weyl's theorem hold for $f(T^*)$.*

Proof. (i) Since T is polaroid by Theorem 3.1. It follows from Lemma 3.11 of [2] that $f(T)$ is polaroid. By Theorem 2.40 of [1], we have $f(T^*)$ has SVEP since T^* has SVEP by Corollary 3.5. Hence from equivalence 3.5 in [28] we conclude that $f(T)$ is a-polaroid and hence it then follows by [28, Theorem 3.5(i)] that property (gt) holds for $f(T)$ and this by (i) of Theorem 4.4 is equivalent to saying that property (gw), property (gm), property (gS), property (gB), property (gb), generalized a-Weyl's theorem hold for $f(T)$.

(ii) By Theorem 3.1, T is polaroid and so T^* is polaroid by the equivalence 3.1 in [28], and this implies by Lemma 3.11 of [2] that $f(T^*)$ is polaroid. By Theorem 2.40 of [1], we have $f(T)$ has SVEP, hence from equivalence 3.6 in [28] we conclude that $f(T^*)$ is a-polaroid and hence it follows by [28, Theorem 3.5(ii)] that property (gt) holds for $f(T^*)$ and this by (ii) of Theorem 4.4 is equivalent to saying that property (gw), property (gm), property (gS), property (gB), property (gb), generalized a-Weyl's theorem hold for $f(T^*)$. □

Theorem 4.6. *If $T \in \mathfrak{C}(n)$ with $\sigma_w(T) = \{0\}$, then T is a compact normal operator.*

Proof. By Theorem 4.2, T satisfies Weyl's theorem and this implies that each element in $\sigma(T) \setminus \sigma_w(T) = \sigma(T) \setminus \{0\}$ is an eigenvalue of T with finite multiplicity, and is isolated in $\sigma(T)$. Hence $\sigma(T) \setminus \{0\}$ is a finite set or a countable set with 0 as its only accumulation point. Put $\sigma(T) \setminus \{0\} = \{\lambda_n\}$, where $\lambda_n \neq \lambda_m$ whenever $n \neq m$ and $\{|\lambda_n|\}$ is a non-increasing sequence. Since T is normaloid, we have $|\lambda_1| = \|T\|$. By Theorem 9 of [13], we have $(T - \lambda_1)x = 0$ implies $(T - \lambda_1)^*x = 0$. Hence $\ker(T - \lambda_1)$ is a reducing subspace of T. Let E_1 be the orthogonal projection onto $\ker(T - \lambda_1)$. Then $T = \lambda_1 \oplus T_1$ on $\mathscr{H} = E_1\mathscr{H} \oplus (1 - E_1)\mathscr{H}$. Since $T_1 \in \mathfrak{C}(n)$ by Theorem 3.2 and $\sigma_p(T) = \sigma_p(T_1) \cup \{\lambda_1\}$, we have $\lambda_2 \in \sigma_p(T_1)$. By the same argument as above, $\ker(T - \lambda_2) = \ker(T_1 - \lambda_2)$ is a finite dimensional reducing subspace of T which is included in $(1 - E_1)\mathscr{H}$. Put E_2 be the orthogonal projection onto $\ker(T - \lambda_2)$.

Then $T = \lambda_1 E_1 \oplus \lambda_2 E_2 \oplus T_2$ on $\mathscr{H} = E_1\mathscr{H} \oplus E_2\mathscr{H} \oplus (1 - E_1 - E_2)\mathscr{H}$. By repeating above argument, each $\ker(T - \lambda_n)$ is a reducing subspace of T and $\|T - \bigoplus_{k=1}^{n} \lambda_k E_k\| = \|T_n\| = |\lambda_{n+1}| \longrightarrow 0$ as $n \to \infty$. Here E_k is the orthogonal projection onto $\ker(T - \lambda_k)$ and $T = (\bigoplus_{k=1}^{n} \lambda_k E_k) \oplus T_n$ on $\mathscr{H} = \bigoplus_{k=1}^{n} E_k\mathscr{H} \oplus (1 - \sum_{k=1}^{n} E_k)\mathscr{H}$. Hence $T = \bigoplus_{k=1}^{\infty} \lambda_k E_k$ is compact and normal because each E_k is a finite rank orthogonal projection which satisfies $E_k E_t = 0$ whenever $k \neq t$ by Corollary 3.4 and $\lambda_n \longrightarrow 0$ as $n \to \infty$. $\square$

