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A SHARP CARATHÉODORY'S INEQUALITY ON THE BOUNDARY

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Abstract. In this paper, a generalized boundary version of Carathéodory's inequality for holomorphic function satisfying $f(z) = f(0) + a_p z^p + \cdots$, and $\Re f(z) \leq A$ for $|z| < 1$ is investigated. Also, we obtain sharp lower bounds on the angular derivative $f'(c)$ at the point c with $\Re f(c) = A$. The sharpness of these estimates is also proved.

1. Introduction

In recent times, a boundary version of Schwarz lemma was investigated in D. Burns and S. G. Krantz [4] and R. Osserman [17] and V. N. Dubinin [6], M. Mateljević ([11], [12], [13] and [14]), M. Jeong ([8], [9]), D. Chelst [5] and other's studies. On the other hand, in [10], Sharp Real-Parts Theorems (in particular Carathéodory's inequalities), which are frequently used in the theory of the entire functions and in the analytic function theory were studied.

The Carathéodory's inequality states that, if the function $f(z) = f(0) + a_p z^p + a_{p+1} z^{p+1} + \cdots$, $p \in \mathbb{N}$ is holomorphic on the unit disc $D = \{z : |z| < 1\}$ and $\Re f \leq A$ in D, then the inequality

$$|f(z) - f(0)| \leq \frac{2\left(A - \Re f(0)\right)|z|^p}{1 - |z|^p} \tag{1.1}$$

holds for all $z \in D$, and moreover

$$|a_p| \leq 2\left(A - \Re f(0)\right). \tag{1.2}$$

Equality is achieved in (1.1) (for some nonzero $z \in D$) or in (1.2) if and only if f is the function of the form

$$f(z) = f(0) + \frac{2\left(A - \Re f(0)\right) z^p e^{i\theta}}{1 + z^p e^{i\theta}},$$

where θ is a real number ([10], pp. 3–4).

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Let f be a holomorphic function in the unit disc $D = \{z : |z| < 1\}$, $f(0) = 0$ and $|f(z)| < 1$ for $|z| < 1$. In accordance with the classical Schwarz lemma, for any point z in the disc D, we have $|f(z)| \leq |z|$ and $|f'(0)| \leq 1$. Equality in these inequalities (in the first one, for $z \neq 0$) occurs only if $f(z) = \lambda z$, $|\lambda| = 1$ [7].

R. Osserman [17] has given the inequalities which are called the boundary Schwarz lemma. He has first showed that

$$|f'(c)| \geq \frac{2}{1 + |f'(0)|} \geq 1 \tag{1.3}$$

under the assumption $f(0) = 0$ where f is a holomorphic function mapping the unit disc into itself and c is a boundary point to which f extends continuously and $|f(c)| = 1$. Subsequently, using the Möbius transformation, he has generalized the inequality on the case of $f(0) \neq 0$.

Dubinin has continued this line and has made a refinement on the boundary Schwarz lemma under the assumption that $f(z) = a_p z^p + a_{p+1} z^{p+1} + \cdots$, with a zero set $\{a_k\}$ (see [6]).

The following lemma, known as the Julia-Wolff lemma, is needed in the sequel (see [18]).

Lemma 1.1 (Julia-Wolff lemma)**.** *Let f be a holomorphic function in D, $f(0) = 0$ and $f(D) \subset D$. If, in addition, the function f has an angular limit $f(c)$ at $c \in \partial D$, $|f(c)| = 1$, then the angular derivative $f'(c)$ exists and $1 \leq |f'(c)| \leq \infty$.*

X. Tang, T. Liu and J. Lu [19] established a new type of the classical boundary Schwarz lemma for holomorphic self-mappings of the unit polydisk D^n in $\mathbb{C}^n$. They extended the classical Schwarz lemma at the boundary to high dimensions.

D. M. Burns and S. G. Krantz [4] and D. Chelst [5] studied the uniqueness part of the Schwarz lemma. In M. Mateljević's papers, for more general results and related estimates, see also ([11], [12], [13] and [14]).

According to M. Mateljević's studies, some other types of results which are related to the subject can be found in (see, e.g., [12], [13]). In addition, (see [14]) was posed on ResearchGate where is discussed concerning results in more general aspects.

Also, M. Jeong [8] showed some inequalities at a boundary point for different form of holomorphic functions and found the condition for equality and in [9] a holomorphic self map defined on the closed unit disc with fixed points only on the boundary of the unit disc.

In [1], in the different class of holomorphic function on the unit disc, assuming the existence of angular limit on the boundary point the estimations below of the modul of angular derivative have been obtained. This different class is as follows:

Let $f(z) = a + a_p z^p + a_{+1} z^{p+1} + \cdots$, $a_p \neq 0$, $p \geq 2$ be a holomorphic function in the unit disc D, and $|f(z) - 1| < 1$ for $|z| < 1$, where $0 < a < 2$. Assume that, for some $c \in \partial D$, f has an angular limit $f(c)$ at c, $f(c) = 2$. Then

$$|f'(c)| \geqslant \frac{2-a}{a}\left(p + \frac{2\left(a(2-a) - |a_p|\right)^2}{a^2(2-a)^2 - |a_p|^2 + a(2-a)\,|a_{p+1}|}\right). \tag{1.4}$$

The equality in (1.4) occurs for the function

$$f(z) = \frac{a\left(1+z^p\right)}{1+(a-1)\,z^p}.$$

In this paper, a boundary version of the known Carathéodory's inequality is examined.