5. An asymmetric Putnam-Fuglede theorem

The classical Puntam-Fuglede theorem asserts that if $T \in \mathscr{B}(\mathscr{H})$ and $S \in \mathscr{B}(\mathscr{H})$ are normal operators and $TX = XS$ for some $X \in \mathscr{B}(\mathscr{H})$, then $T^*X = XS^*$. Let us overwrite the Puntam-Fuglede theorem in an asymmetric form: if $T \in \mathscr{B}(\mathscr{H})$ and $S \in \mathscr{B}(\mathscr{H})$ are normal operators and $TX = XS^*$ for some $X \in \mathscr{B}(\mathscr{H})$, then $T^*X = XS$. In this section, we mainly extend the asymmetric Putnam-Fuglede theorem to the class of n-$*$-paranormal operators.

Theorem 5.1 (**Berberian's Extension**). *Let $\mathscr{H}$ be a complex Hilbert space. Then there exist a Hilbert space $\mathscr{H}^\circ \supset \mathscr{H}$ and $\phi : \mathscr{B}(\mathscr{H}) \longmapsto \mathscr{B}(\mathscr{H})$ $(T \longmapsto T^\circ)$ satisfying*: *ϕ is an $*$-isometric isomorphism preserving the order such that*

(i) *$\phi(T^*) = \phi(T)^*$, $\phi(I) = \phi(I)^\circ$, $\phi(\alpha T + \beta S) = \alpha\phi(T) + \beta\phi(S)$, $\phi(TS) = \phi(T)\phi(S)$, $\|\phi(T)\| = \|T\|$ for all $T, S \in \mathscr{B}(\mathscr{H})$ and $\alpha, \beta \in \mathbb{C}$.*

(ii) *If $T \leq S$, then $\phi(T) \leq \phi(S)$ for all $T, S \in \mathscr{B}(\mathscr{H})$.*

(iii) *$\sigma(T) = \sigma(T^\circ)$ and $\sigma_a(T) = \sigma_a(T^\circ) = \sigma_p(T^\circ)$.*

Lemma 5.2. *Let $T \in \mathscr{B}(\mathscr{H})$ such that $T \in \mathfrak{C}(n)$ and $\mathscr{M} \subset \mathscr{H}$ an invariant subspace of T such that $T|_{\mathscr{M}}$ is normal. Then $\mathscr{M}$ is reducing for T.*

Proof. Let us consider the matrix decomposition

$$T = \begin{pmatrix} N & A \\ 0 & * \end{pmatrix},$$

where $N = T|_{\mathscr{M}}$ is a normal operator. If $T \in \mathfrak{C}(n)$, then

$$\|A^*x\|^2 + \|N^*x\|^2 = \|T^*x\|^2 \leq \|T^n x\|^{\frac{2}{n}} \|x\|^{\frac{2(n-1)}{n}} = \|N^n x\|^{\frac{2}{n}} \|x\|^{\frac{2(n-1)}{n}} \tag{5.1}$$

for all $x \in \mathscr{M}$.

Let us take the Berberian's extension of the operator T. Then the extension T° has the following matrix decomposition

$$T = \begin{pmatrix} N^\circ & A^\circ \\ 0 & * \end{pmatrix},$$

where N° and A° are the Berberian's extension of the operators N and A.

Let $y = [x_n]$ denote the equivalence class of the sequence $\{x_n\}_n \subset \mathscr{M}$. By the inequality (5.1) and Hölder inequality we get

$$\|(A^*)^\circ y\|^2 + \|(N^*)^\circ y\|^2 = \|(T^*)^\circ y\|^2 = \phi(\|T^* x_n\|^2) \leq \phi(\|T^n x\|^{\frac{2}{n}} \|x\|^{\frac{2(n-1)}{n}})$$

$$\leq (\phi(\|T^n x\|))^{\frac{2}{n}} (\phi(\|x\|))^{\frac{2(n-1)}{n}} = \|(T^\circ)^n y\|^{\frac{2}{n}} \|y\|^{\frac{2(n-1)}{n}}$$
$$= \|(N^\circ)^n y\|^{\frac{2}{n}} \|y\|^{\frac{2(n-1)}{n}} .$$

By the [9, Theorem 1] we know that the spectrum of normal operator N° is equal to its point spectrum.