In [16], a weak version of known Carathéodory's inequality was investigated at the boundary of the unit disc. This estimation is as follows:

Let f be a holomorphic function in the unit disc D, $f(0) = 0$ and $\Re f \leq A$ for $|z| < 1$. Further assume that, for some $c \in \partial D$, f has an angular limit $f(c)$ at c, $\Re f(c) = A$. Then

$$|f'(c)| \geqslant \frac{A}{2}. \tag{1.5}$$

The equality in (1.5) holds if and only if

$$f(z) = 2A\frac{ze^{i\theta}}{1+ze^{i\theta}},$$

where θ is a real number.

We have further strengthened the study in [16] by adding the coefficients a_p and a_{p+1} of the function $f(z) = f(0) + a_p z^p + a_{p+1} z^{p+1} + \cdots$. Also, the condition $f(0) = 0$ is removed.

Some other types of strengthening inequalities are obtained in (see [2], [15]).

We studied "Generalized of boundary Carathéodory's inequalities" as analog to the boundary Schwarz lemma. We estimate a module of angular derivative of the functions, that satisfied Carathéodory's inequality, by taking into account their first nonzero two Maclaurin coefficients.

2. Main results

In this section we give the main results of this paper. In the following theorems, new inequalities of Carathéodory's inequality at the boundary are obtained and the sharpness of these inequalities is proved.

Theorem 2.1. *Let $f(z) = f(0) + a_p z^p + a_{p+1} z^{p+1} + \cdots$, $a_p \neq 0$, $p \geq 2$, $p \in \mathbb{N}$ be a holomorphic function in the unit disc D and let $\Re f(z) \leqslant A$ for $|z| < 1$. Further assume that, for some $c \in \partial D$, f has an angular limit $f(c)$ at c, $\Re f(c) = A$. Then*

$$|f'(c)| \geqslant \frac{A-\Re f(0)}{2}\left(p + \frac{2\left(2\left(A-\Re f(0)\right)-|a_p|\right)^2}{4\left(A-\Re f(0)\right)^2-|a_p|^2+2\left(A-\Re f(0)\right)|a_{p+1}|}\right). \tag{1.6}$$

Moreover, the equality in (1.6) *occurs for the function*

$$f(z) = f(0) + 2\left(A - \Re f(0)\right)\frac{z^p}{1+z^p}.$$

Proof. Introducing the notation

$$\alpha = A - \Re f(z),\ \beta = A - \Re f(0).$$

If f is not identically constant, then $\alpha > 0,\ \beta > 0$, $\Re\left(f(z) - f(0)\right) = \beta - \alpha < \beta$ and $4\beta\Re\left(f(z) - f(0)\right) < 4\beta^2$. Therefore, we take

$$\begin{aligned}\left|2\beta - (f(z) - f(0))\right|^2 &= \left|f(z) - f(0) - 2\beta\right|^2 \\ &= \left|f(z) - f(0)\right|^2 - 4\beta\Re\left(f(z) - f(0)\right) + 4\beta^2 \\ &> \left|f(z) - f(0)\right|^2.\end{aligned}$$

Thus, the function

$$\varphi(z) = \frac{f(z) - f(0)}{2\beta - (f(z) - f(0))}$$

is holomorphic in the unit disc D, $|\varphi(z)| < 1$ for $|z| < 1$.

$$B(z) = z^p$$

is a holomorphic functions in D, and $|B(z)| < 1$ for $|z| < 1$. By the maximum principle for each $z \in D$, we have

$$|\varphi(z)| \le |B(z)|.$$

Therefore,

$$h(z) = \frac{\varphi(z)}{B(z)}$$

is a holomorphic function in D and $|h(z)| \le 1$ for $|z| < 1$. If $|h(z)| = 1$, then by the maximum principle we have $\frac{\varphi(z)}{B(z)} = e^{i\theta}$ and $f(z) = f(0) + 2\beta\frac{z^p e^{i\theta}}{1+z^p e^{i\theta}}$, where θ is a real number. In this situation, (1.6) is satisfied with the equality. Thus, from now on we may assume

$$f(z) \not\equiv f(0) + 2\beta\frac{z^p e^{i\theta}}{1+z^p e^{i\theta}},$$

and therefore we obtain $|h(z)| < 1$.

In particular, we have

$$|h(0)| = \frac{|a_p|}{2\beta} \le 1 \tag{1.7}$$

and

$$|h'(0)| = \frac{|a_{p+1}|}{2\beta}.$$

Moreover, since the expression $\frac{c\varphi'(c)}{\varphi(c)}$ is a real number greater than or equal to 1 (see, [2]) and $\Re f(c) = A$ yields $|\varphi(c)| = 1$, we get

$$\frac{c\varphi'(c)}{\varphi(c)} = \left|\frac{c\varphi'(c)}{\varphi(c)}\right| = |\varphi'(c)|.$$

Also, since $|\varphi(z)| \leq |B(z)|$, we take

$$\frac{1-|\varphi(z)|}{1-|z|} \geq \frac{1-|B(z)|}{1-|z|}.$$

Passing to the angular limit in the last inequality yields

$$|\varphi'(c)| \geq |B'(c)|.$$

Therefore, we obtain

$$\frac{c\varphi'(c)}{\varphi(c)} = |\varphi'(c)| \geq |B'(c)| = \frac{cB'(c)}{B(c)}.$$

The function

$$\Theta(z) = \frac{h(z)-h(0)}{1-\overline{h(0)}h(z)}$$

is holomorphic in D, $|\Theta(z)| < 1$, $\Theta(0) = 0$ and $|\Theta(c)| = 1$ for $c \in \partial D$.

From (1.3), we obtain

$$\begin{aligned}
\frac{2}{1+|\Theta'(0)|} \leq |\Theta'(c)| &= \frac{1-|h(0)|^2}{\left|1-\overline{h(0)}h(c)\right|^2} |h'(c)| \\
&= \frac{1-|h(0)|^2}{\left|1-\overline{h(0)}h(c)\right|^2} \left|\frac{\varphi'(c)}{B(c)} - \frac{\varphi(c)B'(c)}{B^2(c)}\right| \\
&= \frac{1-|h(0)|^2}{\left|1-\overline{h(0)}h(c)\right|^2} \left|\frac{\varphi(c)}{cB(c)}\right| \left|\frac{c\varphi'(c)}{\varphi(c)} - \frac{cB'(c)}{B(c)}\right| \\
&\leq \frac{1+|h(0)|}{1-|h(0)|} \left\{|\varphi'(c)| - |B'(c)|\right\},
\end{aligned}$$

and

$$\frac{2}{1+|\Theta'(0)|} \leq \frac{1+|h(0)|}{1-|h(0)|} \left\{|\varphi'(c)| - |B'(c)|\right\}. \tag{1.8}$$

With the simple calculations, we take

$$\Theta'(z) = \frac{1-|h(0)|^2}{\left(1-\overline{h(0)}h(z)\right)^2} h'(z),$$

$$\Theta'(0) = \frac{1-|h(0)|^2}{\left(1-|h(0)|^2\right)^2} h'(0)$$

$$= \frac{h'(0)}{1-|h(0)|^2}$$

and

$$|\Theta'(0)| = \frac{|h'(0)|}{1-|h(0)|^2} = \frac{\frac{|a_{p+1}|}{2\beta}}{1-\left(\frac{|a_p|}{2\beta}\right)^2} = \frac{2\beta\,|a_{p+1}|}{4\beta^2-|a_p|^2}.$$

In addition, we have

$$|\varphi'(c)| = \frac{2\beta\,|f'(c)|}{|2\beta-(f(c)-f(0))|^2}$$

and for $c \in \partial D$

$$|B'(c)| = p.$$

Let's substitute the values of $|\Theta'(0)|$, $|\varphi'(c)|$, $|B'(c)|$ and $|h(0)|$ into (1.8). Therefore, we obtain

$$\frac{2}{1+\frac{2\beta|a_{p+1}|}{4\beta^2-|a_p|^2}} \le \frac{1+\frac{|a_p|}{2\beta}}{1-\frac{|a_p|}{2\beta}}\left\{\frac{2\beta\,|f'(c)|}{|2\beta-(f(c)-f(0))|^2}-p\right\}$$

$$= \frac{2\beta+|a_p|}{2\beta-|a_p|}\left\{\frac{2\beta\,|f'(c)|}{|2\beta-(f(c)-f(0))|^2}-p\right\},$$

$$\frac{2\left(4\beta^2-|a_p|^2\right)}{4\beta^2-|a_p|^2+2\beta\,|a_{p+1}|}\frac{2\beta-|a_p|}{2\beta+|a_p|} \le \frac{2\beta\,|f'(c)|}{|2\beta-(f(c)-f(0))|^2}-p,$$

and

$$p+\frac{2\,(2\beta-|a_p|)^2}{4\beta^2-|a_p|^2+2\beta\,|a_{p+1}|} \le \frac{2\beta\,|f'(c)|}{|2\beta-(f(c)-f(0))|^2}.$$

Since $|2\beta-(f(c)-f(0))|^2 \ge (\Re\,[2\beta-(f(c)-f(0))])^2 = \beta^2$, we get

$$p+\frac{2\,(2\beta-|a_p|)^2}{4\beta^2-|a_p|^2+2\beta\,|a_{p+1}|} \le \frac{2\beta\,|f'(c)|}{|2\beta-(f(c)-f(0))|^2} \le \frac{2\,|f'(c)|}{\beta}.$$

So, we take the inequality (1.6).