If y is an eigenvector of N°, with an eigenvalue λ, then we have

$$\begin{aligned}\|(A^*)^\circ y\|^2 + |\lambda|^2 \|y\|^2 &= \|(A^*)^\circ y\|^2 + \|(N^*)^\circ y\|^2 \\ &\leq \|(N^\circ)^n y\|^{\frac{2}{n}} \|y\|^{\frac{2(n-1)}{n}} = |\lambda|^2 \|y\|^2 .\end{aligned}$$

Thus $(A^\circ)^* y = 0$. But eigenvectors of N° spanned the subspace $\mathscr{M}^\circ$. Hence $A^\circ = 0$ and consequently $A = 0$. □

Proposition 5.3. *Let $T \in \mathscr{B}(\mathscr{H})$. If $T \in \mathfrak{C}(n)$, then the residual spectrum of T^* is empty. In particular, we have $\sigma_a(T^*) = \sigma(T^*)$.*

In order to prove Proposition 5.3, we need the following two lemmas from [13].

Lemma 5.4. *Let $T \in \mathscr{B}(\mathscr{H})$. If T belongs to $\mathfrak{C}(n)$ and $\mathscr{M}$ is an invariant subspace for T, then $T|_{\mathscr{M}}$ belongs to $\mathfrak{C}(n)$.*

Lemma 5.5. *For $T \in \mathscr{B}(\mathscr{H})$, let T belong to $\mathfrak{C}(n)$ and λ be an eigenvalue of T. If $(T - \lambda)x = 0$, then $(T - \lambda)^* x = 0$.*

Proof of Proposition 5.3. It follows from Lemma 5.4 and Lemma 5.5 that each $T \in \mathfrak{C}(n)$ is a direct sum of diagonal operator and n-$*$-paranormal without point spectrum. Let us assume that T has no eigenvalues. Then since $\ker(T - \lambda) = \{0\}$, we have $\overline{(T - \lambda)^* \mathscr{H}} = \mathscr{H}$ for all $\lambda \in \mathbb{C}$. Hence $\sigma_r(T^*) = \emptyset$. Moreover,

$$\sigma(T^*) = \sigma_p(T^*) \cup \sigma_c(T^*) \subset \sigma_a(T^*),$$

where $\sigma_c(T)$ is the continuous spectrum of T. This completes the proof. □

Theorem 5.6. *Let $T, S \in \mathscr{B}(\mathscr{H})$. If $T, S \in \mathfrak{C}(n)$ are such that $TX = XS^*$ for some $X \in \mathscr{B}(\mathscr{H})$, then $T^* X = XS$.*

Proof. Let $X = U|X|$ be a polar decomposition of X, with $U : \overline{|X|\mathscr{H}} \longrightarrow \overline{X\mathscr{H}}$ unitary operator. Then the operator equation $TX = XS^*$ is equivalent to $\widetilde{T}|X| = |X|S^*$, where $\widetilde{T} := U^{-1}TU \oplus 0_{\ker(|X|)}$. The operator $\widetilde{T}$ belongs to class $\mathfrak{C}(n)$. Indeed we have

$$\begin{aligned}\left\|(\widetilde{T})^*(x + y)\right\|^n &= \left\|(U^{-1}T^*U \oplus 0)(x + y)\right\|^n = \|T^*Ux\|^n \\ &\leq \|T^n Ux\| \|Ux\|^{n-1} = \left\|(U^{-1}T^nU \oplus 0)(x + y)\right\| \|x\|^{n-1} \\ &\leq \left\|(\widetilde{T})^n(x + y)\right\| \|x + y\|^{n-1}\end{aligned}$$

for each $x \in \overline{|X|\mathscr{H}}$ and $y \in \ker(|X|)$. Thus it is enough to show that for two n-$*$-paranormal operators T, S and positive operator X such that $TX = XS^*$, the equality $T^*X = XS$ holds true.