Now, we shall show that the inequality (1.6) is sharp. Let

$$f(z) = f(0)+2\beta\frac{z^p}{1+z^p}.$$

Then

$$f'(z) = 2\beta\frac{pz^{p-1}}{(1+z^p)^2}$$

and

$$f'(1) = \frac{p}{2}\beta.$$

Since $|a_p| = 2\beta$, (1.6) is satisfied with equality. □

If $f(z)-f(0)$ has no zeros different from $z=0$ in Theorem 2.1, the inequality (1.6) can be further strengthened. This is given by the following theorem.

Theorem 2.2. *Let $f(z)=f(0)+a_pz^p+a_{p+1}z^{p+1}+\cdots$, $a_p>0$, $p\geq 2$, $p\in\mathbb{N}$ be a holomorphic function in the unit disc D and $f(z)-f(0)$ has no zeros in D except $z=0$, and let $\Re f(z)\leqslant A$ for $|z|<1$. Further assume that, for some $c\in\partial D$, f has an angular limit $f(c)$ at c, $\Re f(c)=A$. Then*

$$|f'(c)|\geqslant\frac{A-\Re f(0)}{2}\left(p-\frac{2\left|a_p\right|\left(\ln\frac{|a_p|}{2(A-\Re f(0))}\right)^2}{2\left|a_p\right|\ln\left(\frac{|a_p|}{2(A-\Re f(0))}\right)-\left|a_{p+1}\right|}\right) \tag{1.9}$$

and

$$\left|a_{p+1}\right|\leq 2\left|a_p\ln\left(\frac{|a_p|}{2\left(A-\Re f(0)\right)}\right)\right|. \tag{1.10}$$

In addition, the equality in (1.9) *occurs for the function*

$$f(z)=f(0)+2\left(A-\Re f(0)\right)\frac{z^p}{1+z^p}$$

and the equality in (1.10) *occurs for the function*

$$f(z)=f(0)+2\left(A-\Re f(0)\right)\frac{z^pe^{\frac{1+z}{1-z}\ln\left(\frac{a_p}{2(A-\Re f(0))}\right)}}{1+z^pe^{\frac{1+z}{1-z}\ln\left(\frac{a_p}{2(A-\Re f(0))}\right)}},$$

where $0<a_p<1$ and $\ln\left(\frac{a_p}{2(A-\Re f(0))}\right)<0$.

Proof. Let $a_p>0$. Let $\varphi(z)$, $h(z)$ and $B(z)$ be as in the proof of Theorem 2.1. From the inequality (1.7) and the function $f(z)-f(0)$ has no zeros in D except $z=0$, we denote by $\ln h(z)$ the holomorphic branch of the logarithm normed by the condition

$$\ln h(0)=\ln\frac{a_p}{2\beta}<0.$$

The function

$$\Phi(z)=\frac{\ln h(z)-\ln h(0)}{\ln h(z)+\ln h(0)}$$

is a holomorphic function in the unit disc D, $|\Phi(z)|<1$, $\Phi(0)=0$ and $|\Phi(c)|=1$ for $c\in\partial D$.

From (1.3), we obtain

$$\begin{aligned}\frac{2}{1+|\Phi'(0)|}\leq|\Phi'(c)|&=\frac{|2\ln h(0)|}{|\ln h(c)+\ln h(0)|^2}\left|\frac{h'(c)}{h(c)}\right|\\&=\frac{|2\ln h(0)|}{|\ln h(c)+\ln h(0)|^2}\left|h'(c)\right|\\&=\frac{|2\ln h(0)|}{|\ln h(c)+\ln h(0)|^2}\left|\frac{\varphi'(c)}{B(c)}-\frac{\varphi(c)B'(c)}{B^2(c)}\right|\end{aligned}$$

$$= \frac{|2\ln h(0)|}{|\ln h(c) + \ln h(0)|^2} \left| \frac{\varphi(c)}{cB(c)} \right| \left| \frac{c\varphi'(c)}{\varphi(c)} - \frac{cB'(c)}{B(c)} \right|$$
$$= \frac{-2\ln h(0)}{\ln^2 h(0) + \arg^2 h(c)} \{|\varphi'(c)| - |B'(c)|\}$$

and

$$\frac{2}{1 + |\Phi'(0)|} \leq \frac{-2\ln h(0)}{\ln^2 h(0) + \arg^2 h(c)} \{|\varphi'(c)| - |B'(c)|\}. \tag{1.11}$$

It can be seen that

$$\Phi'(z) = \frac{2\ln h(0)}{(\ln h(z) + \ln h(0))^2} \frac{h'(z)}{h(z)},$$
$$\Phi'(0) = \frac{2\ln h(0)}{(2\ln h(0))^2} \frac{h'(0)}{h(0)},$$
$$|\Phi'(0)| = \frac{1}{|2\ln h(0)|} \left| \frac{h'(0)}{h(0)} \right| = \frac{1}{-2\ln \frac{|a_p|}{2\beta}} \frac{|a_{p+1}|}{|a_p|}$$

and

$$|\Phi'(0)| = \frac{1}{-2\ln \frac{|a_p|}{2\beta}} \frac{|a_{p+1}|}{|a_p|}.$$

Let's substitute the values of $|\Phi'(0)|$, $|\varphi'(c)|$, $|B'(c)|$ and $\ln h(0)$ into (1.11). Therefore, we obtain

$$\frac{2}{1 - \frac{|a_{p+1}|}{2|a_p| \ln\left(\frac{|a_p|}{2\beta}\right)}} \leq \frac{-2\ln h(0)}{\ln^2 h(0) + \arg^2 h(c)} \left\{ \frac{2\beta\, |f'(c)|}{|2\beta - (f(c) - f(0))|^2} - p \right\}.$$

Replacing $\arg^2 h(c)$ by zero, we take

$$\frac{2}{1 - \frac{|a_{p+1}|}{2|a_p| \ln\left(\frac{|a_p|}{2\beta}\right)}} \leq \frac{-2}{\ln h(0)} \left\{ \frac{2\beta\, |f'(c)|}{|2\beta - (f(c) - f(0))|^2} - p \right\}$$
$$= \frac{-2}{\ln \frac{|a_p|}{2\beta}} \left\{ \frac{2\beta\, |f'(c)|}{|2\beta - (f(c) - f(0))|^2} - p \right\}$$

and

$$\frac{2\,|a_p| \ln\left(\frac{|a_p|}{2\beta}\right)}{2\,|a_p| \ln\left(\frac{|a_p|}{2\beta}\right) - |a_{p+1}|} \leq \frac{-1}{\ln \frac{|a_p|}{2\beta}} \left\{ \frac{2\beta\, |f'(c)|}{|2\beta - (f(c) - f(0))|^2} - p \right\}.$$

Since $|2\beta - (f(c) - f(0))|^2 \geq (\Re\, [2\beta - (f(c) - f(0))])^2 = \beta^2$, we get

$$p - \frac{2\,|a_p| \left(\ln \frac{|a_p|}{2\beta}\right)^2}{2\,|a_p| \ln\left(\frac{|a_p|}{2\beta}\right) - |a_{p+1}|} \leq \frac{2\beta\, |f'(c)|}{|2\beta - (f(c) - f(0))|^2} \leq \frac{2\,|f'(c)|}{\beta}.$$

Thus, we obtain (1.9) with an obvious equality case.

Similarly, $\Phi(z)$ function satisfies the assumptions of the Schwarz lemma, we obtain

$$1 \geq |b'(0)| = \frac{|2\ln h(0)|}{|\ln h(0) + \ln h(0)|^2} \left| \frac{t'(0)}{t(0)} \right|$$
$$= \frac{1}{|2\ln h(0)|} \left| \frac{t'(0)}{t(0)} \right|$$

and

$$1 \geq \frac{-1}{2\ln\left(\frac{|a_p|}{2\beta}\right)} \frac{|a_{p+1}|}{|a_p|}.$$

Therefore, we get the inequality (1.10).

Now, we shall show that the inequality (1.10) is sharp. Let

$$f(z) = z^p g(z),$$

where

$$g(z) = 2\left(A - \Re f(0)\right) \frac{e^{\frac{1+z}{1-z}\ln\left(\frac{a_p}{2(A-\Re f(0))}\right)}}{1 + z^p e^{\frac{1+z}{1-z}\ln\left(\frac{a_p}{2(A-\Re f(0))}\right)}}.$$

Then

$$g'(0) = a_{p+1}.$$

Under the simple calculations, we get

$$a_{p+1} = 2a_p \ln\left(\frac{a_p}{2\left(A - \Re f(0)\right)}\right).$$

Therefore, we obtain

$$|a_{p+1}| = 2\left|a_p \ln\left(\frac{|a_p|}{2\left(A - \Re f(0)\right)}\right)\right|.$$

□

Theorem 2.3. *Under hypotheses of Theorem 2.2, we have*

$$|f'(c)| \geqslant \frac{A - \Re f(0)}{2}\left[p - \frac{1}{2}\ln\frac{|a_p|}{2\left(A - \Re f(0)\right)}\right]. \tag{1.12}$$

The equality in (1.12) *holds if and only if*

$$f(z) = f(0) + 2\left(A - \Re f(0)\right) \frac{z^p e^{\frac{1+ze^{i\theta}}{1-ze^{i\theta}}\ln\left(\frac{a_p}{2(A-\Re f(0))}\right)}}{1 + z^p e^{\frac{1+ze^{i\theta}}{1-ze^{i\theta}}\ln\left(\frac{a_p}{2(A-\Re f(0))}\right)}},$$

where $0 < a_p < 1$, $\ln\left(\frac{a_p}{2(A-\Re f(0))}\right) < 0$ *and* θ *is a real number.*