Let us fix n-$*$-paranormal operators T, S and positive operator X such that $TX = XS^*$. Hence the subspace $\overline{X\mathscr{H}}$ is invariant for T. Since the subspace $\ker(X)$ is invariant for S^*, then $\overline{X\mathscr{H}}$ is also invariant for S. As a consequence we have the following matrices representations with respect to the decomposition $\mathscr{H} = \overline{X\mathscr{H}} \oplus \ker(X)$.

$$X = \begin{pmatrix} A & 0 \\ 0 & 0 \end{pmatrix}, T = \begin{pmatrix} T_1 & T_2 \\ 0 & T_3 \end{pmatrix} \text{ and } S = \begin{pmatrix} S_1 & S_2 \\ 0 & S_3 \end{pmatrix}.$$

It follows from Lemma 5.4 that T_1, S_1 are belong to $\mathfrak{C}(n)$. The equation $TX = XS^*$ implies that $T_1A = AS_1^*$. By Lemma 5.3 we have $\sigma(S_1^*) = \sigma_a(S_1^*)$.

The Berberian's extensions $T_1^\circ, S_1^\circ, A^\circ$ of the operators T_1, S_1, A satisfy the equation

$$T_1^\circ A^\circ = A^\circ (S_1^*)^\circ \tag{5.2}$$

and $\sigma(S_1^*)^\circ = \sigma(S_1^*) = \sigma_a(S_1^*) = \sigma_p((S_1^*)^\circ)$. The equation (5.2) is equivalent to

$$(\lambda - T_1^\circ)A^\circ = A^\circ(\lambda - (S_1^*)^\circ)$$

for $\lambda \in \mathbb{C}$. Thus if $\lambda \in \sigma_r(T_1^\circ)$, then $\lambda \in \sigma_r((S_1^*)^\circ)$, where $\sigma_r(T)$ is the residual spectrum of T. But by Lemma 5.3, we get $\sigma((S_1^*)^\circ) = \emptyset$. As a consequence $\sigma_r(T_1^\circ) = \emptyset$. So $\sigma(T_1^\circ) = \sigma_a(T_1^\circ) = \sigma_p(T_1^\circ)$. Moreover, the operator T_1° belongs to class $\mathfrak{C}(n)$ (see the proof of Lemma 5.2). Thus by Lemma 5.5 the operator T_1° is diagonal, so it is normal. Normality of T_1° shows that T_1 is normal. Hence by Lemma 5.5 we get $T_2 = 0$.

The equation (5.2) is equivalent to $A^\circ(T_1^*)^\circ = S_1^\circ A^\circ$. Thus we can repeat the above argument and show that the operator S_1 is normal and $S_2 = 0$.

Finally, to show that $T^*X = XS$ it is enough to show that $T_1^*A = AS_1$, but it is consequence of the classical Putnam-Fuglede theorem. □

Definition 5.7. We say that the operator $T \in \mathscr{B}(\mathscr{H})$ satisfies the Putnam-Fuglede theorem if and only if for all operators $X, N \in \mathscr{B}(\mathscr{H})$ such that N is normal and $TX = XN$, it holds that $T^*X = XN^*$.

Proposition 5.8 ([8])**.** *The operator $T \in \mathscr{B}(\mathscr{H})$ satisfies the Putnam-Fuglede theorem if and only if each invariant subspace $\mathscr{M} \subset \mathscr{H}$ of T such that $T|_{\mathscr{M}}$ is normal, is reducing for T.*

Theorem 5.9. *Let $T \in \mathscr{B}(\mathscr{H})$. If T belongs to $\mathfrak{C}(n)$ and N normal and $TX = XN$, then $T^*X = XN^*$.*