Proof. From proof of Theorem 2.2, using the inequality (1.3) for the function $\Phi(z)$, we obtain

$$1 \leq |\Phi'(c)| = \frac{|2\ln h(0)|}{|\ln h(c) + \ln h(0)|^2} \left| \frac{h'(c)}{h(c)} \right|$$

$$
\begin{aligned}
&= \frac{|2\ln h(0)|}{|\ln h(c)+\ln h(0)|^2}\,|h'(c)| \\
&= \frac{|2\ln h(0)|}{|\ln h(c)+\ln h(0)|^2}\left|\frac{\varphi'(c)}{B(c)}-\frac{\varphi(c)B'(c)}{B^2(c)}\right| \\
&= \frac{|2\ln h(0)|}{|\ln h(c)+\ln h(0)|^2}\left|\frac{\varphi(c)}{cB(c)}\right|\left|\frac{c\varphi'(c)}{\varphi(c)}-\frac{cB'(c)}{B(c)}\right| \\
&= \frac{-2\ln h(0)}{\ln^2 h(0)+\arg^2 h(c)}\left\{|\varphi'(c)|-|B'(c)|\right\} \\
&= \frac{-2\ln h(0)}{\ln^2 h(0)+\arg^2 h(c)}\left\{\frac{2\beta\,|f'(c)|}{|2\beta-(f(c)-f(0))|^2}-p\right\}.
\end{aligned}
$$

Replacing $\arg^2 h(c)$ by zero and since

$$|2\beta-(f(c)-f(0))|^2 \geq (\Re\,[2\beta-(f(c)-f(0))])^2=\beta^2,$$

we get

$$1 \leq |\Phi'(c)| = \frac{-2}{\ln\frac{|a_p|}{2\beta}}\left\{\frac{2\,|f'(c)|}{\beta}-p\right\}. \tag{1.13}$$

Therefore, we have the inequality (1.12).

If $|f'(c)|=\frac{\beta}{2}\left(p-\frac{1}{2}\ln\frac{|a_p|}{2\beta}\right)$ from (1.13) and $|\Phi'(c)|=1$, we obtain

$$\Phi(z)=ze^{i\theta}$$

and

$$\frac{\ln h(z)-\ln h(0)}{\ln h(z)+\ln h(0)}=ze^{i\theta}.$$

So, we take

$$\ln h(z)=\frac{1+ze^{i\theta}}{1-ze^{i\theta}}\ln h(0)=\frac{1+ze^{i\theta}}{1-ze^{i\theta}}\ln\frac{a_p}{2\beta},$$

$$h(z)=e^{\frac{1+ze^{i\theta}}{1-ze^{i\theta}}\ln\frac{a_p}{2\beta}},$$

$$\frac{\varphi(z)}{B(z)}=e^{\frac{1+ze^{i\theta}}{1-ze^{i\theta}}\ln\frac{a_p}{2\beta}},$$

$$\frac{f(z)-f(0)}{2\beta-(f(z)-f(0))}=z^p e^{\frac{1+ze^{i\theta}}{1-ze^{i\theta}}\ln\frac{a_p}{2\beta}}$$

and

$$f(z)=f(0)+2\,(A-\Re f(0))\,\frac{z^p e^{\frac{1+ze^{i\theta}}{1-ze^{i\theta}}\ln\left(\frac{a_p}{2(A-\Re f(0))}\right)}}{1+z^p e^{\frac{1+ze^{i\theta}}{1-ze^{i\theta}}\ln\left(\frac{a_p}{2(A-\Re f(0))}\right)}}. \qquad \square$$

If $f(z)-f(0)$ have zeros different from $z=0$, taking into account these zeros, the inequality (1.6) can be strengthened in another way. This is given by the following theorem.

Theorem 2.4. *Let $f(z) = f(0) + a_p z^p + a_{p+1} z^{p+1} + \cdots$, $a_p \neq 0$, $p \geq 2$, $p \in \mathbb{N}$ be a holomorphic function in the unit disc D, and let $\Re f(z) \leqslant A$ for $|z| < 1$. Assume that for some $c \in \partial D$, f has an angular limit $f(c)$ at c, $\Re f(c) = A$. Let $z_1, z_2, \ldots, z_n$ be zeros of the function $f(z) - f(0)$ in D that are different from zero. Then we have the inequality*

$$|f'(c)| \geqslant \frac{A - \Re f(0)}{2} \left(p + \sum_{k=1}^{n} \frac{1 - |z_k|^2}{|c - z_k|^2} \right. \\ \left. + \frac{2\left(2((A-\Re f(0))) \prod_{k=1}^{n} |z_k| - |a_p|\right)^2}{4((A-\Re f(0)))^2 \left(\prod_{k=1}^{n} |z_k|\right)^2 - |a_p|^2 + 2(A-\Re f(0))|a_{p+1}| \prod_{k=1}^{n} |z_k|} \right). \tag{1.14}$$

In addition, the equality in (1.14) *occurs for the function*

$$f(z) = f(0) + 2\left(A - \Re f(0)\right) \frac{z^p \prod_{k=1}^{n} \frac{z - z_k}{1 - \overline{z_k} z}}{1 + z^p \prod_{k=1}^{n} \frac{z - z_k}{1 - \overline{z_k} z}},$$

where $z_1, z_2, \ldots, z_n$ are positive real numbers.

Proof. Let $\varphi(z)$ be as in the proof of Theorem 2.1 and $z_1, z_2, \ldots, z_n$ zeros of the function $f(z) - f(0)$ in D that are different from zero. Let

$$B_0(z) = z^p \prod_{k=1}^{n} \frac{z - z_k}{1 - \overline{z_k} z}.$$

$B_0(z)$ is a holomorphic function in D and $|B_0(z)| < 1$ for $|z| < 1$. By the maximum principle for each $z \in D$, we have

$$|\varphi(z)| \leq |B_0(z)|.$$

The function

$$k(z) = \frac{\varphi(z)}{B_0(z)}$$

is a holomorphic function in D, and $|k(z)| < 1$ for $|z| < 1$. In particular, we have

$$|k(0)| = \frac{|a_p|}{2\beta \prod_{k=1}^{n} |z_k|} \leq 1$$

and

$$|k'(0)| = \frac{|a_{p+1}|}{2\beta \prod_{k=1}^{n} |z_k|}.$$

Moreover, it can be seen that

$$\frac{c\varphi'(c)}{\varphi(c)} = |\varphi'(c)| \geq |B_0'(c)| = \frac{cB_0'(c)}{B_0(c)}.$$

It is obviously that

$$|B_0'(c)| = \frac{cB_0'(c)}{B_0(c)} = p + \sum_{k=1}^{n} \frac{1-|z_k|^2}{|c-z_k|^2}.$$

Let

$$\Omega(z) = \frac{k(z)-k(0)}{1-\overline{k(0)}k(z)}.$$

$\Omega(z)$ is a holomorphic function in the unit disc D, $|\Omega(z)| < 1$, $\Omega(0) = 0$ and $|\Omega(c)| = 1$ for $c \in \partial D$.

From (1.3), we obtain

$$\begin{aligned}
\frac{2}{1+|\Omega'(0)|} \le |\Omega'(c)| &= \frac{1-|k(0)|^2}{\left|1-\overline{k(0)}k(c)\right|^2} |k'(c)| \\
&= \frac{1-|k(0)|^2}{\left|1-\overline{k(0)}k(c)\right|^2} \left| \frac{\varphi'(c)}{B_0(c)} - \frac{\varphi(c)B_0'(c)}{B_0^2(c)} \right| \\
&= \frac{1-|k(0)|^2}{\left|1-\overline{k(0)}k(c)\right|^2} \left| \frac{\varphi(c)}{cB_0(c)} \right| \left| \frac{c\varphi'(c)}{\varphi(c)} - \frac{cB_0'(c)}{B_0(c)} \right| \\
&\le \frac{1+|k(0)|}{1-|k(0)|} \left\{ |\varphi'(c)| - |B_0'(c)| \right\}
\end{aligned}$$

and

$$\frac{2}{1+|\Omega'(0)|} \le \frac{1+|k(0)|}{1-|k(0)|} \left\{ |\varphi'(c)| - |B_0'(c)| \right\}. \tag{1.15}$$

It can be seen that

$$\Omega'(z) = \frac{1-|k(0)|^2}{\left(1-\overline{k(0)}k(z)\right)^2} k'(z),$$

$$\Omega'(0) = \frac{1-|k(0)|^2}{\left(1-|k(0)|^2\right)^2} k'(0)$$

and

$$|\Omega'(0)| = \frac{|k'(0)|}{1-|k(0)|^2} = \frac{\frac{|a_{p+1}|}{2\beta \prod_{k=1}^{n} |z_k|}}{1-\left(\frac{|a_p|}{2\beta \prod_{k=1}^{n} |z_k|}\right)^2} = \frac{2\beta |a_{p+1}| \prod_{k=1}^{n} |z_k|}{4\beta^2 \left(\prod_{k=1}^{n} |z_k|\right)^2 - |a_p|^2}.$$

Let's substitute the values of $|\Omega'(0)|$, $|\varphi'(c)|$, $|B_0'(c)|$ and $|k(0)|$ into (1.15). Therefore, we obtain

$$\frac{2}{1+\frac{2\beta|a_{p+1}|\prod_{k=1}^{n}|z_k|}{4\beta^2\left(\prod_{k=1}^{n}|z_k|\right)^2-|a_p|^2}} \leq \frac{2\beta\prod_{k=1}^{n}|z_k|+|a_p|}{2\beta\prod_{k=1}^{n}|z_k|-|a_p|}\left(\frac{2A\left|f'(c)\right|}{|2\beta-(f(c)-f(0))|^2}-p-\sum_{k=1}^{n}\frac{1-|z_k|^2}{|c-z_k|^2}\right),$$