Proof. The proof follows immediately from Lemma 5.2 and Proposition 5.8. □

6. Finite operators and orthogonality

Let $T, S \in \mathscr{B}(\mathscr{H})$ we define the generalized derivation $\delta_{T,S} : \mathscr{B}(\mathscr{H}) \longmapsto \mathscr{B}(\mathscr{H})$ by

$$\delta_{T,S}(X) = TX - XS,$$

we note $\delta_{T,T} = \delta_T$. Let Y be a complex Banach space. We say that $b \in Y$ is orthogonal to $a \in Y$ if for all complex λ there holds

$$\|a + \lambda b\| \geq \|a\| . \tag{6.1}$$

This definition has a natural geometric interpretation. Namely, $b \perp a$ if and only if the complex line $\{a + \lambda b \mid \lambda \in \mathbb{C}\}$ is disjoint with the open ball $\mathbf{B}(0, \|a\|)$, i.e., if and only if this complex line is a tangent one. Note that if b is orthogonal to a, then a need not be orthogonal to b. If Y is a Hilbert space, then from (6.1) follows $\langle a, b\rangle = 0$, i.e., orthogonality in the usual sense.

Definition 6.1. We say that the operator $T \in \mathscr{B}(\mathscr{H})$ is finite if for all $X \in \mathscr{B}(\mathscr{H})$, we have $\|I - (TX - XT)\| \geq 1$.

J. H. Anderson and C. Foias [6] have shown that if T, S are normal operators, then

$$\|K - (TX - XS)\| \geq \|K\| \tag{6.2}$$

for all $X \in \mathscr{B}(\mathscr{H})$ and for all $K \in \delta_{T,S}$. Hence the range of $\delta_{T,S}$ is orthogonal to the null space of $\delta_{T,S}$. In particular the inequality $\|K - (TX - XT)\| \geq \|K\|$ means that the range of δ_T is orthogonal to $\ker(\delta_T)$ in the sense of Birkhoff. It is easy to see that if the range of δ_T is orthogonal to δ_T, then T is finite. Indeed, we have $K = I \in \ker(\delta_T)$. In this paper we prove that n-$*$-paranormal operator is finite. An extension of inequality (6.2) is also given.

An operator $T \in \mathscr{B}(\mathscr{H})$ is said to be spectraloid if $w(T) = r(T)$, where $w(T)$ is the numerical radius of T. Hence the following inclusions hold:

hyponormal $\subset$ p-hyponormal $\subset$ $*$-paranormal $\subset$ normaloid $\subset$ spectraloid.

Let $T \in \mathscr{B}(\mathscr{H})$, the approximate reduced spectrum of T, $\sigma_{ar}(T)$ is defined as

$$\sigma_{ar}(T) := \{\lambda \in \mathbb{C} : \text{there exists a normed sequence } \{x_n\} \subset \mathscr{H} \text{ satisfying}$$
$$(T - \lambda)x_n \longrightarrow 0 \text{ and } (T - \lambda)^* x_n \longrightarrow 0\} .$$

In [30], J. P. Williams proved that the class of finite operator, $\mathbb{F}$, contains every normal, hyponormal operators. We will show that operators in $\mathfrak{C}(n)$ are finite. The following lemmas was proved in [22].

Lemma 6.2. *Let $T \in \mathscr{B}(\mathscr{H})$.*

(i) *If $\sigma_{ar}(T) \neq \emptyset$, then T is finite.*
(ii) $\partial W(T) \cap \sigma(T) \subset \sigma_{ar}(T)$.

Lemma 6.3. *Let $T \in \mathscr{B}(\mathscr{H})$. If T is spectraloid, then T is finite.*

Proof. Suppose that T is spectraloid. Then we have $w(T) = r(T)$. Then there exists $\lambda \in \sigma(T) \subset \overline{W(T)}$ such that $|\lambda| = w(T)$. Thus $\lambda \in \partial W(T)$. This implies that $\partial W(T) \cap \sigma(T) \neq \emptyset$. Hence $\sigma_{ar}(T) \neq \emptyset$ by (ii) of Lemma 6.2 and so T is finite by (i) of Lemma 6.2. □

In the following theorem we will show that if $T \in \mathfrak{C}(n)$, then T is finite.