$$\frac{2\left[4\beta^2\left(\prod_{k=1}^{n}|z_k|\right)^2-|a_p|^2\right]}{4\beta^2\left(\prod_{k=1}^{n}|z_k|\right)^2-|a_p|^2+2\beta|a_{p+1}|\prod_{k=1}^{n}|z_k|}\,\frac{2\beta\prod_{k=1}^{n}|z_k|-|a_p|}{2\beta\prod_{k=1}^{n}|z_k|+|a_p|}$$
$$\leq \left(\frac{2A\left|f'(c)\right|}{|2\beta-(f(c)-f(0))|^2}-p-\sum_{k=1}^{n}\frac{1-|z_k|^2}{|c-z_k|^2}\right)$$

and

$$\frac{2\left(2\beta\prod_{k=1}^{n}|z_k|-|a_p|\right)^2}{4\beta^2\left(\prod_{k=1}^{n}|z_k|\right)^2-|a_p|^2+2\beta|a_{p+1}|\prod_{k=1}^{n}|z_k|}+p+\sum_{k=1}^{n}\frac{1-|z_k|^2}{|c-z_k|^2}\leq\frac{2A\left|f'(c)\right|}{|f(c)-f(0)-2\beta|^2}.$$

Therefore, we have

$$|f'(c)| \geqslant \frac{A-\Re f(0)}{2}\left\{p+\sum_{k=1}^{n}\frac{1-|z_k|^2}{|c-z_k|^2}\right\}$$
$$+\frac{A-\Re f(0)}{2}\left\{\frac{2\left(2((A-\Re f(0)))\prod_{k=1}^{n}|z_k|-|a_p|\right)^2}{4((A-\Re f(0)))^2\left(\prod_{k=1}^{n}|z_k|\right)^2-|a_p|^2+2(A-\Re f(0))|a_{p+1}|\prod_{k=1}^{n}|z_k|}\right\}.$$

Now, we shall show that the inequality (1.14) is sharp. Let

$$f(z)=f(0)+2\beta\frac{z^p\prod_{k=1}^{n}\frac{z-z_k}{1-\overline{z_k}z}}{1+z^p\prod_{k=1}^{n}\frac{z-z_k}{1-\overline{z_k}z}}.$$

Then

$$f'(z)=2\beta\frac{\left(pz^{p-1}\prod_{k=1}^{n}\frac{z-z_k}{1-\overline{z_k}z}+\sum_{k=1}^{n}\frac{1-|z_k|^2}{(1-\overline{z_k}z)^2}\prod_{\substack{k\neq i\\ i=1}}^{n}\frac{z-z_i}{1-\overline{z_i}z}z^p\right)\left(1+z^p\prod_{k=1}^{n}\frac{z-z_k}{1-\overline{z_k}z}\right)}{\left(1+z^p\prod_{k=1}^{n}\frac{z-z_k}{1-\overline{z_k}z}\right)^2}$$

$$- 2\beta \frac{\left(pz^{p-1} \prod_{k=1}^{n} \frac{z-z_k}{1-\overline{z_k}z} + \sum_{k=1}^{n} \frac{1-|z_k|^2}{(1-\overline{z_k}z)^2} \prod_{\substack{k \neq i \\ i=1}}^{n} \frac{z-z_i}{1-\overline{z_i}z} z^p \right) z^p \prod_{k=1}^{n} \frac{z-z_k}{1-\overline{z_k}z}}{\left(1 + z^p \prod_{k=1}^{n} \frac{z-z_k}{1-\overline{z_k}z}\right)^2}$$

and

$$f'(1) = 2\beta \frac{\left(p \prod_{k=1}^{n} \frac{1-z_k}{1-\overline{z_k}} + \sum_{k=1}^{n} \frac{1-|z_k|^2}{(1-\overline{z_k})^2} \prod_{\substack{k \neq i \\ i=1}}^{n} \frac{1-z_i}{1-\overline{z_i}} \right) \left(1 + \prod_{k=1}^{n} \frac{1-z_k}{1-\overline{z_k}}\right)}{\left(1 + \prod_{k=1}^{n} \frac{1-z_k}{1-\overline{z_k}}\right)^2}$$

$$- 2\beta \frac{\left(p \prod_{k=1}^{n} \frac{1-z_k}{1-\overline{z_k}} + \sum_{k=1}^{n} \frac{1-|z_k|^2}{(1-\overline{z_k})^2} \prod_{\substack{k \neq i \\ i=1}}^{n} \frac{1-z_i}{1-\overline{z_i}} \right) \prod_{k=1}^{n} \frac{1-z_k}{1-\overline{z_k}}}{\left(1 + \prod_{k=1}^{n} \frac{1-z_k}{1-\overline{z_k}}\right)^2}.$$

Since $z_1, z_2, \ldots, z_n$ are positive real numbers, we take

$$f'(1) = \frac{\beta}{2} \left(p + \sum_{k=1}^{n} \frac{1+z_k}{1-z_k} \right).$$

Since $|a_p| = 2\beta \prod_{k=1}^{n} |z_k|$, (1.14) is satisfied with equality. □

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