Theorem 6.4. *Let $T \in \mathscr{B}(\mathscr{H})$. If $T \in \mathfrak{C}(n)$, then $T \in \mathbb{F}$.*

Proof. Let $T \in \mathfrak{C}(n)$. Then T is normaloid and hence spectraloid. But a spectraloid operator T is finite by Lemma 6.3. Consequently, T is finite. $\square$

Lemma 6.5. *Let $T, N \in \mathscr{B}(\mathscr{H})$ such that $T \in \mathfrak{C}(n)$ and N is normal. If $TN = NT$, then for every $\lambda \in \sigma_p(T)$, we have*

$$|\lambda| \leq \|N - (TX - XT)\|$$

for all $X \in \mathscr{B}(\mathscr{H})$.

Proof. Let $\lambda \in \sigma_p(T)$ and M_λ the eigenspace associate to λ. Since $TN = NT$, we have $T^*N = NT^*$ by the Fuglede-Putnam's Theorem 5.9. Hence M_λ reduces both T and N. According to the decomposition of $\mathscr{H} = M_\lambda \oplus M_\lambda^\perp$, we can write T, N and X as follows:

$$N = \begin{pmatrix} \lambda & 0 \\ 0 & N_2 \end{pmatrix}, T = \begin{pmatrix} T_1 & 0 \\ 0 & T_2 \end{pmatrix} \text{ and } X = \begin{pmatrix} X_1 & X_2 \\ X_3 & X_4 \end{pmatrix}.$$

Since the restriction of n-$*$-paranormal operator to an invariant subspace is n-$*$-paranormal, we have

$$\begin{aligned} \|N - (TX - XT)\| &= \left\| \begin{pmatrix} \lambda - (T_1X_1 - X_1T_1) & * \\ * & * \end{pmatrix} \right\| \\ &\geq \|\lambda - (T_1X_1 - X_1T_1)\| \\ &\geq |\lambda| \left(1 - T_1 \left(\frac{X_1}{\lambda} \right) - \left(\frac{X_1}{\lambda} \right) T_1 \right) \\ &\geq |\lambda|. \end{aligned}$$

$\square$

Theorem 6.6. *If T belongs to class $\mathfrak{C}(n)$, then for every normal operator N such that $TN = NT$, we have*

$$\|N - (TX - XT)\| \geq \|N\| \text{ for all } X \in \mathscr{B}(\mathscr{H}).$$

Proof. Since N is normal, we have $\sigma(N) = \sigma_a(N)$. Let $\lambda \in \sigma(T)$, then it follows by Berberian's extension that N° is normal, $T^\circ \in \mathfrak{C}(n)$, $T^\circ N^\circ = N^\circ T^\circ$ and $\lambda \in \sigma_p(T^\circ)$. By applying Lemma 6.5, we get

$$|\lambda| \leq \|N^\circ - (T^\circ X^\circ - X^\circ T^\circ)\| = \|N - (TX - XT)\|$$

for all $X \in \mathscr{B}(\mathscr{H})$.

$$\sup_{\lambda \in \sigma(T^\circ)} |\lambda| = \|T^\circ\| = \|T\| = r(T) \leq \|N - (TX - XT)\|$$

for all $X \in \mathscr{B}(\mathscr{H})$. $\square$

Theorem 6.7. *Let $\mathcal{A}$ be a C^*-algebra and let $a \in \mathcal{A}$ belongs to $\mathfrak{C}(n)$. Then a is finite.*

Proof. It is known from [18, Page 91] that there exist a $*$-isometric homomorphism ϕ and a Hilbert space $\mathscr{H}$ ($\phi : \mathcal{A} \longmapsto \mathscr{B}(\mathscr{H})$). Then $\phi(a)$ belongs to class $\mathfrak{C}(n)$. Since ϕ is isometric it results from Theorem 6.4 that a is finite. $\square$

Corollary 6.8. *Let $T \in \mathscr{B}(\mathscr{H})$ belongs to class $\mathfrak{C}(n)$. Then $A = T + K$ is finite, where K is a compact operator.*

Proof. Since the Calkin algebra $\mathscr{B}(\mathscr{H})/K\mathscr{H}$ is a C^*-algebra, $[T]$ belongs to $\mathfrak{C}(n)$. Hence it follows from Theorem 6.7 that $[T] = T + K$ is finite and we have

$$\|I - (AX - XA)\| \geq \|[I] - ([T][X] - [X][T])\| \geq \|[I]\| = 1. \qquad \square$$

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